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TOPICS IN

NONLINEAR ANALYSIS

AND APPLICATIONS TO

MATHEMATICAL ECONOMICS

House of the Book of Science 2007

Reviewer:

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to Lenuţa, Anca, Silvia, Gabriela, Mara and Lisa

with love

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Preface

Mathematical Economics is one of the most dynamic domain in the field of Applied Mathematics, with a impetuous development in the last years. Multivalued Analysis Theory and Fixed Point Theory are two of the today's strong tools for new investigations in Nonlinear Analysis, in general, and in Mathematical Economics area, in particular.

This new book is based, to a great extent, on the first two authors former book "Multivalued Analysis and Mathematical Economics", published in 2003.

Since then, the second author used some parts of the above mentioned book during the one semester course in Mathematical Economics for the Applied Mathematics Master Program in the Faculty of Mathematics and Computer Science from Babeş-Bolyai University Cluj-Napoca. This new book reflects this experience. Also, the first author presented some chapters of the book, during his visits to some USA universities. Some remarks of the audience are here included. Last but not least, this book contains several new results, part of the Ph.D. Dissertation, of the third author.

Finally, we would like to mention that we took into account of the comments and remarks from the reviews of the 2003 book, from Zentralblatt fur Mathematik (European Mathematical Society) and from Mathematical Reviews (American Mathematical Society). We thank all these colleagues for their help.

We do hope that this new book will be useful for researchers and graduate, postgraduate or Ph.D. students in nonlinear analysis and mathematical economics.

The Authors October 2006

Introduction

The main aim of this monograph is to give an outline of various formal models of games and economies that have been developed in order to rigorously and formally govern the economic processus.

We would like to show how the purely mathematical results, especially those in connection with nonlinear analysis, are relevant to the economic topics. The tools we will use in this respect are **fixed point theory** and **multivalued analysis theory**. An important approach in the same direction is based on K^2M **operator technique**. The book ends with some mathematical economics results based on a **topological approach**.

A. Arrow-Debreu model of an economy. Let us consider first the so-called Arrow-Debreu model. The presentation will be brief. A more detailed description and several justifications can be found in Debreu [54], Border [28] or Isac [86]. Let's start by presenting the main elements of an abstract economy.

The fundamental idealization made in modeling an economy is the notion of commodity. We suppose that it is possible to classify all the different goods and services in the world economy into a finite number. Let say m commodities, which are available in infinitely divisible units. The commodity space is $\mathbb{R}^m$. A vector $x \in \mathbb{R}^m$ specifies a list of quantities of each commodity. There are commodity vectors that are exchanged or manufactured or consumed in economic activities and not individual commodities. Of course, if $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ it is possible that some quantities x_i, $i \in \{1, \cdots, m\}$ to be equal to zero. We will denote by E the set of all available commodities.

A price vector p lists the value of a unit of each commodity and so $p \in \mathbb{R}^m$.

The value of the commodity vector x, when on the market acts the price system p is the inner product $p \cdot x = \sum_{i=1}^{m} p_i x_i$.

Let us make now an important remark: the assumption of the existence of only a finite number of distinct commodities can be eliminated. So, it is possible to consider economies with an infinite number of distinct commodities. In this case the commodity space is an infinite-dimensional vector space and the price vector belongs to the dual space of the commodity space. For some references of this topic, see, for example, the book of Aliprantis, Brown and Burkinshaw [3].

The consumers are the main actors of an economy. The ultimate purpose of an economic organization is to provide commodity vectors for final consumption by consumers. We will assume that there exists a finite given number of consumers.

It is quite obviously that not every commodity vector is admissible as a final consumption for a consumer. We will denote by $X \subset \mathbb{R}^m$ the set of all admissible consumption vectors for a given consumer. (or $X_i \subset \mathbb{R}^m$ if we discuss about the consumer i) So, X (or X_i) is the consumption set. What restrictions can be placed on the consumption set ?

A first restriction is that the admissible consumption vectors are non-negative.

An alternative restriction is that the consumption set is bounded below. Under this interpretation, negative quantities of a commodity in a final consumption vector mean that the consumer is supplying the commodity as a service. The lower bound puts a limit in the services that a consumer can provide. Also, the lower bound could be interpreted as a minimum requirement of some commodity for the consumer.

In a private ownership economy consumers are also characterized by their initial endowment of commodities. This is an element w (or w_i) in the commodity space. These are the resources the consumer owns.

In a market economy, a consumer must purchase his consumption vector at the market prices. The set of all admissible commodity vectors that he can afford at prices p, given an income M (or M_i) is called the budget set and will be denoted by A (or A_i). The budget set can be represented as:

$$A = \{x \in X | p \cdot x \leq M\}.$$

Of course, the budget set can be also empty.

The problem faced by a consumer in a market economy is to choose a consumption vector or a set of them from the budget set. To do this, the consumer must have some criteria for choosing. A first method to formalize the criterion is to assume that the consumer has a utility index, that is a real-valued function u (or u_i) defined on the set of consumption vectors. The idea is that a consumer would prefer to consume vector x rather than vector y if $u(x) > u(y)$ and it would be indifferent if $u(x) = u(y)$. A solution to the consumer's problem is to find all the vectors x which maximize u on the budget set. This kind of problem is not so easy like it seems. But, if some restrictions are placed on the utility index, for example if the function u is continuous and the budget set A is compact, then from the well-known theorem of Weierstrass, we get that there exist vectors that maximize the value of u over the budget set, and so the proposed problem has at least a solution. Unfortunately, these assumptions on the consumer's criterion are somewhat severe, because we would like that the consumer's preferences to mirror the order properties of real numbers for example, if $u(x^1) = u(x^2), u(x^2) = u(x^3), \cdots, u(x^{n-1}) = u(x^n)$ then $u(x^1) = u(x^n)$, but on the other hand one can easily imagine situations where a consumer is indifferent between x^1 and x^2, between x^2 and x^3, etc but not between x^1 and x^n. Of course, there are weaker assumptions we can make about the preferences. These approaches involve multivalued operators, in order to describe a consumer's preferences. To do this, let us denote by $U(x)$ the set of all consumption vectors which consumer strictly prefer to x, i. e.

$$U(x) = \{y \in A | y \text{ is strictly preferred to } x\},\ x \in A.$$

Obviously, $U : A \multimap A$ and it is called the preference multifunction or the multivalued operator of preferences. (for example, in terms of the utility function, we have $U(x) = \{y \in A | u(y) > u(x)\}$.)

If we consider the abstract preference multifunction U then a vector $x^* \in A$ is an optimal preference for a given consumer if and only if $U(x^*) = \emptyset$. Such elements x^* are also called U-maximal or simply maximal. It is easy to see that any fixed point result for a multifunction generate an existence result for an U-maximal element of the above preference multifunction. Indeed, let us suppose that $U : A \to \mathcal{P}(A)$ is a multivalued operator such that $U : A \to P(A)$ satisfies to a fixed point theorem. If $y \notin U(y)$, for each $y \in A$ then there exists at least one U-maximal element of U. In order to justify the above assertion, let us suppose by contradiction, that $U(y) \neq \emptyset$, for any $y \in A$. From the fixed point theorem we obtain the existence of an element $x^* \in A$ such that $x^* \in U(x^*)$, which is a contradiction with the hypothesis. Hence, any fixed point result for a multivalued operator is an U-maximal existence theorem for the preference multifunction.

On the other hand, if we a preference multifunction defined by the relation:

$$U(x) = \{y \in A | y \text{ is preferred to } x\}, \ x \in A,$$

then a vector $x^* \in A$ is an optimal preference for the consumer if and only if $\{x^*\} = U(x^*)$. Such points are, by definition, strict fixed points of U. They are also called end points for the multivalued dynamical system (A, U) generated by the multivalued operator U (see also section D. of this Introduction). Hence, any strict fixed point theorem is, in fact, an existence result for an optimal preference.

So, more general the consumer's problem is to find all vectors which are optimal preferences with respect to U. The set of solution to a consumer's problem for given price system p is called the demand set.

Let us discuss now something about the supplier's problem. This is much simpler, because the suppliers are motivated by profit. Each supplier j has a production set Y (or Y_j) of technologically feasible supply vectors. A supply vector y specifies the quantities of each commodity supplied and the amount of each commodity used as an input. Inputs are denoted by negative quantities and outputs by positive ones. The profit (net income) associated with a supply vector y at prices p is just $p \cdot y = \sum_{i=1}^{m} p_i y_i$. The supplier's problem is then to choose an element y from the set of technologically feasible supply vectors which maximizes the associated profit. As in the consumer's problem, there may be no solution, as it may pay to increase the outputs and inputs indefinitely at ever increasing profits. The set of all solutions of the supplier's problem is called the supply set.

Thus, for a given price vector p, there is a set of supply vectors y_j, for each supplier j (determined by maximizing the profit) and a set of demand vectors x_i, for each consumer i (determined by preference optimality). The excess demand multifunction with respect to a given price system p, is defined as the set of sums of demand vectors minus the set of sums of supply vectors (i. e. the demand set minus the supply set) and it is denoted by $E(p)$. Obviously, E is a multivalued operator $E : \mathbb{R}^m \multimap \mathbb{R}^m$.

The notion of equilibrium that I am now recalling was basically formalized by Leon Walras in 1874. So, by definition, a price vector $p^* \in \mathbb{R}^m$ is a Walrasian equilibrium price if $0 \in E(p^*)$. This means that some combinations of supply and demand vectors adds up to zero. We may say that p^* clears the market.

There exists another situation called a Walrasian free disposal equilibrium. That is the following situation: some commodities might be allowed to be in excess supply at equilibrium provided their price is zero. So, the price p^* is a Walrasian free disposal equilibrium price if there exists $z \in E(p^*)$ such that $z \leq 0$ and whenever $z_i < 0$ then $p_i^* = 0$.

Of fundamental importance to this approach is a property of the excess demand multifunction known as Walras' law. Shortly, Walras' law says that if the profits of all suppliers are returned to consumers as dividends, then the value at prices p of any excess demand vector must be non-positive. This happens because the value of each consumer's demand must be no more than his income and the sum of all incomes must be the sum of all profits from suppliers. Thus, the value of total supply must be at least as large as the value of total demand. If each consumer spends all his income, then these two values are equal and the value of excess demand multifunction must be zero.

Let us present now briefly an example of how the excess demand multifunction can be expressed. We will consider, for simplicity, the problem of sharing between "n" consumers a commodity bundle w, i. e. the supply. So, the problem is to find n commodity bundles x_i, such that $\sum_{i=1}^{n} x_i \leq w$. A solution to this problem is called an allocation of w. The solution proposed by Walras and his followers consists in letting price systems play a crucial role. Namely, a consumer i is defined as an automaton associating to every price vector p and every income r (in monetary units) its demand $d_i(p,r)$, which is the commodity bundle that he buys when the price system is p and its income is r. So it is assumed that demand operator d_i describes the behavior of the consumer i. Let us recall that, neoclassical economists assume that demand operators derive from the maximization of an utility function. But, in what follows, we assume that consumers are just demand operators $d_i(\cdot,\cdot)$ independent of the supply bundle w.

We also assume that an income allocation of the gross income w is given. This means the following: if p is the price vector, the gross income is the value $p \cdot w$ of the supply w. We then assume that gross income $r(p) = p \cdot w$ is allocated among consumers in incomes $r_i(p)$ and hence $r(p) = \sum_{i=1}^{n} r_i(p)$. We must observe that the model does not provide this allocation of income, but assumes that it is given. An example of such an income allocation is supplied by the so-called exchange economies, where the supply w is the sum of n supply bundles w_i brought to the market by n consumers. So, in this case $r(p) = p \cdot w$ and $r_i(p) = p \cdot w_i$ is the income derived by consumer i from its supply bundle w_i. In summary, the mechanism we are about to describe depends upon:

1) the description of each consumer i by its demand operator $d_i(\cdot,\cdot)$

2) an allocation $r(p) = \sum_{i=1}^{n} r_i(p)$ of the gross income.

