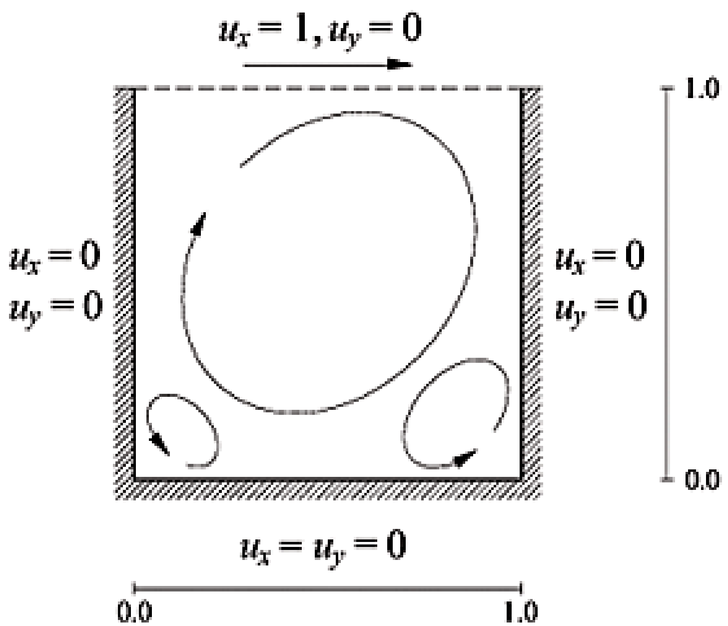
FINITE DIFFERENCES METHOD - TYPICAL PROBLEMS (part I)

Lid-Driven Problem

(K.A.Hoffman, S.T. Chiang, Computational Fluid Dynamics, EES, Wichita, 2000)

(see videos)

Introduction



Mathematical model

Navier-Stokes equations (incompressible viscous flow) using primitive variables formulation

Vector form

$$\nabla \cdot \vec{V} = 0$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} + \frac{\nabla p}{\rho} = \nu \nabla^2 \vec{V}$$

Two-dimensional Cartesian coordinate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Dimensionless

Two-dimensional Cartesian coordinate

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{8-18}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \frac{\partial p^*}{\partial x^*} = \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{8-19}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + \frac{\partial p^*}{\partial y^*} = \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{8-20}$$

The variables in the equations above are nondimensionalized as follows,

$$t^* = \frac{t u_\infty}{L} \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L}$$
$$u^* = \frac{u}{u_\infty} \qquad v^* = \frac{v}{u_\infty} \qquad p^* = \frac{p}{\rho_\infty u_\infty^2} \tag{8-21}$$

where L is a characteristic length, and ρ_∞ and u_∞ are the reference (e.g., freestream) density and velocity, respectively. The nondimensional parameter Reynolds number is defined as

$$Re = \frac{\rho_\infty u_\infty L}{\mu_\infty}$$

Indeed

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t^*}\left(U_\infty \cdot u^*\right)\frac{\partial t^*}{\partial t} = \frac{U_\infty^2}{L}\cdot\frac{\partial u^*}{\partial t^*}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x^*}\left(u_\infty \cdot u\right)\cdot\frac{\partial x^*}{\partial x} = \frac{u_\infty}{L}\cdot\frac{\partial u^*}{\partial x^*}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x^*}\left(\frac{u_\infty}{L}\cdot\frac{\partial u^*}{\partial x^*}\right)\cdot\frac{\partial x^*}{\partial x} = \frac{u_\infty}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x^*}\left(p^* \rho_\infty u_\infty^2\right)\cdot\frac{\partial x^*}{\partial x} = \rho_\infty u_\infty^2\cdot\frac{1}{L}\cdot\frac{\partial p^*}{\partial x^*}$$

Similar equations for the derivative with respect to *y* and the derivatives of *v* are obtained.