The mechanism works if and only if demand balances supply, i. e. if and only if

$$\sum_{i=1}^{n} d_i(p, r_i(p)) \leq w. \qquad (*)$$

A solution p^* to this problem is a Walrasian equilibrium price.

There is no doubt that Adam Smith (1776) is at the origin of what we now call descentralization, i. e. the ability of a complex system, moved by different actions to pursuit of different objectives to achieve an allocation of scarce resources: " Every individual endeavors to employ his capital so that its produce may be of greatest value. He generally neither intends to promote the public security, nor knows how much he is promoting it. He intends only his own security, only his own gain. And he is in this led by an invisible hand to promote and end which has no part of his intention. By pursuing his own interest, he frequently thus promotes that of society more effectively that when he really intends to promote it". However, Adam Smith did not provide a careful statement of what the invisible hand manipulates, nor a fortiori, a rigorous argument for its existence. We had to wait a century for Leon Walras to recognize that price systems are the elements on which the invisible hand acts and that actions of different agents are guided by those price systems, providing enough information to all the agents for guaranteeing the consistency of their actions with the scarcity of available commodities. (see Aubin and Cellina [14] or Aubin [16], for more comments and details.)

Hence, if Adam Smith's invisible hand does provide a Walras equilibrium p^*, then the consumers i are led to demand commodities $d_i(p^*, r_i(p^*))$, that permits to share w according to the desire of everybody.

So, the task is to solve problem $(*)$.

It is remarkable that a sufficient condition with a clear economic interpretation is the following financial constraint on the behavior of the consumers (the so-called individual Walras law:

$$p \cdot d_i(p, r_i) \le r_i, \text{ for each } i \in \{1, \cdots, n\}.$$

The individual Walras law forbids consumers to spend more than their incomes.

Another hypothesis which appear is the so-called collective Walras law:

$$\sum_{i=1}^{n} p \cdot d_i(p, r_i) \le \sum_{i=1}^{n} r_i.$$

This law allows financial transactions among consumers.

Both laws do not involve the supply bundle w. A more general model suppose that the supply is not given, but has to be chosen in a set X^* of available commodity bundles supplied to the market. Thus, the income derived from this set X^* is $r(p) = \sup_{w \in X^*} p \cdot w$. When X^* is reduced to one supply vector w, we fall back to the case we have considered above.

The mechanism is described by:

i) the "n" demand operators $d_i(\cdot, \cdot)$

ii) an income allocation $r(p) = \sum_{i=1}^{n} r_i(p)$, which depends upon X^* via the above formula.

The problem is to find a price p^* (a Walrasian equilibrium), cleaning the market in the sense that:

$$\sum_{i=1}^{n} d_i(p^*, r_i(p^*)) \in X^*.$$

This means that the sum of the demands lies among the set of available supplies. If we define the excess demand multifunction E by:

$$E(p) = \sum_{i=1}^{n} d_i(p^*, r_i(p^*)) - X^*,$$

then a Walrasian equilibrium p^* is a solution of the following inclusion:

$$0 \in E(p^*).$$

Hence, an existence result for the zero-point element of the multivalued operator E (i. e. an element $p^* \in X$ with $0 \in E(p^*)$) is, basically, an existence theorem for a Walrasian equilibrium price of the market.

Of course, there are also many bad points of these models. The first is that the fundamental nature of Walras world is static, while we live in a dynamical environment, where no equilibria have been observed. There exist also several dynamical models built on the ideas of the Walras hypothesis. More precisely, one regard the price system not as a state of a dynamical system whose evolution law is known, but as a control which evolves as an operator of the consumptions according to a feedback law.

B. Equilibrium price, variational inequalities and the complementarity problem. A particular case of the above model is when the excess demand multifunction is a singlevalued operator. We will consider now the case when excess demand set is a singleton for each price vector p and the price vectors are non-negative. So, for each price vector p, there is a vector $f(p)$ of excess demands for each commodity. We assume that f is continuous. A very important property of market excess demand operator is the individual Walras law. The mathematical statement of Walras' law for this singlevalued case can take either two forms. The strong form of Walras' law is:

$$p \cdot f(p) = 0, \text{ for all p },$$

while the weak form of Walras law replaces the equality by the weak inequality:

$$p \cdot f(p) \leq 0, \text{ for all p }.$$

The economic meaning of Walras' law is that in a closed economy, at most all of everyone's income is spent. To see how the mathematical statement follows from the economic hypothesis, first consider the case of a pure exchange economy. The k-th consumer comes to market with a vector w_k of commodities and leaves with a vector x_k of commodities. If all the consumers face the price vector p, then their individual budgets require that $p \cdot x_k \leq p \cdot w_k$, that is they cannot spend more than they earn. In this case, the excess demand operator is: $f(p) = \sum x_k - \sum w_k$, i. e. the sum of total demands minus the sum of total supply. Summing up the individual budget constraints and rearranging terms we obtain that: $\sum p \cdot (x_k - w_k) \leq 0$ or equivalently $p \cdot \sum (x_k - w_k) \leq 0$. Hence we have obtained: $p \cdot f(p) \leq 0$, the weak form of Walras law. The strong form obtains if each consumer spends all his income.

The case of a production economy is similar. The j-th supplier produces a net output vector y_j, which yields a net income of $p \cdot y_j$. In a private ownership economy this net income is redistributed to consumers. The new budget constraint form for a consumer is :

$$p \cdot x_k \leq p \cdot w_k + \sum_j \alpha_j^k p \cdot y_j,$$

where α_j^k is consumers' k's share of profits of firm j. Thus $\sum_k \alpha_j^k = 1$, for each j. So, the excess demand operator $f(p) = \sum_k x_k - \sum_k w_k - \sum_j y_j$. Again adding up the budget constraints and rearranging terms yields $p \cdot f(p) \leq 0$. The law remains true even if consumers may borrow from each other, as long as, no borrowing from outside the economy takes place. Also, we can restrict the prices to belong to the standard simplex because both constraints and the profit functions are positively homogeneous in prices. Thus we can normalize prices.

By definition, $p^* \in \mathbb{R}^m_+$ is said to be an equilibrium price if $f(p^*) = 0$. A free disposal equilibrium price is a price vector $p^* \in \mathbb{R}^m_+$ satisfying $f(p^*) \leq 0$.

Let us remark that, if $p^* \in \mathbb{R}^m_+$ is a free disposal equilibrium price and the weak form of Walras law take place (i. e. $p \cdot f(p) \leq 0$), then $f_i(p^*) < 0$ for some i necessarily implies $p^*_i = 0$, i. e. if a commodity is in excess, then the price must be zero.

A mathematical more general problem is what is known as the nonlinear complementarity problem. The function f is assumed to be continuous and its domain is a closed convex cone C in $\mathbb{R}^m$. The problem is:

$$\text{find } p^* \in C \text{ such that } f(p^*) \in C^* \text{ and } p^* \cdot f(p^*) = 0.$$

If in particular, C is the non-negative cone $\mathbb{R}^m_+$, then its dual $C^* = \mathbb{R}^n_-$ and so $f(p^*) \in C^*$ becomes $f(p^*) \leq 0$. In this case, since $f(p^*) \leq 0$ can be also writen $p \cdot f(p^*) \leq 0$, for each $p \in \mathbb{R}^m_+$, then we immediately get that $p \cdot f(p^*) \leq p^* \cdot f(p^*) = 0$ and so the problem becomes:

$$\text{find } p^* \in \mathbb{R}^m_+ \text{ such that } p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for each } p \in \mathbb{R}^m_+.$$

Of course, the complementarity problem could be formulated in a more general setting, for example in a Hilbert space or in a dual system of locally convex spaces (E, E^*), see Isac [86].

So, in both, the price problem and the complementarity problem there is a cone C and a function f defined on C and we are looking for a $p^* \in C$ satisfying $f(p^*) \in C^*$. As we already mentioned above, another way to write the condition $f(p^*) \in C^*$ is the following:

$$p \cdot f(p^*) \leq 0, \text{ for all } p \in C.$$

Since in both problems (in the price problem, on the assumption of the strong Walras' law, while in the complementarity problem, by definition) $p^* \cdot f(p^*) = 0$, we can rewrite this as:

$$p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for all } p \in C.$$

A system of inequalities of the above form is called a system of variational inequalities, because it compares expressions involving $f(p^*)$ and p^* with expressions involving $f(p^*)$ and p, where p can be viewed as a variation of p^*. The intuition involved in these situation is the following: if a commodity is in excess demand, then its price should be raised and if it in excess supply, then its price should be lowered. This increases the value of demand. Let us say that price p is better than price p^* if p gives a higher value to p^*'s excess demand than p^* does. The variational inequalities tell us that we are looking for a maximal element of this binary relation. Of course, a multivalued operator is then involved, namely

$$U(p) = \{q \in C | q \cdot f(p) > p \cdot f(p)\}, \ p \in C,$$

and, as we mentioned above, we are looking for an element $p^* \in C$ such that $U(p^*) = \emptyset$.

If we consider $f : \mathbb{R}^m_+ \to \mathbb{R}^m$ and we denote by (VIP) the variational inequalities problem and by (CP) the complementarity problem then:

$$(VIP) \text{ find } p^* \in \mathbb{R}^m_+ \text{ such that } p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for each } p \in \mathbb{R}^m_+.$$

$$(CP) \text{ find } p^* \in \mathbb{R}^m_+ \text{ such that } f(p^*) \in \mathbb{R}^m_- \text{ and } p^* \cdot f(p^*) = 0$$

are equivalent. Indeed, if $p^* \in \mathbb{R}^m_+$ is a solution of (CP) then $f(p^*) \in \mathbb{R}^m_-$ and $p^* \cdot f(p^*) = 0$. Then $p \cdot f(p^*) \leq 0 = p^* \cdot f(p^*)$, for each $p \in \mathbb{R}^m_+$ and so p^* is a solution of (VIP). For the reverse implication, let $p^* \in \mathbb{R}^m_+$ is a solution of (VIP). Then $f(p^*) \cdot (p - p^*) \leq 0$, for each $p \in \mathbb{R}^m_+$. By taking $p = 0$ and $p = 2p^*$ in the above relation, we immediately get that $f(p^*) \cdot p^* = 0$. we need to show now that $f(p^*) \in \mathbb{R}^m_-$. If we suppose by contradiction that here exists $i \in \{1, 2, \cdots, m\}$ such that $f_i(p^*) > 0$ then, by a suitable choice for the vector p (with a large $p_i > 0$) we obtain a contradiction with $f(p^*) \cdot (p - p^*) \leq 0$.

This shows that $f_i(p^*) > 0$, for each $i \in \{1, 2, \cdots, m\}$. See also G. Isac [85], pp. 63.

Finally, we would like to point out another (obvious) connection with fixed point theory. If we denote by f the operator defining a complementarity problem, then x^* is a solution for the complementarity problem if and only if x^* is a fixed point of the operator $1_C + f$.

For important contributions in the field of complementarity theory and connections to mathematical economics and variational inequalities theory see Isac [85], [86], Isac, Bulavski, Kalashnikov [87] and S. P. Singh, B. Watson, P. Srivastava [196].

C. Optimization problems. Let X, Y be topological vector spaces. Consider $A \subset X$, $B, C \subset Y$, $f : X \to Y$ a singlevalued operator and $F : Y \to P(Y)$ a multivalued operator.

Let us show now that maximization with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a fixed point problem of the following type:

$$\text{find } y \in Y \text{ such that } \{y\} = F(y).$$

Recall that a a set $C \subset Y$ is a cone if $\lambda y \in C$, for all $y \in C$ and each $\lambda \geq 0$. A convex cone is a cone for which $\lambda_1 y_1 + \lambda_2 y_2 \in C$, for all $y_1, y_2 \in C$ and each $\lambda_1, \lambda_2 \geq 0$. A cone is called pointed if $C \cap (-C) = \{\theta\}$. For a pointed cone we write $y \geq z$ if and only if $y - z \in C$ and $y > z$ if and only if $y - z \in C - \{\theta\}$.