Use the above expressions in the dimensional equations

Continuity

$$\frac{U_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{U_\infty}{L}\cdot\frac{\partial v^*}{\partial y^*} = 0 \implies \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum: (Ox)

$$\frac{U_\infty^2}{L}\cdot\frac{\partial u^*}{\partial t^*} + U_\infty\cdot u^*\frac{U_\infty}{L}\cdot\frac{\partial u^*}{\partial x^*} + U_\infty\cdot v^*\cdot\frac{U_\infty}{L}\cdot\frac{\partial u^*}{\partial y^*} + \frac{1}{\rho}\cdot\rho_\infty U_\infty^2\cdot\frac{1}{L}\frac{\partial p^*}{\partial x^*} =$$

$$= \nu\left(\frac{U_\infty}{L^2}\cdot\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U_\infty}{L^2}\cdot\frac{\partial^2 u^*}{\partial y^{*2}}\right) \Bigg| : \frac{U_\infty^2}{L}$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\cdot\frac{\partial u^*}{\partial y^*} + \frac{\partial p^*}{\partial x^*} = \underbrace{\frac{\nu}{L\cdot U_\infty}}_{=1/Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$Re = \frac{U_\infty\cdot L}{\nu} = \frac{\frac{m}{s}\cdot m}{\frac{m^2}{s}}$$

In a similar way the momentum equation in *y* direction is obtained. One have to notice that Re is dimensionless.

Primitive variable formulation

(a) The governing equations are a mixed elliptic-parabolic system of equations which are solved simultaneously. The unknowns in the equations are velocity and pressure.

(b) There is no direct link for the pressure between the continuity and momentum equations. To establish a connection between the two equations, mathematical manipulations are introduced. Generally speaking there are two procedures for this purpose. The first is that of the Poisson equation for pressure which is developed in the next section; and the second is the introduction of artificial compressibility into the continuity equation.

(c) Specification of boundary conditions and in particular for pressure may be nonexistent. To overcome this difficulty, a special procedure must be introduced.

(d) Extension to three dimensions is straightforward with the least amount of complications.

Poisson Equation for Pressure: Primitive Variables

In this section an equation is developed which may be used for the computation of the pressure field. The reason for incorporating the Poisson equation for pressure, which is usually used in lieu of the continuity equation, is the lack of a direct link for pressure between continuity and momentum equations. A typical numerical scheme for the solution of the Poisson equation for pressure is investigated in the subsequent sections. For the time being, the steps required to obtain such a formulation are illustrated. The conservative form of the x- and y-components of the momentum equation obtained previously are

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(uv) = \frac{1}{Re}\nabla^2 u \tag{8-41}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial p}{\partial y} = \frac{1}{Re}\nabla^2 v \tag{8-42}$$

Equations (8-41) and (8-42) are differentiated with respect to x and y, respectively, to provide

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2}{\partial x \partial y}(uv) = \frac{1}{Re}\frac{\partial}{\partial x}(\nabla^2 u) \tag{8-43}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial x \partial y}(uv) + \frac{\partial^2}{\partial y^2}(v^2) + \frac{\partial^2 p}{\partial y^2} = \frac{1}{Re}\frac{\partial}{\partial y}(\nabla^2 v) \tag{8-44}$$

Addition of equations (8-43) and (8-44) yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial x^2}(u^2) + 2\frac{\partial^2}{\partial x \partial y}(uv) + \frac{\partial^2}{\partial y^2}(v^2) + \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}$$
$$= \frac{1}{Re}\left[\frac{\partial}{\partial x}(\nabla^2 u) + \frac{\partial}{\partial y}(\nabla^2 v)\right] \tag{8-45}$$

The right-hand side is rearranged as

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Finally, Equation (8-45) can be rewritten in the form of Poisson equation

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=-\frac{\partial D}{\partial t}-\frac{\partial^2}{\partial x^2}(u^2)-2\frac{\partial^2}{\partial x \partial y}(uv)-\frac{\partial^2}{\partial y^2}(v^2)+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}(D)+\frac{\partial^2}{\partial y^2}(D)\right] \tag{8-46}$$

where

$$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

is known as dilatation.

It is obvious that for an incompressible flow, the dilatation term is zero by continuity. However, due to numerical considerations, this term will not be set to zero.

Vorticity-Stream Function Formulations

The vorticity at a fluid point is defined as twice the angular velocity and is

$$\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{V}$$

which, for a two-dimensional flow, is reduced to

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{8-23}$$

Now, for a two-dimensional, incompressible flow, a function may be defined which satisfies the continuity equation. Such a function is known as the *stream function* and, in Cartesian coordinate system, is given by

$$u = \frac{\partial \psi}{\partial y} \tag{8-24}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{8-25}$$

From a physical point of view, the lines of constant ψ represent stream lines, and the difference in the values of ψ between two streamlines gives the volumetric flow rate between the two.

In order to derive the vorticity transport equation, the pressure is eliminated from the momentum equations by cross-differentiation. Differentiation with respect to y of Equation (8-9) yields

$$\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \nu\left(\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3}\right) \quad (8\text{-}26)$$

whereas the differentiation with respect to x of Equation (8-10) yields

$$\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \nu\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial y^2}\right) \quad (8\text{-}27)$$

Subtract Equation (8-27) from Equation (8-26) to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$
$$= \nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right]$$

Note that the fourth term on the left-hand side is zero by continuity. Now, upon substitution of the vorticity defined by (8-23), one obtains

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \nu\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right) \quad (8\text{-}28)$$

where the subscript z is dropped from Ω_z. Thus, in the remainder of this section, Ω will designate the z-component of the vorticity, unless otherwise specified. Equation (8-28) is known as the *vorticity transport equation* and is classified as a parabolic equation with the unknown being the vorticity Ω.

Now, reconsider the definition of vorticity given by

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{8-29}$$

Substitution of relations (8-24) and (8-25) yields

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega \tag{8-30}$$

This equation is known as the *stream function equation* and is classified as an elliptic PDE. The unknown is the stream function ψ, whose Ω is provided from the solution of Equation (8-28). Once the stream function has been computed, the velocity components may be determined from relations (8-24) and (8-25).

The vorticity equation may be expressed in a nondimensional form by using the nondimensional quantities defined previously and a nondimensional vorticity defined as

$$\Omega^* = \frac{\Omega L}{u_\infty}$$

The nondimensional form of the vorticity equation can be expressed as

$$\frac{\partial \Omega^*}{\partial t^*} + u^* \frac{\partial \Omega^*}{\partial x^*} + v^* \frac{\partial \Omega^*}{\partial y^*} = \frac{1}{Re_\infty} \left(\frac{\partial^2 \Omega^*}{\partial x^{*2}} + \frac{\partial^2 \Omega^*}{\partial y^{*2}} \right) \tag{8-31}$$

Similarly, the nondimensional form of the stream function equation given by Equation (8-30) is

$$\frac{\partial^2 \psi^*}{\partial x^{*2}} + \frac{\partial^2 \psi^*}{\partial y^{*2}} = -\Omega^* \tag{8-32}$$

where

$$\psi^* = \frac{\psi}{u_\infty L}$$

A summary of the vorticity-stream function formulations is provided below.

$$\frac{\partial \Omega^*}{\partial t^*} + u^* \frac{\partial \Omega^*}{\partial x^*} + v^* \frac{\partial \Omega^*}{\partial y^*} = \frac{1}{Re} \left(\frac{\partial^2 \Omega^*}{\partial x^{*2}} + \frac{\partial^2 \Omega^*}{\partial y^{*2}} \right) \tag{8-39}$$

$$\frac{\partial^2 \psi^*}{\partial x^{*2}} + \frac{\partial^2 \psi^*}{\partial y^{*2}} = -\Omega^* \tag{8-40}$$

Vorticity-stream function formulation

(a) By introduction of new variables, namely the vorticity and the stream function, the incompressible Navier-Stokes equations are decoupled into one elliptic equation and one parabolic equation which can be solved sequentially.

(b) Vorticity-stream function formulation does not include the pressure term. Therefore, the velocity field is determined initially and, subsequently, the Poisson equation for pressure (which is described in the next section) is employed to solve for the pressure field.