An element $y^* \in B$ is a maximal element of B with respect to C (we will denote this by: $y^* = \max(B; C)$) if and only if there is no $y \in B$ for which $y^* < y$.

Now, for a specified pointed cone C we consider the problem:

$$\text{maximize } f(x) \text{ subject to } x \in A, \qquad (*)$$

of determining all $x^* \in A$ for which $f(x^*) \in \max[f(A); C]$. Such an element x^* is said to be a maximal point for the considered problem.

This abstract problem has been studied in several papers by Borwein and others. When $X = \mathbb{R}^n, Y = \mathbb{R}^m, f_1, \cdots, f_m : \mathbb{R}^n \to \mathbb{R}$, with $f(x) =$

$(f_1(x), \cdots, f_m(x))$ and $C = \mathbb{R}^m_+$, then the previous abstract problem becomes a Pareto maximization problem, which has been considered by numerous authors.

Let us show now that the considered problem is equivalent cu a strict fixed point problem.

Theorem. *Let $f : X \to Y$ and $F : Y \to \mathcal{P}(Y)$, be defined by $F(y) = \{f(x)|x \in A, f(x) \in C + y\}$.*

Then x^* is a maximal element for problem (*) if and only if $\{f(x^*)\} = F(f(x^*))$.

Proof. First suppose that x^* is a maximal element for $(*)$. Then, there is no $x \in A$ such that $f(x^*) < f(x)$, i. e. there is no $x \in A$ such that $f(x) - f(x^*) \in C - \{\theta\}$. Also, we can observe that $\{f(x^*)\} \in F(f(x^*))$. We have to show now that $\{f(x^*)\} = F(f(x^*))$. If there exists another element $f(x)$ of $F(f(x^*))$, with $f(x) \neq f(x^*)$, then since x is feasible to $(*)$ it satisfies $\theta \neq f(x) - f(x^*) \in C$, contrary to our assumption. Thus the equality $\{f(x^*)\} = F(f(x^*))$ is established. Next, suppose that $\{f(x^*)\} = F(f(x^*))$ holds. Then, there is no $x \in A$ such that $f(x) \in F(f(x^*))$, with $f(x) \neq f(x^*)$. So, there is no $x \in A$ such that $f(x) \in C + f(x^*)$, with $f(x) \neq f(x^*)$. As consequence, there is no $x \in A$ such that $f(x) - f(x^*) \in C - \{\theta\}$. Since $f(x) - f(x^*) \in C - \{\theta\}$ cannot hold for any feasible x to $(*)$, we get the desired conclusion: x^* is a maximal point. □

D. Multivalued dynamic systems. Let us consider now the so-called multivalued dynamic systems. We follow the notations and terminologies in Aubin-Siegel [17] and Yuan [217].

Definition. If (X, d) is a metric space and $T : X \to P(X)$ a multivalued operator then the pair (X, T) is said to be a multivalued dynamic system (briefly MDS).

A sequence $(x_n)_{n \in \mathbb{N}}$, with $x_0 = x$, $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$, is called, in this framework, a motion of x throughout the MDS (X, T). The set $\mathcal{T}(T, x) = \{x_n |\ n \in \mathbb{N} \text{ and } x_0 = x,\ x_{n+1} \in T(x_n)\}$ is called the trajectory of this motion.

A fixed point of T is called a stationary point of the MDS (X, T), while a strict fixed point of T is called an endpoint for (X,T).

An important problem of the theory of multivalued dynamic systems is the existence of stationary points and of endpoints of MDS.

E. Game theory. Roughly speaking, a game is a situation where a number of players, having absolutely independent interests, must each choose a strategy of a certain action and, then, based on these choices, some consequences appears. If we suppose that there are n game participants, with absolutely independent interests, then the game is said to be a noncooperative n-person game.

Let us present now the elements that characterize the noncooperative n-person game. Denote by X_i the set of all strategies of the i player, where $i \in \{1,2,...,n\}$. Then, $X := \prod_{i=1}^{n} X_i$ is the set of all strategy vectors. Each $x = (x_1, x_2, ..., x_n) \in X$ induces an outcome.

Players preferences are described using the preference multifunction $\widetilde{U}_i : X \multimap X$, defined by $\widetilde{U}_i(x) := \{y \in X | y \text{ is preferred to x }\}$.

We also define, the good reply multifunction.

Denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$, where $X_{-i} := \prod_{k=1, k\neq i}^{n} X_k$. and $x|y_i := (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \in X$.

Then, by definition, y_i is a good reply for the player i with respect to the strategy vector x if $x|y_i \in \widetilde{U}_i(x)$.

In this setting, the good reply multifunction for the player i is $U_i : X_{-i} \multimap X_i$ defined by

$$U_i(x_{-i}) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x|u_i), \text{ for each } u_i \in X_i\}.$$

A game in strategic form or an abstract economy is the pair $(X_i, U_i)_{i\in\{1,2,...,n\}}$.

For example, if we consider $p_i : X \to \mathbb{R}$, for $i \in \{1,2,...,n\}$, the pay-off function of the i player, then the good reply multifunction can be expressed by:

$$U_i(x_{-i}) := \{y_i \in X_i | p_i(x|y_i) \geq p_i(x|z_i), \text{ for each } z_i \in X_i\}.$$

By definition, $x^* \in X$ is a (noncooperative) Nash equilibrium point for an abstract economy if $x_i^* \in U_i(x_{-i}^*)$, for $i \in \{1, 2, ..., n\}$.

Let us observe that the above Nash equilibrium problem is equivalent to the following fixed point problem: $x^* \in U(x^*)$, where $U(x) := \prod_{i=1}^{n} U_i(x_{-i})$.

If $x^* = (x_1^*, \cdots, x_n^*) \in X$ is a (noncooperative) Nash equilibrium then each player of the game reckons his choice as acceptable and does't want to change it.

Let us consider now the case of a 2-person game (or an abstract economy with neighborhood effects) given by $(X_1, U_1), (X_2, U_2)$, where X_1, X_2 denote the set of strategies of the player 1, respectively player 2, and $U_1 : X_2 \multimap X_1$, $U_2 : X_1 \multimap X_2$ are the good reply multifunctions for each player.

By definition, (x_1^*, x_2^*) is a Nash equilibrium point if $x_1^* \in U_1(x_2^*)$ and $x_2^* \in U_2(x_1^*)$.

Another possibility is to define the good reply multifunction $U_i : X \multimap X_i$ as follows:

$$U_i(x) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x)\}.$$

Then, by definition, $x^* \in X$ is a Nash equilibrium point if $U_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$. In what follows we will consider this definition for the good reply multifunction.

Another important concept in game theory is the constraint (feasible) multifunction. It happens frequently that the choices of the players cannot be made independently. Two simple examples are the case of a mineral water exploitation from several springs, by several economic agents or the case of a fish exploitation from a lake by a number of fishers. Each participant has partial control of the price and the strategy x_i of the i player cannot be chosen independently because their sum cannot exceed the total amount of the exploitation. These situations can be, from the mathematical point of view, modelled by introducing the feasibility or constraint multivalued operator $F_i : X \multimap X_i$, which tell us which strategies are actually feasible for the player i, with respect to the strategy vector x.

So, let us denote by $F_i : X \multimap X_i$, the constraint (feasibility) multifunction for the i player, where $i \in \{1, 2, ..., n\}$. Then define

$$F := \prod_{i=1}^{n} F_i : X \multimap X, \text{ by } F(x) := \prod_{i=1}^{n} F_i(x)$$

.

Obviously, the feasible strategy vectors are the fixed points of F, i. e. elements $x \in X$ with $x \in F(x)$.

By definition, a generalized game or a generalized abstract economy is a strategic game (or an abstract economy), which also includes the constraint multifunction F_i, i.e. $(X_i, U_i, F_i)_{i\in\{1,2,...,n\}}$.

A Nash equilibrium point for a generalized abstract economy is a strategy vector $x^* \in X$ such that $x^* \in F(x^*)$ and $U_i(x^*) \cap F_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$.

As a conclusion, if $F : X \to \mathcal{P}(X)$ is a multivalued operator, then **fixed points** (i.e. $x \in X$ with $x \in F(x)$), **strict fixed points** (i.e. $x \in X$ with $\{x\} = F(x)$), **maximal elements** (i.e. $x \in X$ with $F(x) = \emptyset$) and **zero points** (i.e. $x \in X$ with $0 \in F(x)$, where $F : X \to \mathcal{P}(\mathbb{E})$, E is a linear space) of the multifunction F have important meanings in the abstract mathematical economics theory.

It is in our intention to report several results in these four directions.

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Part I

An Introduction to Multivalued Analysis

Chapter 1

Pompeiu-Hausdorff metric

The aim of this section is to present the main properties of some (generalized) functionals defined on the space of all subsets of a metric space. A special attention is paid to gap functional, excess functional and to Pompeiu-Hausdorff functional.

Let (X, d) be a metric space. Sometimes we will need to consider infinite-valued metrics, also called generalized metrics $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$, see Luxemburg [107] and Jung [91].

Throughout this book, we denote by $\mathcal{P}(X)$ the set of all subsets of a nonempty set X. If X is a metric space, $x \in X$ and $R > 0$, then $B(x, R)$ and respectively $\widetilde{B}(x, R)$ denote the open, respectively the closed ball of radius R centered in x. If X is a topological space and Y is a subset of X, then we will denote by $\overline{Y}$ the closure and by $intY$ the interior of the set Y. Also, if X is a normed space and Y is a nonempty subset of X, then coY respectively $\overline{co}Y$ denote the convex hull, respectively the closed convex hull of the set Y.

We consider, for the beginning, the generalized diameter functional defined on the space of all subsets of a metric space X.

Definition 1.1. Let (X, d) be a metric space. The generalized diameter functional $diam : \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by:

$$diam(Y) = \begin{cases} \sup\{d(a,b)|\ a \in Y,\ b \in Y\}, & \text{if } Y \neq \emptyset \\ 0, & \text{if } Y = \emptyset \end{cases}$$

Definition 1.2. The subset Y of X is said to be bounded if and only if $diam(Y) < \infty$.

Lemma 1.3. *Let (X, d) be a metric space and Y, Z nonempty bounded subsets of X. Then:*

i) $diam(Y) = 0$ if and only if $Y = \{y_0\}$.

ii) If $Y \subset Z$ then $diam(Y) \leq diam(Z)$.

iii) $diam(\overline{Y}) = diam(Y)$.

iv) If $Y \cap Z \neq \emptyset$ then $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) If X is a normed space then:

a) $diam(x + Y) = diam(Y)$, for each $x \in X$.

b) $diam(\alpha Y) = |\alpha| diam(Y)$, where $\alpha \in \mathbb{R}$.

c) $diam(Y) = diam(co\,Y)$.

d) $diam(Y) \leq diam(Y + Z) \leq diam(Y) + diam(Z)$.

Proof. iii) Because $Y \subseteq \overline{Y}$ we have $diam(Y) \leq diam(\overline{Y})$. For the reverse inequality, let consider $x, y \in \overline{Y}$. Then there exist $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. It follows that $d(x_n, y_n) \stackrel{\mathbb{R}}{\to} d(x, y)$. Because $d(x_n, y_n) \leq diam(Y)$, for all $n \in \mathbb{N}$ we get by passing to limit $d(x, y) \leq diam(Y)$. Hence $diam(\overline{Y}) \leq diam(Y)$.

iv) Let $u, v \in Y \cup Z$. We have the following cases:

a) If $u, v \in Y$ then $d(u, v) \leq diam(Y) \leq diam(Y) + diam(Z)$ and so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

b) If $u, v \in Z$ then by an analogous procedure we have $d(u, v) \leq diam(Z) \leq diam(Y) + diam(Z)$ and so $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

c) If $u \in Y$ and $v \in Z$ then choosing $t \in Y \cap Z$ we have that $d(u, v) \leq d(u, t) + d(t, v) \leq diam(Y) + diam(Z)$. Hence, $diam(Y \cup Z) \leq diam(Y) + diam(Z)$.

v) c) Let us prove that $diam(coY) \leq diam(Y)$. Let $x, y \in coY$. Then there exist $x_i, y_j \in Y$, $\lambda_i, \mu_j \in \mathbb{R}_+$, such that

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad y = \sum_{j=1}^{m} \mu_j y_j, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \sum_{j=1}^{m} \mu_j = 1.$$

From these relations we have:

$$\|x-y\| = \left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{m} \mu_j y_j \right\| = \left\| \left(\sum_{j=1}^{m} \mu_j \right) \sum_{i=1}^{n} \lambda_i x_i - \left(\sum_{i=1}^{n} \lambda_i \right) \sum_{j=1}^{m} \mu_j y_j \right\|$$

$$\leq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \|x_i - y_j\| \leq \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \mu_j \right) diam(Y) = diam(Y).$$

□

Let us consider now the following sets of subsets of a metric space (X, d):

$$P(X) = \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};\ P_b(X) = \{Y \in P(X) |\ diam(Y) < \infty\};$$

$$P_{op}(X) = \{Y \in P(X) |\ Y \text{ is open}\};\ P_{cl}(X) = \{Y \in P(X) |\ Y \text{ is closed}\};$$

$$P_{b,cl}(X) = P_b(X) \cap P_{cl}(X);\ P_{cp}(X) = \{Y \in P(X) |\ Y \text{ is compact}\};$$

$$P_{cn}(X) = \{Y \in P(X) |\ Y \text{ is connex}\}.$$

If X is a normed space, then we denote:

$$P_{cv}(X) = \{Y \in P(X) |\ Y \text{ convex}\};\ P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Let us define the following generalized functionals:

(1) $D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A, B) = \begin{cases} \inf\{d(a, b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

D is called the gap functional between A and B.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

(2) $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A, B) = \begin{cases} \sup\{d(a, b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

(3) $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) |\ a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

ρ is called the excess functional of A over B.

(4) $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

H is called the generalized Pompeiu-Hausdorff functional of A and B.

Let us prove now that the functional H is a metric on the space $P_{b,cl}(X)$. First we will prove the following auxiliary result:

Lemma 1.4. *$D(b,A) = 0$ if and only if $b \in \overline{A}$.*

Proof. We shall prove that $\overline{A} = \{x \in X |\ D(x,A) = 0\}$. For this aim, let $x \in \overline{A}$ be arbitrarily. It follows that for each $r > 0$ and for each $B(x,r) \subset X$ we have $A \cap B(x,r) \neq \emptyset$. Then for each $r > 0$ there exists $a_r \in A$ such that $d(x,a) < r$. It follows that for each $r > 0$ we have $D(x,A) < r$ and hence $D(x,A) = 0$. □

Theorem 1.5. *Let (X,d) be a metric space. Then the pair $(P_{b,cl}(X), H)$ is a metric space.*

Proof. We shall prove that the three axioms of the metric hold:

a) $H(A,B) \geq 0$, for all $A, B \in P_{b,cl}(X)$ is obviously.

$H(A,B) = 0$ is equivalent with $\rho(A,B) = 0$ and $\rho(B,A) = 0$, that means $\sup\limits_{a \in A} D(a,B) = 0$ and $\sup\limits_{b \in B} D(b,A) = 0$. Hence $D(a,B) = 0$, for each $a \in A$ and $D(b,A) = 0$, for each $b \in B$. Using Lemma 1.4. we obtain that $a \in B$, for all $a \in A$ and $b \in A$, for all $b \in B$, proving that $A \subseteq B$ and $B \subseteq A$.

b) $H(A,B) = H(B,A)$ is quite obviously.

c) For the third axiom of the metric, let consider $A, B, C \in P_{b,cl}(X)$. For each $a \in A,\ b \in B$ and $c \in C$ we have $d(a,c) \leq d(a,b) + d(b,c)$. It follows that $\inf_{c \in C} d(a,c) \leq d(a,b) + \inf_{c \in C} d(b,c)$, for all $a \in A$ and $b \in B$. We get $D(a,C) \leq d(a,b) + D(b,C)$, for all $a \in A,\ b \in B$. Hence $D(a,C) \leq D(a,B) + H(B,C)$, for all $a \in A$ and so $D(a,C) \leq H(A,B) + H(B,C)$, for all $a \in A$. In conclusion, we have proved that $\rho(A,C) \leq H(A,B) + H(B,C)$. Similarly, we get $\rho(C,A) \leq H(A,B) + H(B,C)$, and so $H(A,C) \leq H(A,B) + H(B,C)$. □

Remark 1.6. H is called the Pompeiu- Hausdorff metric induced by the metric d. Occasionally, we will denote by H_d the Pompeiu-Hausdorff functional generated by the metric d of the space X.

Remark 1.7. H is a generalized metric on $P_{cl}(X)$.

Lemma 1.8. *Let (X,d) a metric space. Then we have:*

i) $D(\cdot, Y) : (X,d) \to \mathbb{R}_+$, $x \mapsto D(x,Y)$, (where $Y \in P(X)$) is nonexpansive.

ii) $D(x, \cdot) : (P_{cl}(X), H) \to \mathbb{R}_+$, $Y \mapsto D(x,Y)$, (where $x \in X$) is nonexpansive.

Proof. i) We shall prove that for each $Y \in P(X)$ we have

$$|D(x_1,Y) - D(x_2,Y)| \leq d(x_1,x_2), \text{for all } x_1, x_2 \in X.$$

Let $x_1, x_2 \in X$ be arbitrarily. Then for all $y \in Y$ we have

$d(x_1,y) \leq d(x_1,x_2) + d(x_2,y)$. Then $\inf_{y \in Y} d(x_1,y) \leq d(x_1,x_2) + \inf_{y \in Y} d(x_2,y)$ and so $D(x_1,Y) \leq d(x_1,x_2) + D(x_2,y)$. We have proved that $D(x_1,y) - D(x_2,Y) \leq d(x_1,x_2)$. Interchanging the roles of x_1 and x_2 we obtain $D(x_2,Y) - D(x_1,Y) \leq d(x_1,x_2)$, proving the conclusion.

ii) We shall prove that for each $x \in X$ we have:

$$|D(x,A) - D(x,B)| \leq H(A,B), \text{ for all } A, B \in P_{cl}(X).$$

Let $A, B \in P_{cl}(X)$ be arbitrarily. Let $a \in A$ and $b \in B$. Then we have $d(x,a) \leq d(x,b) + d(b,a)$. It follows $D(x,A) \leq d(x,b) + D(b,A) \leq d(x,b) + H(B,A)$ and hence $D(x,A) - D(x,B) \leq H(A,B)$. By a similar procedure we get $D(x,B) - D(x,A) \le H(A,B)$ and so $|D(x,A) - D(x,B)| \le H(A,B)$, for all $A, B \in P_{b,cl}(X)$. $\square$

Lemma 1.9. *Let (X,d) be a metric space. Then the generalized functional $diam : (P_{cl}(X), H) \to \mathbb{R}_+ \cup \{+\infty\}$ is continuous.*

Lemma 1.10. *Let (X,d) be a metric space. Then we have:*
i) $Y \subset Z$ implies $D(x,Z) \le D(x,Y)$.
ii) $D(\overline{Y}, \overline{Z}) = D(Y,Z)$, for all $Y, Z \in P(X)$.
iii) $D(Y,Z) \le D(Y,W) + D(W,Z) + diam(W)$, for all $Y, Z, W \in P(X)$.
iv) $D(Y, Z \cup W) = \min\{D(Y,Z), D(Y,W)\}$, for all $Y, Z, W \in P(X)$.
v) $D(x,Y) = D(x,\overline{Y})$, where $x \in X$ and $Y \in P(X)$.
vi) [For each $x \in X$ $D(x,Y) = D(x,Z)$] if and only if $\overline{Y} = \overline{Z}$.
vii) If $Y, Z \in P(X)$ such that $Y \subset Z \subset \overline{Y}$ then

$$D(x_0, Y) = D(x_0, Z) = D(x_0, \overline{Y}), \text{for all } x_0 \in X.$$

viii) If $Y, Z \in P(X)$ then $C := \{x \in X | D(x,Y) = D(x,Z)\}$ is closed and $E := \{x \in X | D(x,Y) < D(x,Z)\}$ is open.

Proof. i) For each $\epsilon > 0$ there exists $d(x,y) < D(x,Y) + \epsilon$. Since $y \in Y \subset Z$ we have $D(x,Z) \le d(x,y)$. Then for each $\epsilon > 0$ $D(x,Z) \le d(x,y) < D(x,Y) + \epsilon$. Letting $\epsilon \searrow 0$ we get the desired conclusion $D(x,Z) \le D(x,Y)$.

ii) Because $Y \subseteq \overline{Y}$ and $Z \subseteq \overline{Z}$ the inequality $D(\overline{Y}, \overline{Z}) \le D(Y,Z)$ is obviously. For the reverse inequality let us consider $u \in \overline{Y}$, $v \in \overline{Z}$. Then there exists $(x_n)_{n\in\mathbb{N}} \subset Y$ and $(y_n)_{n\in\mathbb{N}} \subset Z$ such that $\lim\limits_{n\to\infty} x_n = u$, $\lim\limits_{n\to\infty} y_n = v$. Because $D(Y,Z) \le d(x_n, y_n) \le d(x_n, u) + d(u,v) + d(v, y_n)$ it follows, for $n \to \infty$, that: $D(Y,Z) \le d(u,v)$, for all $u \in \overline{Y}, v \in \overline{Z}$. Hence $D(Y,Z) \le D(\overline{Y}, \overline{Z})$.

iii) We have $d(y,z) \le d(y,w_1) + d(w_1,w_2) + d(w_2,z)$, for all $y \in Y, z \in Z$, and for all $w_1, w_2 \in W$. We get $D(y,Z) \le d(y,w_1) + d(w_1,w_2) + D(w_2,Z)$, for all $y \in Y, w_1, w_2 \in W$. Then $D(Y,Z) \le D(y,Z) \le d(y,w_1) + d(w_1,w_2) + D(w_2,Z)$, for all $y \in Y$ and $w_1, w_2 \in W$. We have now $D(Y,Z) \le d(y,w_1) + diam(W) + D(w_2,Z)$, for all $y \in Y, w_1, w_2 \in W$. So $D(Y,Z) \le D(y,W) + diam(W) + D(W,Z)$, for all $y \in Y$. Finally $D(Y,Z) \le D(Y,W) + D(W,Z) + diam(W)$.

v) Because $Y \subset \overline{Y}$ we obtain $D(x, \overline{Y}) \leq D(x, Y)$. On the other side, for each $\epsilon > 0$ there is $y \in Y$ such that $d(x, y) \leq D(x, \overline{Y}) + \epsilon$. But $y = \lim_{n \longrightarrow \infty} y_n$, $y_n \in Y$. But d is a continuous function and then $D(x, Y) \leq d(x, y_n)$, for each $n \in \mathbb{N}$ implies $D(x, Y) \leq d(x, y)$. Hence, for each $\epsilon > 0$ $D(x, Y) < D(x, \overline{Y}) + \epsilon$. When $\epsilon \searrow 0$ we have $D(x, Y) \leq D(x, \overline{Y})$.

vi) From v) the following implication holds:

$\overline{Y} = \overline{Z}$ implies that for each $x \in X$, $D(x, Y) = D(x, \overline{Y}) = D(x, \overline{Z}) = D(x, Z)$.

For the reverse implication, let us suppose that for each $x \in X$ $D(x, Y) = D(x, Z)$. Suppose by contradiction that $\overline{Y} \neq \overline{Z}$. Let $x \in \overline{Y}$ and $x \notin \overline{Z}$. Then $x \in \overline{Y}$ implies $D(x, Y) = 0$ and $x \notin \overline{Z}$ implies $D(x, Z) \neq 0$. This is the desired contradiction.

vii) Because the function $x \rightarrow D(x, Z) - D(x, Y)$ is continuous we have that $C := f^{-1}(\{0\})$ is closed and $E := f^{-1}(]0, +\infty[)$ is open. □

Let us define now the notion of neighborhood for a nonempty set.