(c) Due to lack of a simple stream function in three dimensions, extension of the vorticity-stream function formulation to three dimensions loses its attractiveness.

Poisson Equation for Pressure: Vorticity-Stream Function Formulation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(uv) = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (8\text{-}47)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial p}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (8\text{-}48)$$

Equation (8-47) is now differentiated with respect to x to provide

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) = \frac{1}{Re}\frac{\partial}{\partial x}(\nabla^2 u)$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2u\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 p}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{Re}\frac{\partial}{\partial x}(\nabla^2 u)$$

The second term (only one) and the fifth term are combined to provide

$$\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

due to continuity. Similarly, term three (only one) and term six are added to yield

$$u\frac{\partial^2 v}{\partial x\partial y}+u\frac{\partial^2 u}{\partial x^2}=u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Thus, one has

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 p}{\partial x^2}=\frac{1}{Re}\frac{\partial}{\partial x}(\nabla^2 u) \qquad (8\text{-}49)$$

Similarly, the y-component of momentum becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)+\left(\frac{\partial v}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 p}{\partial y^2}=\frac{1}{Re}\frac{\partial}{\partial y}(\nabla^2 v) \qquad (8\text{-}50)$$

Addition of Equations (8-49) and (8-50) yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)$$

$$+v\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)+u\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\frac{1}{Re}\left[\frac{\partial}{\partial x}(\nabla^2 u)+\frac{\partial}{\partial y}(\nabla^2 v)\right] \qquad (8\text{-}51)$$

Note that the first, fifth, and sixth terms each contain the continuity equation and therefore disappear

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

and

$$\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

The right-hand side is now rearranged to provide

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y \partial x^2}+\frac{\partial^3 v}{\partial x^3}$$

$$=\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Therefore Equation (8-51) is reduced to

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=-\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right) \tag{8-52}$$

Now, the left-hand side can be further reduced by considering the continuity equation as follows

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)=0$$

from which

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=-2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)$$

Substituting into Equation (8-52) yields

$$-\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) \tag{8-53}$$

This equation can be written in terms of the stream function by using relations (8-24) and (8-25)

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=2\left[\left(\frac{\partial^2 \psi}{\partial x^2}\right)\left(\frac{\partial^2 \psi}{\partial y^2}\right)-\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right] \tag{8-54}$$

A numerical scheme

$$\frac{\Omega_{i,j}^{n+1} - \Omega_{i,j}^{n}}{\Delta t} + u_{i,j}^{n} \frac{\Omega_{i+1,j}^{n} - \Omega_{i-1,j}^{n}}{2\Delta x} + v_{i,j}^{n} \frac{\Omega_{i,j+1}^{n} - \Omega_{i,j-1}^{n}}{2\Delta y}$$

$$= \frac{1}{Re}\left[\frac{\Omega_{i+1,j}^{n} - 2\Omega_{i,j}^{n} + \Omega_{i-1,j}^{n}}{(\Delta x)^2} + \frac{\Omega_{i,j+1}^{n} - 2\Omega_{i,j}^{n} + \Omega_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[(\Delta x)^2 \Omega_{i,j}^{n+1} + \psi_{i+1,j}^{k} + \psi_{i-1,j}^{k+1} + \beta^2(\psi_{i,j+1}^{k} + \psi_{i,j-1}^{k+1})\right]$$

where $\beta = \dfrac{\Delta x}{\Delta y}$.

Boundary conditions

The stream function equation is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega \qquad (8\text{-}105)$$

A solid surface can be considered as a stream line and, therefore, the stream function is constant and its value may be assigned arbitrarily. As mentioned previously, boundary conditions for the vorticity do not exist. Therefore, a set of boundary conditions must be constructed. The procedure involves the stream function equation along with Taylor series expansion of the stream function. As a result, a different formulation with various orders of approximation can be derived. At this point, the construction of a first-order expression is illustrated. Subsequently, a second-order relation is provided. For illustration purposes, assume non-porous and stationary surfaces and a rectangular domain, as shown in Fig. 8-9. The ex-

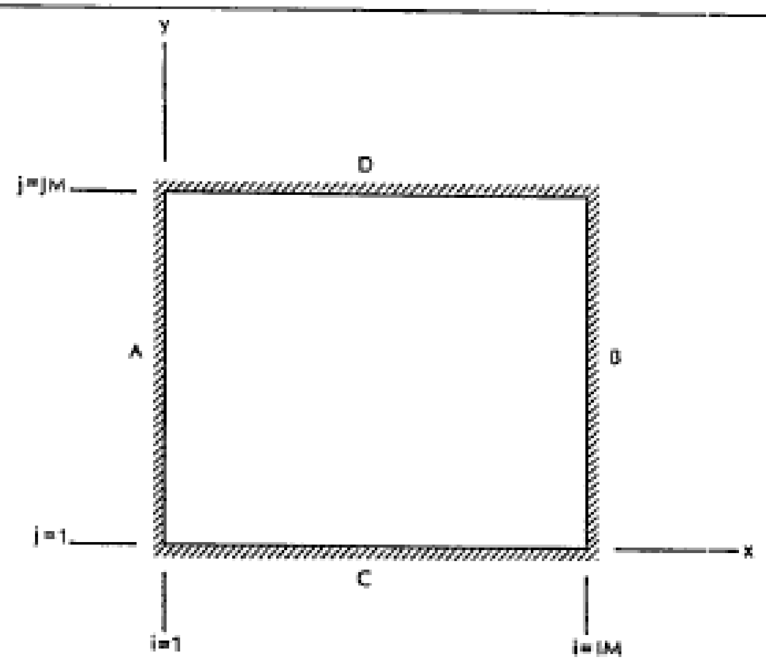


Figure 8-9. The rectangular cavity with fixed surfaces.

pression for the vorticity to be applied at boundary A is determined initially and, subsequently, the result is extended to the other boundaries at B, C, and D.

Consider Equation (8-105) at point $(1, j)$, i.e.,

$$\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)_{1,j} = -\Omega_{1,j} \tag{8-107}$$

Along the surface, the stream function is constant, and its value may be specified arbitrarily; for example, $\psi_{1,j} = \psi_1$. Then, along A,

$$\left.\frac{\partial^2\psi}{\partial y^2}\right|_{1,j} = 0$$

and Equation (8-107) is then reduced to

$$\left.\frac{\partial^2\psi}{\partial x^2}\right|_{1,j} = -\Omega_{1,j} \tag{8-108}$$

To obtain an expression for the second-order derivative in the equation above, consider the Taylor series expansion

$$\psi_{2,j} = \psi_{1,j} + \left.\frac{\partial\psi}{\partial x}\right|_{1,j}\Delta x + \left.\frac{\partial^2\psi}{\partial x^2}\right|_{1,j}\frac{(\Delta x)^2}{2} + \cdots$$

Along boundary A

$$v_{1,j} = -\frac{\partial \psi}{\partial x}\bigg|_{1,j} = 0$$

Therefore,

$$\psi_{2,j} = \psi_{1,j} + \frac{\partial^2 \psi}{\partial x^2}\bigg|_{1,j} \frac{(\Delta x)^2}{2} + O(\Delta x)^3$$

from which

$$\frac{\partial^2 \psi}{\partial x^2}\bigg|_{1,j} = \frac{2(\psi_{2,j} - \psi_{1,j})}{(\Delta x)^2} + O(\Delta x) \qquad (8\text{-}109)$$

Substitution of (8-109) into (8-108) yields

$$\Omega_{1,j} = \frac{2(\psi_{1,j} - \psi_{2,j})}{(\Delta x)^2} \qquad (8\text{-}110)$$