Definition 1.11. Let (X, d) be a metric space, $Y \in P(X)$ and $\varepsilon > 0$. An open neighborhood of radius ε for the set Y is the set denoted $V^0(Y, \varepsilon)$ and defined by $V^0(Y, \varepsilon) = \{x \in X | \ D(x, Y) < \varepsilon\}$. We also consider the closed neighborhood for the set Y, defined by $V(Y, \varepsilon) = \{x \in X | \ D(x, Y) \leq \varepsilon\}$.

Remark 1.12. From the above definition we have that, if (X, d) is a metric space, $Y \in P(X)$ then:

a)$\bigcup\{B(y, r) : y \in Y\} = V^0(Y, r)$, where $r > 0$.

b)$\bigcup\{\widetilde{B}(y, r) : y \in Y\} \subset V(Y, r)$, where $r > 0$.

c)$V^0(Y, r + s) \supset V^0(V^0(Y, s), r)$, where $r, s > 0$.

d)$V^0(Y, r)$ is an open set, while $V(A, r)$ is a closed set.

e) If (X, d) is a normed space, then:

i) $V^0(Y, r + s) = V^0(V^0(Y, s), r)$, where $r, s > 0$

ii) $V^0(Y, r) = Y + int(r\widetilde{B}(0, 1))$.

Proof. d) $V^0(Y, r) = f^{-1}(] - \infty, r[)$ and $V(Y, r) = f^{-1}([0, r])$, where $f(x) = D(x, Y)$, $x \in X$ is a continuous function.

Remark 1.13. If (X, d) is a metric space and $Y, Z \in P(X)$ then $D(Y, Z) =$

$\inf\{\varepsilon > 0 |\ Y \bigcap V(Z,\varepsilon) \neq \emptyset\}$.

Lemma 1.14. *a) Let (X, d) be a metric space and $Y, Z \in P(X)$. Then $D(Y, Z) = \inf_{x\in X} D(x, Y) + D(x, Z)$.*

b) Let (X, d) be a metric space and $(A_{i)})_{i\in I}$, B nonempty subsets of X. Then $D(\bigcup_{i\in I} A_i, B) = \inf_{i\in I} D(A_i, B)$

c) Let X be a normed space and $A, B, C \in P(X)$. If A is a convex set, then we have:

$$D(\lambda B + (1-\lambda)C, A) \leq \lambda D(B, A) + (1-\lambda)D(C, A), \text{ for each } \lambda \in [0, 1].$$

Proof. a) We denote by $u = \inf\{D(x, Z) + D(x, Y) : x \in X\}$. Because $D(Y, Z) = \inf\{D(x, Y) + D(x, Z) : x \in Y\}$ we have that $u \leq D(Y, Z)$. For the reverse inequality, let $x \in X$ and $y \in Y, z \in Z$ having the property $d(x, y) \leq D(x, Y) + \varepsilon$ and $d(x, z) \leq D(x, Z) + \varepsilon$. Then we have: $D(Y, Z) \leq d(y, z) \leq D(x, Y) + D(x, Z) + 2\varepsilon$. But ε was arbitrarily chosen, and so $D(Y, Z) \leq u$. □

Lemma 1.15. *Let (X, d) a metric space. Then we have:*

i) If $Y, Z \in P(X)$ then $\delta(Y, Z) = 0$ if and only if $Y = Z = \{x_0\}$

ii) $\delta(Y, Z) \leq \delta(Y, W) + \delta(W, Z)$, for all $Y, Z, W \in P_b(X)$.

iii) Let $Y \in P_b(X)$ and $q \in]0, 1[$. Then, for each $x \in X$ there exists $y \in Y$ such that $q\delta(x, Y) \leq d(x, y)$.

Proof. ii) Let $Y, Z, W \in P_b(X)$. Then we have:

$d(y, z) \leq d(y, w) + d(w, z)$, for all $y \in Y, z \in Z, w \in W$. Then $\sup_{z\in Z} d(y, z) \leq d(y, w) + \sup_{z\in Z} d(w, z)$, for all $y \in Y, w \in W$. So $\delta(y, Z) \leq \delta(y, w) + \delta(w, Z) \leq \delta(y, W) + \delta(W, Z)$ and hence $\delta(Y, Z) \leq \delta(Y, W) + \delta(W, Z)$.

iii) Suppose, by absurdum, that there exists $x \in X$ and there exists $q \in]0, 1[$ such that for all $y \in Y$ to have $q\delta(x, Y) > d(x, y)$. It follows that $q\delta(x, Y) \geq \sup_{y\in Y} d(x, y)$ and hence $q\delta(x, Y) \geq \delta(x, Y)$. In conclusion, $q \geq 1$, a contradiction. □

Lemma 1.16. *Let (X, d) be a metric space, $Y, Z, W \in P(X)$. Then:*

i) $\rho(Y, Z) = 0$ if and only if $Y \subset \overline{Z}$

ii) $\rho(Y,Z) \le \rho(Y,W) + \rho(W,Z)$

iii) If $Y, Z \in P(X)$ *and* $\varepsilon > 0$ *then:*

a) $\rho(Y,Z) \le \varepsilon$ *if and only if* $Y \subset V(Z;\varepsilon)$.

b) $\rho(Y,Z) = \inf\{\varepsilon > 0 |\ Y \subset V^0(Z,\varepsilon)\}$. *(we consider* $\inf \emptyset = \infty$*)*

c) If Y *is closed, then* $\rho(Y,Z) = \sup_{x \in X} D(x,Z) - D(x,Y)$

d) $\rho(Y,Z) = \rho(\overline{Y},\overline{Z})$

iv) Let $\varepsilon > 0$. *If* $Y, Z \in P(X)$ *such that for each* $y \in Y$ *there exists* $z \in Z$ *such that* $d(y,z) \le \varepsilon$ *then* $\rho(Y,Z) \le \varepsilon$.

v) Let $\varepsilon > 0$ *and* $Y, Z \in P(X)$. *Then for each* $y \in Y$ *there exists* $z \in Z$ *such that* $d(y,z) \le \rho(Y,Z) + \varepsilon$.

vi) Let $q > 1$ *and* $Y, Z \in P(X)$. *Then, for each* $y \in Y$ *there exists* $z \in Z$ *such that* $d(y,z) \le q\rho(Y,Z)$.

Proof. i) Suppose that $\rho(Y,Z) = 0$ and let $y \in Y$ be arbitrary. Then $0 \le \inf\{d(y,z)|\ z \in Z\} = D(y,Z) \le \rho(Y,Z) = 0$ implies that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ such that $d(y,z_n) \to 0$, when $n \to \infty$. It follows $z_n \to y$ when $n \to \infty$ and so $y \in \overline{Z} \Rightarrow Y \subset \overline{Z}$.

Reversely, suppose that $Y \subset \overline{Z}$ with $\alpha = \dfrac{1}{2}\rho(Y,Z) > 0$. Then there exists $y_0 \in Y$ with $D(y_0,Z) > \alpha$. For $y_0 \in Y \subset \overline{Z}$ we find a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ such that $z_n \to y_0$, when $n \to \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $d(z_n,y_0) \le \alpha$, for all $n \ge n_0$, a contradiction with: for all $n \ge n_0 : \ \alpha \ge d(z_n,y_0) \ge \inf\{d(z,y_0)|\ z \in Z\} = D(y_0,Z) > \alpha$.

ii) Let $\varepsilon > 0$ and $y \in Y$. Because $D(y,W) = \inf\{d(y,w)|\ w \in W\}$ we have that there exists $w \in W$ such that $d(y,w) < D(y,W) + \varepsilon$. For each $z \in Z$ we have: $D(y,Z) \le d(y,z) \le d(y,w) + d(w,z) < d(w,z) + D(y,W) + \varepsilon$.

So $D(y,Z) - D(y,W) - \varepsilon < d(z,w)$, for all $z \in Z$ proving that $D(y,Z) - D(y,W) - \varepsilon \le D(w,Z)$.

Hence $D(y,Z) \le \rho(W,Z) + \rho(Y,W) + \varepsilon$, for all $y \in Y$.

Finally, $\rho(Y,Z) \le \rho(Y,W) + \rho(W,Z) + \varepsilon$ and so we get the desired conclusion.

iii) a) $\rho(Y,Z) \le \varepsilon$ is equivalent with: for all $y \in Y, D(y,Z) \le \varepsilon$ and equivalent with $Y \subset V(Z,\varepsilon)$.

If Z is compact, then $Y \subset V(Z,\varepsilon)$ is equivalent with the fact that for all $y \in Y$ we have $D(y,Z) \leq \varepsilon$ and equivalent with: for all $y \in Y$ there exists $z_0 \in Z$ such that $d(y,z_0) \leq \varepsilon$, meaning that for all $y \in Y$ there exists $z_0 \in Z \cap \widetilde{B}(y;\varepsilon)$ and hence for all $y \in Y:\ Z \cap \widetilde{B}(y,\varepsilon) \neq \emptyset$.

c) Denote $u = \sup_{x\in X} D(x,Z) - D(x,Y)$. We shall prove that $\rho(Y,Z) \leq u$. If $u = \infty$ then the inequality is obviously. Let us consider $u < \infty$. Let $y \in Y$ and $v > u$. We have: $D(y,Z) = D(y,Z) - D(y,Y) \leq u < v$ and so $y \in V^0(Z,v)$. Hence we have proved that $Y \subseteq V^0(Z,v)$ and so we get that $\rho(Y,Z) \leq u$. We will prove now that $\rho(Y,Z) \geq u$. Let $\varepsilon > 0$ and $x \in X$. We can choose $y \in Y$ such that $d(x,y) < D(x,Y) + \varepsilon$. Let $z \in Z$ be such that $d(y,z) < D(y,Z) + \varepsilon \leq \rho(Y,Z) + \varepsilon$. We have $D(x,Z) \leq d(x,z) \leq d(x,y) + d(y,z) < D(x,Y) + \rho(Y,Z) + 2\varepsilon$ and so $D(x,Z) - D(x,Y) \leq \rho(Y,Z) + 2\varepsilon$. Because x was arbitrarily we obtain that $\sup_{x\in X} D(x,Z) - D(x,Y) \leq \rho(Y,Z) + 2\varepsilon$. For $\varepsilon \searrow 0$, we have $u \leq \rho(Y,Z)$. □

Lemma 1.17. *Let (X,d) be a metric space, $A,B \in P(X)$ and $(A_{i)})_{i\in I}$ a family of nonempty subsets of X. Then:*

a) $\rho(\bigcup_{i\in I} A_i, B) = \sup_{i\in I} \rho(A_i,B)$

b) If $A \in P_{cl}(X)$ then:

i) $\rho(A,\cdot) : (P_{cl}(X),H) \to \mathbb{R}_+$ *is nonexpansive.*

ii) $\rho(\cdot,A) : (P_{cl}(X),H) \to \mathbb{R}_+$ *is nonexpansive.*

Proof. b) ii) Let us consider $B,C \in P_{cl}(X)$ with $H(B,C) < +\infty$. Then $\rho(B,A) \leq \rho(B,C) + \rho(C,A)$ and $\rho(C,A) \leq \rho(C,B) + \rho(B,A)$. Since $\rho(C,B) < +\infty$ it is clear that $\rho(B,A) = +\infty$ if and only if $\rho(C,A) = +\infty$. If both are finite then $|\rho(C,A) - \rho(B,A)| \leq max\{\rho(B,C),\rho(C,B)\} = H(B,C)$. □

Lemma 1.18. *Let X be a normed space, A,B,C nonempty subsets of X and $r \in [0,1]$. Then:*

a) If A is convex, then $\rho(\overline{co}B, A) = \rho(A,B)$

b) If A is convex, then $\rho(rB + (1-r)C, A) \leq r\rho(B,A) + (1-r)\rho(C,A)$

c) $\rho(A, rB + (1-r)C) \leq r\rho(A,B) + (1-r)\rho(A,C)$

If (X,d) is a metric space, we have defined the generalized Pompeiu-Hausdorff functional $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ and we have shown that H is a generalized metric on $P_{cl}(X)$. Other important properties of the functional H are as follows.