A similar procedure is used to derive the boundary conditions at boundaries B, C, and D. The appropriate expressions are, respectively,

$$\Omega_{IM,j} = -\frac{\partial^2 \psi}{\partial x^2}\bigg|_{IM,j} = \frac{2(\psi_{IM,j} - \psi_{IMM1,j})}{(\Delta x)^2} \qquad (8\text{-}111)$$

$$\Omega_{i,1} = -\frac{\partial^2 \psi}{\partial y^2}\bigg|_{i,1} = \frac{2(\psi_{i,1} - \psi_{i,2})}{(\Delta y)^2} \qquad (8\text{-}112)$$

$$\Omega_{i,JM} = -\frac{\partial^2 \psi}{\partial y^2}\bigg|_{i,JM} = \frac{2(\psi_{i,JM} - \psi_{i,JMM1})}{(\Delta y)^2} \qquad (8\text{-}113)$$

Now suppose a boundary is moving with some specified velocity. For example, assume that the upper surface is moving to the right with a constant velocity u_0. Following the procedure described previously, the Taylor series expansion yields

$$\psi_{i,j-1} = \psi_{i,j} - \left.\frac{\partial \psi}{\partial y}\right|_{i,j} \Delta y + \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,j} \frac{(\Delta y)^2}{2} + \cdots$$

or

$$\psi_{i,JMM1} = \psi_{i,JM} - u_0 \Delta y - \Omega_{i,JM} \frac{(\Delta y)^2}{2}$$

from which

$$\Omega_{i,JM} = \frac{2(\psi_{i,JM} - \psi_{i,JMM1})}{(\Delta y)^2} - \frac{2u_0}{\Delta y} \qquad (8\text{-}114)$$

A second-order equivalent of (8-110) can be expressed as

$$\Omega_{1,j} = \frac{\psi_{3,j} - 8\psi_{2,j} + 7\psi_{1,j}}{2(\Delta x)^2} + O(\Delta x)^2 \qquad (8\text{-}115)$$

For a moving boundary with a constant velocity of u_0 at $j = JM$, one has

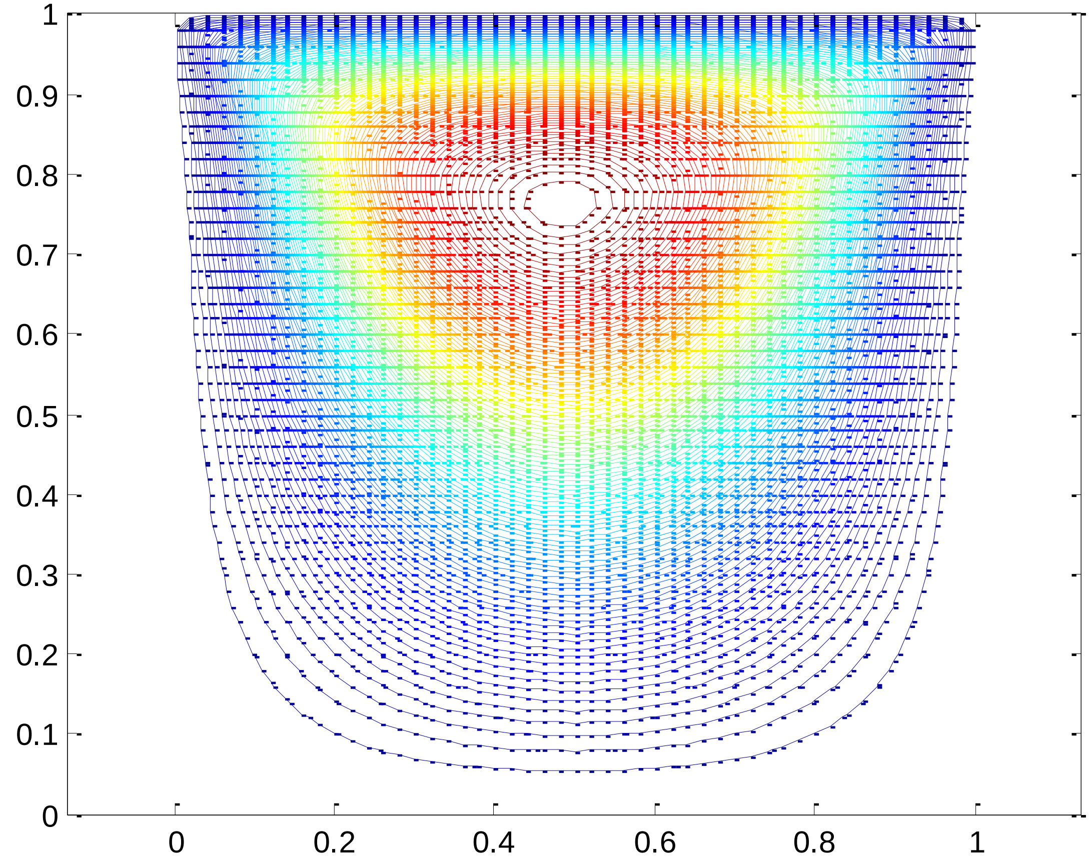
$$\Omega_{i,JM} = \frac{-\psi_{i,JM} + 8\psi_{i,JMM1} - 7\psi_{i,JMM2}}{2(\Delta y)^2} - \frac{3u_0}{\Delta y} \qquad (8\text{-}116)$$