Lemma 1.19. *Let (X,d) be a metric space and $Y,Z,V,W \in P(X)$. Then we have:*

i) $H(Y,Z)=0$ if and only if $\overline{Y}=\overline{Z}$

ii) $H(Y,Z)=H(Y,\overline{Z})$.

iii) $H(Y\bigcup V, Z\bigcup W) \le max\{H(Y,Z),H(V,W)\}$.

Proof. iii) From the definition of ρ we have:

$$\rho(Y\cup V, Z\cup W)=\sup\{D(x,Z\cup W)|\ x\in Y\cup V\}=$$

$$=\max\{\rho(Y,Z\cup W),\rho(V,Z\cup W)\}\le \max\{\rho(Y,Z),\rho(V,W)\}.$$

By a similar procedure we also get:

$$\rho(Z\cup W, Y\cup V)\le \max\{\rho(Z,Y),\rho(W,V)\}.$$

Hence

$$H(Y\cup V,Z\cup W)\le \max\{\rho(Y,Z),\rho(V,W),\rho(Z,Y),\rho(W,V)\}$$

$=\max\{H(Y,Z),H(V,W)\}$. $\square$

Lemma 1.20. *Let (X,d) be a metric space. Then we have:*

i) Let $Y,Z\in P(X)$. Then $H(Y,Z)=\sup\limits_{x\in X} D(x,Y)-D(x,Z)$

ii) The operator $I(x)=\{x\}$ is an isometry of (X,d) into $(P_{cl}(X),H_d)$

iii) Let $Y,Z\in P(X)$ and $\varepsilon>0$. Then for each $y\in Y$ there exists $z\in Z$ such that $d(y,z)\le H(Y,Z)+\varepsilon$.

iv) Let $Y,Z\in P(X)$ and $q>1$. Then for each $y\in Y$ there exists $z\in Z$ such that $d(y,z)\le qH(Y,Z)$.

v) If $Y,Z\in P_{cp}(X)$ then for each $y\in Y$ there exists $z\in Z$ such that $d(y,z)\le H(Y,Z)$.

vi) If $Y,Z\in P(X)$ then $H(Y,Z)\le\varepsilon$ is equivalent with the following assertion: for each $y\in Y$ there exists $z\in Z$ such that $d(y,z)\le\varepsilon$ and for each $z\in Z$ there exists $y\in Y$ with $d(y,z)\le\varepsilon$.

vii) Let $\varepsilon > 0$. If $Y, Z \in P(X)$ are such that $H(Y,Z) < \varepsilon$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) < \varepsilon$.

Proof. iii) Supposing contrary, there exists $\varepsilon > 0$ and exists $y \in Y$ such that for all $z \in Z$ we have $d(y,z) > H(Y,Z) + \varepsilon$. It follows that $D(y,Z) \geq H(Y,Z) + \varepsilon$ and so $H(Y,Z) \geq D(y,Z) \geq H(Y,Z) + \varepsilon$, proving that $\varepsilon \leq 0$, a contradiction.

iv) Supposing again contrary: there exists $q > 1$ and there exists $y \in Y$ such that for all $z \in Z$ we have $d(y,z) > qH(Y,Z)$. Then we have: $D(y,Z) \geq qH(Y,Z)$. But $H(Y,Z) \geq D(Y,Z) \geq qH(Y,Z)$. Hence $q \leq 1$, a contradiction. □

Remark 1.21. Using the above result (vi) it follows that the Pompeiu-Hausdorff functional can be also defined by the following formula:

$$H(A,B) = \inf\{\varepsilon > 0 |\ A \subset V(B,\varepsilon) \text{ and } B \subset V(A,\varepsilon)\},$$

for all $A, B \in P(X)$.

Lemma 1.22. *Let X be a Banach space. Then:*

i) $H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) \leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n)$, for all $Y_i, Z_i \in P(X)$, $i = 1, 2, \ldots, n$ $(n \in \mathbb{N}^*)$

ii) $H(Y + Z, Y + W) \leq H(Z,W)$, for all $Y, Z, W \in P(X)$

iii) $H(Y + Z, Y + W) = H(Z,W)$, for all $Y \in P_b(X)$ and for all $Z, W \in P_{b,cl,cv}(X)$

iv) $H(co\, Y, co\, Z) \leq H(Y,Z)$, for all $Y, Z \in P_b(X)$

v) $H(\overline{co}\, Y, \overline{co}\, Z) \leq H(Y,Z)$, for all $Y, Z \in P_{b,cl}(X)$.

Proof. i) Let $\varepsilon > 0$. From the definition of H it follows that there exists $(y_1 + \cdots + y_n) \in Y_1 + \cdots + Y_n$ such that $D(y_1 + \cdots + y_n, Z_1 + \cdots + Z_n) \geq H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon$ or exists $(z_1 + \cdots + z_n) \in Z_1 + \cdots + Z_n$ such that $D(z_1 + \cdots + z_n, Y_1 + \cdots + Y_n) \geq H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon$. Let us consider the first situation.

For $y_1, \ldots, y_n$ we get $z_1 \in Z_1, \ldots, z_n \in Z_n$ such that $\|y_1 - z_1\| \leq H(Y_1, Z_1) + \frac{\varepsilon}{4}, \ldots, \|y_n - z_n\| \leq H(Y_n, Z_n) + \frac{\varepsilon}{4}$. Then

$$\|(y_1 + \cdots + y_n) - (z_1 + \cdots + z_n)\| \leq \|y_1 - z_1\| + \cdots + \|y_n - z_n\| \leq$$

$$\leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n) + \varepsilon.$$

Because

$$H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon \leq D(y_1 + \cdots + y_n, z_1 + \cdots + z_n) \leq$$

$$\leq \|(y_1 + \cdots + y_n) - (z_1 + \cdots + z_n)\|$$

we obtain that

$$H(Y_1 + \cdots + Y_n, Z_1 + \cdots + Z_n) - \varepsilon \leq H(Y_1, Z_1) + \cdots + H(Y_n, Z_n) + \varepsilon,$$

proving the desired inequality.

iii) From ii) we have $H(Y + Z, Y + W) \leq H(Z, W)$. For the equality, let us suppose contrary: $H(Y + Z, Y + W) < H(Z, W)$. Let $t \in \mathbb{R}_+^*$ such that $H(Y + Z, Y + W) < t < H(Z, W)$. Then

$$Y + Z \subset Y + W + B_X(0;t) \subset Y + \overline{W + B_X(0;t)}$$

$$Y + W \subset Y + Z + B_X(0;t) \subset Y + \overline{Z + B_X(0;t)}.$$

Because $\overline{W + B_X(0;t)}$, $\overline{Z + B_X(0;t)} \in P_{cl,cv}(X)$ and $Y \in P_m(X)$ it follows from Lemma 4.1.7(i) that

$$Z \subset \overline{W + B_X(0;t)} \quad \text{and} \quad W \subset \overline{Z + B_X(0;t)}.$$

On the other side,

$$\overline{W + B_X(0;t)} = \bigcap_{n=1}^{n} [(W + B_X(0;t) + 2^{-n} B_X(0;1)]$$

$$\overline{Z + B_X(0;t)} = \bigcap_{n=1}^{n} [(Z + B_X(0;t) + 2^{-n} B_X(0;1)]$$

and choosing n such that $t + 2^{-n} < H(Z, W)$ we get

$$Z \subset W + (t + 2^{-n}) B_X(0;1) \quad \text{and} \quad W \subset Z + (t + 2^{-n}) B_X(0;1).$$

Hence we obtain $H(Z, W) \leq t + 2^{-n}$, a contradiction.

iv) Because $Y \subseteq co\,Y$ it follows that $D(z, co\,Y) \leq D(z, Y)$, for all $z \in Z$. Let $A = \{a \in X |\ D(a, co\,Y) \leq H(Y, Z)\}$. Of course A is convex and $A \supseteq Z$. we can write $co\, Z \subset A$ and hence for all $v \in co\, Z$ we have $D(v, co\, Y) \leq H(Y, Z)$. A similar procedure produces that for all $u \in co\, Y$ we have $D(u, co\, Z) \leq H(Y, Z)$. In conclusion: $H(co\, Y, co\, Z) \leq H(Y, Z)$.

v) Let $Y, Z \in P_{m,cl}(X)$ and $\varepsilon > 0$. Let $p \in \overline{co}\, Y$. Then there exist $y_1, y_2, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\left\| p - \sum_{i=1}^{n} \lambda_i y_i \right\| < \frac{\varepsilon}{2}.$$

For each $i = 1, 2, \ldots, n$ and $y_1, \ldots, y_n \in A$ there exist (see Lemma 1.1.21. iii)) $z_1, \ldots, z_n \in Z$ such that $\|y_i - z_i\| \leq H(Y, Z) + \frac{\varepsilon}{2}$. Let $q = \sum_{i=1}^{n} \lambda_i z_i$. Obviously $q \in \overline{co}\, Z$ and we also have:

$$\|p - q\| \leq \left\| p - \sum_{i=1}^{n} \lambda_i g_i \right\| + \left\| \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \lambda_i z_i \right\| <$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{n} \lambda_i \|y_i - z_i\| < H(Y, Z) + \varepsilon.$$

Hence

$$p \in V(\overline{co}\, Z; H(Y, Z) + \varepsilon) \;\Rightarrow\; \overline{co}\, Y \subseteq V(\overline{co}\, Z, H(Y, Z) + \varepsilon).$$

Similarly, we can be prove $\overline{co}\, Z \subseteq V(\overline{co}\, Y, H(Y, Z) + \varepsilon)$. In conclusion we obtain that $H(\overline{co}\, Y, \overline{co}\, Z) \leq H(Y, Z) + \varepsilon$, proving the conclusion. □

Remark 1.23. Let X be a normed space and $A \in P_{cp}(X)$. We denote $\|A\| = H(A, \{0\})$.

Some very important properties of the metric space $(P_{cl}(X), H_d)$ are contained in the following result:

Theorem 1.24. *i) If (X, d) is a complete metric space, then $(P_{cl}(X), H_d)$ is a complete metric space.*

ii) If (X, d) is a totally bounded metric space, then $(P_{cl}(X), H_d)$ is a totally bounded metric space.

iii) If (X,d) is a compact metric space, then $(P_{cl}(X),H_d)$ is a compact metric space.

iv) If (X,d) is a separable metric space, then $(P_{cp}(X),H_d)$ is a separable metric space.

v) If (X,d) is a ε-chainable metric space, then $(P_{cp}(X),H_d)$ is also an ε-chainable metric space.

Proof. i) We will prove that each Cauchy sequence in $(P_{cl}(X),H_d)$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(P_{cl}(X),H_d)$. Let us consider the set A defined as follows:

$$A=\bigcap_{n=1}^{\infty}\left(\overline{\bigcup_{m=n}^{\infty}A_m}\right).$$

We have two steps in the proof:

1) $A\neq\emptyset$.

In this respect, consider $\varepsilon>0$. Then for each $k\in\mathbb{N}$ there is $N_k\in\mathbb{N}$ such that for all $n,m\geq N_k$ we have $H(A_n,A_m)<\frac{\varepsilon}{2^{k+1}}$. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_k\geq N_k$. Let $x_0\in A_{n_0}$. Let us construct inductively a sequence $(x_k)_{k\in\mathbb{N}}$ having the following properties:

$\alpha)$ $x_k\in A_{n_k}$, for each $k\in\mathbb{N}$

$\beta)$ $d(x_k,x_{k+1})<\frac{\varepsilon}{2^{k+1}}$, for each$k\in\mathbb{N}$.