which is a second-order equivalence of (8-114). Higher order implementation of the boundary conditions in general will increase the accuracy of the solution. However, it has been shown that it may cause instabilities in high Reynolds number flow.

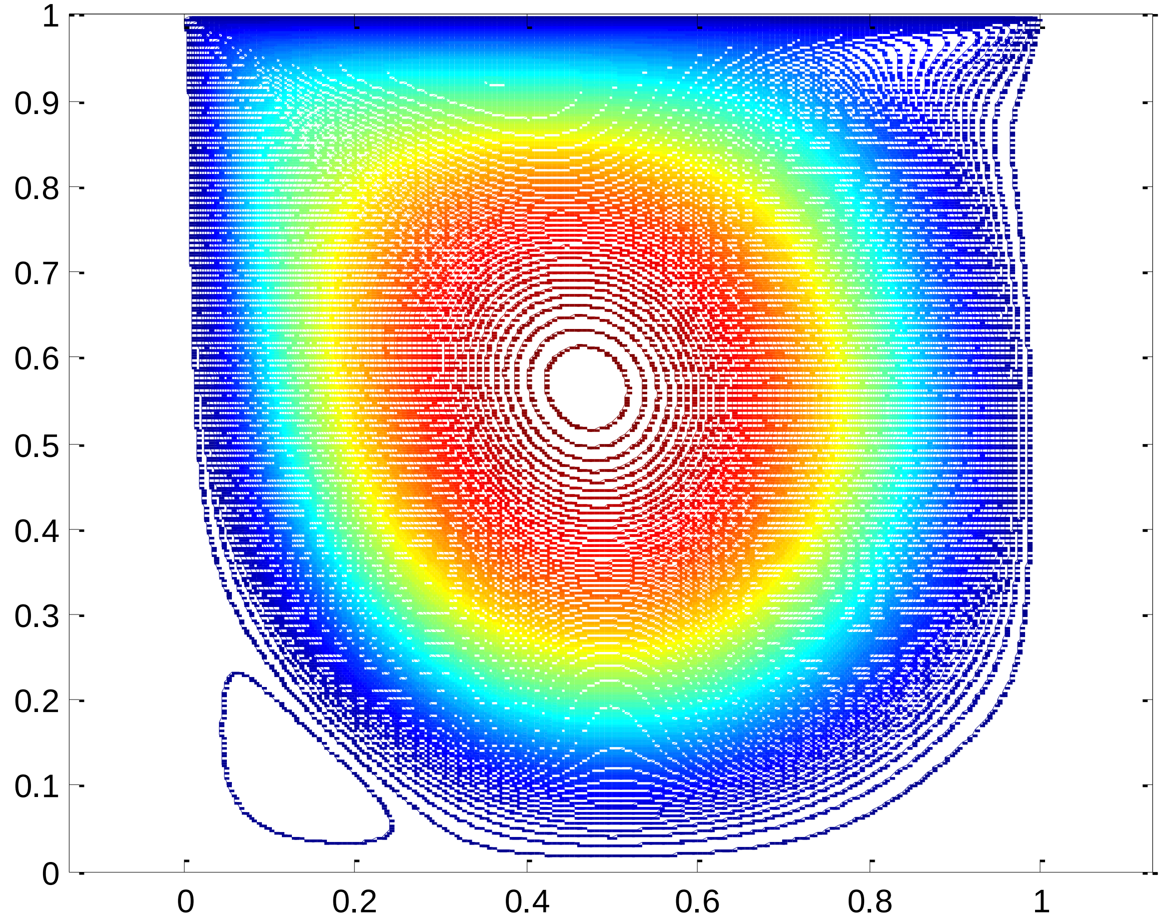
Matlab program

```
format long g;
tic
N=201;h=1/(N-1);
xplot=0:h:1;yplot=0:h:1;

Re=1000;

u=zeros(N,N);O=zeros(N,N);uo=zeros(N,N);Oo=zeros(N,N);
Utop=-1;

Oo(:,N)=2*(u(:,N)-u(:,N-1))/h/h-2*Utop/h;

stop=1;nr_it=0;rO=0.9;ru=1;
while stop==1
 nr_it=nr_it+1;

 for i=2:N-1
  for j=2:N-1
   O(i,j)=0.0625*Re*((Oo(i,j+1)-O(i,j-1))*(uo(i+1,j)-u(i-1,j))-(Oo(i+1,j)-
         O(i-1,j))*(uo(i,j+1)-u(i,j-1)))+0.25*(Oo(i+1,j)+O(i-1,j)+Oo(i,j+1)+O(i,j-1));
   O(i,j)=rO*O(i,j)+(1-rO)*Oo(i,j);
   u(i,j)=0.25*(uo(i+1,j)+u(i-1,j)+uo(i,j+1)+u(i,j-1))+0.25*h*h*O(i,j);
  end;
 end;
     O(1,:)=-2*(u(2,:)-u(1,:))/h/h;
     O(N,:)=2*(u(N,:)-u(N-1,:))/h/h;
     O(:,1)=-2*(u(:,2)-u(:,1))/h/h;
```

```
     O(:,N)=2*(u(:,N)-u(:,N-1))/h/h-2*Utop/h;

    errU=norm(u-uo);errO=norm(O-Oo);

      if mod(nr_it,1000)==0