Suppose that we have $x_0,x_1,\ldots,x_k$ satisfying $\alpha)$ and $\beta)$ and we will generate x_{k+1} in the following way.

We have:

$$D(x_k,A_{n_{k+1}})\leq H(A_{n_k},A_{n_{k+1}})<\frac{\varepsilon}{2^{k+1}}.$$

It follows that there exists $x_{k+1}\in A_{n_{k+1}}$ such that $d(x_k,x_{k+1})<\frac{\varepsilon}{2^{k+1}}$.

Hence, we have proved that there exist a sequence $(x_k)_{k\in\mathbb{N}}$ satisfying $\alpha)$ and $\beta)$.

From $\beta)$ we get that $(x_k)_{k\in\mathbb{N}}$ is Cauchy in (X,d). Because (X,d) is complete it follows that there exists $x\in X$ such that $x=\lim\limits_{k\to\infty}x_k$. I need to show now that $x\in A$. Since $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence it follows that for $n\in\mathbb{N}^*$ there exists $k_n\in\mathbb{N}^*$ such that $n_{k_n}\geq n$. Then $x_k\in\bigcup\limits_{m\geq n}A_m$, for $k\geq k_n$, $n\in\mathbb{N}^*$

implies that $x \in \overline{\bigcup_{m \geq n} A_m}$, $n \in \mathbb{N}^*$. Hence $x \in A$.

2) In the second step of the proof, we will establish that $H(A_n, A) \to 0$ as $n \to \infty$.

The following inequalities hold:

$$d(x_k, x_{k+p}) \leq d(x_k, x_{k+1}) + \cdots + d(x_{k+p-1}, x_{k+p}) <$$

$$< \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+2}} + \cdots + \frac{\varepsilon}{2^{k+p}} < \varepsilon \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \ldots \right) =$$

$$= \varepsilon \frac{1}{1 - \frac{1}{2}} = 2\varepsilon, \text{ for all } p \in \mathbb{N}^*.$$

If in $d(x_k, x_{k+p}) < 2\varepsilon$ we are letting $p \to \infty$ we obtain $d(x_k, x) < 2\varepsilon$, for each $k \in \mathbb{N}$. In particular $d(x_0, x) < 2\varepsilon$. So, for each $n_0 \in \mathbb{N}$, $n_0 \geq N_0$ and for $x_0 \in A_{n_0}$ there exists $x \in A$ such that $d(x_0, x) \leq 2\varepsilon$, which imply

$$\rho(A_{n_0}, A) \leq 2\varepsilon, \text{ for all } n_0 \geq N_0 \quad (1).$$

On the other side, because the sequence $(A_n)_{n \in \mathbb{N}}$ is Cauchy, it follows that there exists $N_\varepsilon \in \mathbb{N}$ such that for $m, n \geq N_\varepsilon$ we have $H(A_n, A_m) < \varepsilon$. Let $x \in A$ be arbitrarily. Then $x \in \overline{\bigcup_{m=n}^{\infty} A_m}$, for $n \in \mathbb{N}^*$, which implies that there exist $n_0 \in \mathbb{N}$, $n_0 \geq N_\varepsilon$ and $y \in A_{n_0}$ such that $d(x, y) < \varepsilon$. Hence, there exists $m \in \mathbb{N}$, $m \geq N_\varepsilon$ and there is $y \in A_m$ such that $d(x, y) < \varepsilon$.

Then, for $n \in \mathbb{N}^*$, with $n \geq N_\varepsilon$ we have:

$$D(x, A_n) \leq d(x, y) + D(y, A_n) \leq d(x, y) + H(A_m, A_n) < \varepsilon + \varepsilon = 2\varepsilon.$$

So,

$$\rho(A, A_n) < 2\varepsilon, \text{ for each } n \in \mathbb{N} \text{ with } n \geq N_\varepsilon. \quad (2)$$

From (1) and (2) and choosing $n_\varepsilon := \max\{N_0, N_\varepsilon\}$ it follows that $H(A_n, A) < 2\varepsilon$, for each $n \geq n_\varepsilon$. Hence $H(A_n, A) \to 0$ as $n \to \infty$.

v) (X, d) being an ε-chainable metric space (where $\varepsilon > 0$) it follows, by definition, that for all $x, y \in X$ there exists a finite subset (the so-called ε-net) of X, let say $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{k-1}, x_k) < \varepsilon$, for all $k = 1, 2, \ldots, n$.

Let $y \in X$ arbitrary and $Y = \{y\}$. Obviously, $Y \in P_{cp}(X)$. Because the ε-chainability property is transitive, it is sufficient to prove that for all $A \in P_{cp}(X)$ there exist an ε-net in $P_{cp}(X)$ linking Y with A. We have two steps in our proof:

a) Let suppose first that A is a finite set, let say $A = \{a_1, a_2, ..., a_n\}$ We will use the induction method after the number of elements of A. If $n = 1$ then $A = \{a\}$ and the conclusion follows from the ε-chainability of (X, d). Let suppose now that the conclusion holds for each subsets of X consisting of at most n elements. Let A be a subset of X with $n+1$ points, $A = \{x_1, x_2, \ldots, x_{n+1}\}$. Using the ε-chainability of the space (X, d) it follows that there exist an ε-net in X, namely $x_1 = u_0, u_1, \ldots, u_m = x_2$ linking the points x_1 and x_2. We obtain that the following finite set: A, $\{u_1, x_2, \ldots, x_{n+1}\}, \ldots,$ $\{u_{m-1}, x_2, \ldots, x_{n+1},$ $\{x_2, \ldots, x_{n+1}\}$ is an ε-net in $P_{cp}(X)$ from A to $B := \{x_2, \ldots, x_{n+1}\}$. But, from the hypothesis B is ε-chainable with Y, and hence A is ε-chainable with Y in $P_{cp}(X)$.

b) Let consider now $A \in P_{cp}(X)$ be arbitrary.

A being compact, there exists a finite family of nonempty compact subsets of A, namely $\{A_k\}_{k=1}^{n}$, having $diam(A_k) < \varepsilon$ such that $A = \bigcup_{k=1}^{n} A_k$. For each $k = 1, 2, \ldots n$ we can choose $x_k \in A_k$ and define $C = \{x_1, \ldots, x_n\}$. Then for all $z \in A$ there exists $k \in \{1, 2, \ldots, n\}$ such that $D(z, C) \leq \delta(A_k)$. We obtain:

$$H(A, C) = \max\left\{\sup_{z \in A} D(z, C), sup_{y \in C} D(y, A)\right\} =$$

$$= \sup_{z \in A} D(z, C) \leq \max_{i \leq k \leq n} \delta(A_k) < \varepsilon,$$

meaning that A is ε-chainable by C in $P_{cp}(X)$. Using the conclusion a) of this proof, we get that C is ε-chainable by Y in $P_{cp}(X)$ and so we have proved that A is ε-chainable by Y in $P_{cp}(X)$. □

Bibliographical comments.

Other results and related notions can be found in books and papers on multivalued analysis such as: Aubin-Cellina [14], Aubin-Frankowska [15], Beer [24], Berge [26], Deimling [58], Hu-Papageorgiou [84], [93], Kirk-Sims (eds.) [97], Kisielewicz [100], G. Moţ, [119], Petruşel A. [149], I. A. Rus [172], etc.

Chapter 2

Basic notions and results

In this section, we describe some basic concepts and results for multivalued operators.

Let X and Y two nonempty sets. A multivalued operator (or a multifunction) from X into Y is a correspondence which associates to each element $x \in X$ a subset $F(x)$ of Y. Hence, a multivalued operator mens $F : X \to \mathcal{P}(Y)$. Occasionally we will denote it by: $F : X \multimap Y$. Throughout this book we denote single-valued operators by small letters and multivalued operators by capital letters.

Multivalued operators arises in various branches of pure and applied mathematics, as we can see from the following examples:

i) The metric projection multifunction. Let (X, d) be a metric space and $Y \in P(X)$. Then the metric projection on Y is the multifunction $P_Y : X \to \mathcal{P}(Y)$ defined by:

$$P_Y(x) = \{y \in Y |\ D(x, Y) = d(x, y)\}.$$

If X is a Hilbert space and Y is a closed convex set, then P_Y becomes a single-valued operator.

ii) Implicit differential equations. Consider the implicit differential equation:

$$f(t, x, x') = 0, x(0) = x^0.$$

This problem may be reduced to a multivalued initial value problem:

$$x'(t) \in F(t, x(t)), x(0) = x^0$$

involving the multivalued operator $F(t, x) := \{v | f(t, x, v) = 0\}$.

iii) Differential inequalities. The differential inequality:

$$\|x'(t) - g(t, x)\| \leq f(t, x), x(0) = x^0$$

may be recast into the form:

$$x'(t) \in F(t, x(t)), x(0) = x^0$$

with $F(t, x) = \tilde{B}(g(t, x), f(t, x))$, where $\tilde{B}$ denotes the closed ball.

iv) Control theory. If $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and it determines the dynamics of a system having the equations of motion given by:

$$x'(t) = f(t, x(t), u(t)), x(0) = x^0,$$

where u is the so-called "control operator" and it may be chosen as any measurable operator from $U(t, x(t))$ (denote by $U : \mathbb{R} \times \mathbb{R}^n \to P(\mathbb{R}^m)$ the feedback multifunction), then we have, by definition, a control theory problem.

The description of this control system can be presented in a differential inclusion form:

$$x'(t) \in F(t, x(t)), x(0) = x^0,$$

involving the multivalued operator $F(t, x) = \{f(t, x(t), u(t)) |\ u \in U(t, u(t)\}$.

v) Optimal preference and equilibrium of an abstract economy. Let us consider now the Arrow-Debreu model of an economy. Recall that $\mathbb{R}^n$ is the commodity space. A vector $x \in \mathbb{R}^n$ specifies a list of quantities of each commodity. A price p is also an element of $\mathbb{R}^n$, because p lists the value of an unit of each commodity. The main "actors" in a economy are the consumers. We assume that there is a given finite number of consumers. If M is the income of the consumer, then his budget set is $A = \{x \in X | p \cdot x \leq M\}$, where X denotes the consumption set (i.e. the set of all admissible consumption vectors of the consumer). The problem faced by a consumer is to choose a consumption vector or a set of them from the budget set. In order to do this, the consumer must have some criterion for choosing. Let us denote by U the preferences multivalued operator for our consumer: $U : X \to \mathcal{P}(X)$, $U(x) = \{y \in X|$ y is strictly prefered to x$\}$.

An element $x^* \in X$ is an optimal preference for the consumer if $U(x^*) = \emptyset$. This is the so-called consumer's problem.

Another important question from mathematical economics is the equilibrium price problem. The set of sums of demand vectors minus sums of supply vectors is, by definition, the excess-demand multifunction, denoted by $E(p)$. A Walrasian equilibrium price problem means the following:

$$\text{find a price } p^* \in \mathbb{R}^n \text{ such that } 0 \in E(p^*).$$

v) Multivalued fractals. Let (X, d) be a metric space and $F_1, \ldots, F_m : X \to P_{cl}(X)$ be u.s.c. multivalued operators. The system $F = (F_1, \ldots, F_m)$ is called an iterated multifunction system (briefly IMS).

In the theory of multivalued fractals appears the following concept.

The multivalued operator

$$\tilde{F} : X \to P_{cl}(X), \tilde{F}(x) := \overline{\bigcup_{i=1}^{m} F_i(x)}, \text{ for each } x \in X,$$

is called the Barnsley-Hutchinson multifunction generated by the IMS F.

Let us remark that $\tilde{F}$ is well defined and if $F_i : X \to P_{b,cl}(X)$, for $i \in \{1, \ldots, m\}$, then $\tilde{F} : X \to P_{b,cl}(X)$.