         fprintf('nr_it=%d  errU=%g  errO=%g\n', nr_it, errU, errO);
         if (errU<1e-6)&(errO<1e-6)
             stop=0;
         end
      end
      uo=u;Oo=O;

end;
contour(xplot,yplot,u',100);
axis equal
max(max(abs(u)))
toc
```



$Ra = 10, \qquad \psi_{max} = 0.099981, \qquad 51\ x\ 51\ grid, \ \ 17\ seconds\ CPU\ time$



$Ra = 1000,\qquad \psi_{max} = 0.117519,\qquad 201\ x\ 201\ grid,\ \ 4954\ seconds\ CPU\ time$

Table 2
Comparison of the four third-order schemes on the primary vortex at $Re = 1000$: maximum of the stream-function, vorticity and location

Scheme	Grid	ψ_{max}	ω	x	y
Present	128×128	0.11786	2.0508	0.46875	0.5625
Upwind 3	128×128	0.11796	2.0549	0.46875	0.5625
Kawamura	128×128	0.11790	2.0557	0.46875	0.5625
Quickest	128×128	0.11503	1.9910	0.46875	0.5625
Present	256×256	0.11865	2.0634	0.46875	0.5664
Upwind 3	256×256	0.11870	2.0644	0.46875	0.5664
Kawamura	256×256	0.11867	2.0636	0.46875	0.5664
Quickest	256×256	0.11599	2.0069	0.46875	0.5664

C-H Bruneau, M. Saad, The 2D lid-driven cavity problem revisited, Computers&Fluids 35 (2006) 326-348.