In the same setting, the operator

$$\widetilde{T_F} : P_{cl}(X) \to P_{cl}(X), \widetilde{T_F}(Y) = \overline{\bigcup_{i=1}^{m} F_i(Y)},$$

is well defined and it is called the extended multi-fractal operator generated by the IMS F. A fixed point of $\widetilde{T_F}$ is called a multivalued large fractal. For other details we refer to Chifu-Petruşel A. [45].

Let us recall now some basic notions in the analysis of multivalued operators.

Definition 2.1. Let X, Y be two nonempty sets. For the multivalued operator $F : X \to \mathcal{P}(Y)$ we define:

i) the effective domain: $Dom\, F := \{x \in X|\ F(x) \neq \emptyset\}$

ii) the graph: $Graf\, F := \{(x,y) \in X \times Y|\ y \in F(x)\}$

iii) the range: $F(X) := \bigcup\limits_{x\in X} F(x)$

iv) the image of the set $A \in P(X)$: $F(A) := \bigcup\limits_{x\in A} F(x)$

v) the inverse image of the set $B \in P(Y)$:

$$F^-(B) := \{x \in X|\ F(x) \cap B \neq \emptyset\}$$

vi) the strict inverse image of the set $B \in P(Y)$:

$$F^+(B) := \{x \in Dom\, F|\ F(x) \subset B\}.$$

vii) the inverse multivalued operator, denoted $F^{-1} : Y \to \mathcal{P}(X)$ and defined by $F^{-1}(y) := \{x \in X|\ y \in F(x)\}$. The set $F^{-1}(y)$ is called the fibre of F at the point y.

Remark 2.2. We consider, by convention: $F^-(\emptyset) = \emptyset$ and $F^+(\emptyset) = \emptyset$.

Definition 2.3. Let $F, G : X \to \mathcal{P}(Y)$ be multivalued operators. Then:

i) If $\otimes$ defines a certain operation between sets, then we will use the same symbol $\otimes$ for the corresponding operation between multifunctions, namely: $F \otimes G : X \to \mathcal{P}(Y)$, $(F \otimes G)(x) := F(x) \otimes G(x)$, $\forall\, x \in X$. (where $\otimes$ could be $\cap$, $\cup$, $+$, etc.)

iii) If $\eta : \mathcal{P}(Y) \to \mathcal{P}(Y)$, then we define $\eta(F) : X \to \mathcal{P}(Y)$ by $\eta(F)(x) := \eta(F(x))$, for all $x \in X$. In such way, we are able to define in topological spaces, for example, $\overline{F} : X \to \mathcal{P}(Y)$, $\overline{F}(x) = \overline{F(x)}$, for all $x \in X$ or $co\, F : X \to \mathcal{P}(Y)$, $(co\, F)(x) := co(F(x))$, for all $x \in X$ in linear spaces, etc.

Definition 2.4. Let X, Y, Z be nonempty sets and $F : X \multimap Y$, $G : Y \multimap Z$ be multivalued operators. The composite of G and F is the multivalued operator $H = G \circ F$, defined by the relation $H : X \multimap Z$, $H(x) := \bigcup\limits_{y \in F(x)} G(y)$.

If X is a nonempty set, then $Y \in P(X)$ is said to be invariant with respect to a multivalued operator $F : X \to P(X)$ if $F(Y) \subset Y$. The family of all invariant subsets of F will be denoted by $I(F)$. Also, if $f : X \to \mathbb{R}$, then Z_f denotes the set of all zero point of f, i. e. $Z_f = \{x \in X | f(x) = 0\}$.

Definition 2.5. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then, F is called:

i) a-Lipschitz if $a \geq 0$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2)$, for all $x_1, x_2 \in X$.

ii) a-contraction if it is a-Lipschitz, with $a < 1$.

iii) contractive if $H(F(x_1), F(x_2)) < d(x_1, x_2)$, for all $x_1, x_2 \in X,\ x_1 \neq x_2$.

Lemma 2.6. *Let (X, d), (Y, d') and (Z, d'') be metric spaces. Then:*

i) If $F : X \to P_{b,cl}(Y)$ is a-Lipschitz and $G : X \to P_{b,cl}(Y)$ is b-Lipschitz, then $F \cup G$ is $\max\{a, b\}$-Lipschitz.

ii) If $F : X \to P_{cp}(Y)$ is a-Lipschitz and $G : Y \to P_{cp}(Z)$ is b-Lipschitz, then $G \circ F$ is ab-Lipschitz.

Lemma 2.7. *Let X be a Banach space and $F : X \to P_{b,cl}(X)$ be a-Lipschitz. Then $\overline{co}\, F : X \to P_{b,cl}(X)$ defined by $(\overline{co}\, F)(x) = \overline{co}(F(x))$, for all $x \in X$ is a-Lipschitz. Moreover, if $F : X \to P_{cp}(X)$ then $\overline{co}\, F : X \to P_{cp}(X)$.*

Let us remark now that, if (X, d) is a metric space and Y is a Banach space, then a multifunction $F : X \to \mathcal{P}(Y)$ is said to be α -Lipschitz on the set $K \in P(X)$ if $\alpha \geq 0$ and

$$F(x_1) \subseteq F(x_2) + \alpha d(x_1, x_2)\widetilde{B}(0; 1),\ \text{for all } x_1, x_2 \in K.$$

It is quite obviously that, if there exists $a > 0$ such that F is a-Lipschitz in the sense of Definition 2.5., then F is α-Lipschitz in the above mentioned sense with any $\alpha > a$ and also reversely.

Bibliographical comments.

For further results and more details see Aubin-Frankowska [15], Beer [24], Deimling [58], Hu-Papageorgiou [84], Kamenskii-Obuhovskii-Zecca [93], Kirk-Sims (eds.) [97], Kisielewicz [100], Petruşel A. [149], I. A. Rus [172].

Chapter 3

Continuity concepts

Let us consider, for the beginning, the following characterization theorem of the continuity of a singlevalued operator.

Theorem. *Let X, Y be topological spaces and $f : X \to Y$. Then the following assertions are equivalent:*

i) f is a continuous operator on X.

ii) for each $x_0 \in X$ and each open neighborhood V of $f(x_0)$ there is an open neighborhood U of x_0 such that $f(U) \subset V$.

iii) for each $x_0 \in X$ and each net $(x_i)_{i\in I} \subset X$ which converges to x_0, we have that $(f(x_i))_{i\in I} \subset Y$ converges to $f(x_0)$.

iv) The set $f^{-1}(V) = \{x \in X | \ f(x) \leq V\}$ is open, for each open set $V \subset Y$.

v) The set $f^{-1}(W) = \{x \in X | \ f(x) \in W\}$ is closed, for each closed set $W \subset Y$.

If $F : X \to P(Y)$ is a multivalued operator then the following conditions are no longer equivalent:

a) for each $x_0 \in X$ and each open set V in Y such that $F(x_0) \subset V$ there exists an open neighborhood U of x_0 having the property $F(U) \subset V$.

b) for each $x_0 \in X$ and each open set V in Y with $F(x_0) \cap V \neq \emptyset$ there exists an open neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$, for each $x \in U$.

c) for each $x_0 \in X$ and each net $(x_i)_{i\in I} \subset X$ which converges to x_0 and for each $(y_i)_{i\in I} \subset Y$, $y_i \in F(x_i)$, $i \in I$ that converges to an element $y_0 \in Y$

we have $y_0 \in F(x_0)$.

d) $F^-(V) := \{x \in X|\ F(x) \cap V \neq \emptyset\}$ is an open set, for each open set $V \subset Y$.

e) $F^+(V) := \{x \in X|\ F(x) \subset V\}$ is open, for each open set $V \subset Y$.

f) $F^-(W) := \{x \in X|\ F(x) \cap W \neq \emptyset\}$ is closed, for each closed set $W \subset Y$.

g) $F^+(W) := \{x \in X|\ F(x) \subset W\}$ is closed, for each closed set $W \subset Y$.

Hence, it is quite natural the fact that we will discern several notions of continuity for multifunctions.

Let us consider, for the beginning, the notion of upper semi-continuity of a multifunction.

Definition 3.1. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is said to be upper semi-continuous in $x_0 \in X$ (briefly u.s.c.) if and only if for each open subset U of Y with $F(x_0) \subset U$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $F(x) \subset U$.

F is u.s.c. on X if it is u.s.c. in each $x_0 \in X$.

Remark 3.2. If $x_0 \in X$ has the property $F(x_0) = \emptyset$ then F is u.s.c. in x_0 if and only if there exists a neighborhood V of x_0 such that $F(V) = \emptyset$.

Remark 3.3. If X, Y are metric spaces, then $F : X \to P(Y)$ is u.s.c. in $x_0 \in X$ if and only if for all $U \subset Y$ open, with $F(x_0) \subset U$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset U$.

Definition 3.4. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then F is called H-upper semi-continuous in $x_0 \in X$ (briefly H-u.s.c.) if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset V(F(x_0); \varepsilon)$.

F is H-u.s.c. on X if it is H-u.s.c. in each $x_0 \in X$.

Remark 3.5. If $F : X \to P_{b,cl}(Y)$ then F is H-u.s.c. in $x_0 \in X$ if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $\rho_{d'}(F(x), F(x_0)) \leq \varepsilon$.

Lemma 3.6. *Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P(Y)$. If F is u.s.c. in $x_0 \in X$ then F is H-u.s.c. in $x_0 \in X$.*

For a reverse implication, we have:

Lemma 3.7. *Let $(X,d),(Y,d')$ be metric spaces. If $F : X \to P_{cp}(Y)$ is H-u.s.c. in $x_0 \in X$ then F is u.s.c. in $x_0 \in X$.*

Remark 3.8. $F : X \to P_{b,cl}(X)$ is H-u.s.c. in $x_0 \in X$ if and only if for each sequence $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} \rho(F(x_n), F(x_0)) = 0$.

For Hausdorff topological spaces, we have the following characterization of global upper semi-continuity:

Theorem 3.9. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. The following assertions are equivalent:*

i) F is u.s.c. on X

ii) $F^+(V) = \{x \in X |\ F(x) \subset V\}$ is open, for each open set $V \subset Y$.

iii) $F^-(W) = \{x \in X |\ F(x) \cap W \neq \emptyset\}$ is closed, for each closed set $W \subset Y$.

Lemma 3.10. *a) Let X, Y, Z be Hausdorff topological spaces and $F : X \to P(Y)$, $G : Y \to P(Z)$ be u.s.c. on X respectively on Y. Then $G\circ F : X \to P(Z)$ is u.s.c. on X.*

b) If X, Y are Hausdorff topological spaces and $F : X \to P_{cl}(Y)$ is u.s.c. on X, then $Graf\, F$ is a closed set in $X \times Y$.

Lemma 3.11. *Let $(X,d),(Y,d')$ be metric spaces, $f : X \to Y$ be a continuous operator and $F : X \to P_{b,cl}(Y)$ be a multivalued operator H-u.s.c. on X. then the functional $p : X \to \mathbb{R}_+$, defined by $p(x) := D(f(x), F(x))$, for all $x \in X$ is lower semi-continuous on X.*

Proof. Let $x \in X$ be a fixed point and $(x_n)_{n\in\mathbb{N}} \subset X$ convergent to x. It follows that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(f(x), f(x_n)) < \frac{\varepsilon}{2}$, for all $n \geq N_\varepsilon$. From the H-u.s.c. of F in x we have that $\rho(F(x_n), F(x)) < \frac{\varepsilon}{2}$, for all $n \geq N_\varepsilon$. Hence, for each $n \geq N_\varepsilon$ we have: $p(x) = D(f(x), F(x)) \leq d(f(x), f(x_n)) + D(f(x_n), F(x_n)) + \rho(F(x_n), F(x)) < \varepsilon + p(x_n)$. If $p^* = \liminf\limits_{n\to\infty} p(x_n)$ then for each $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}^*$ such that $p(x_n) < p^* + \varepsilon$, for each $n \geq N_\varepsilon$. So, for each $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}^*$ such that $p(x) < \varepsilon + p(x_n) <$

$\varepsilon + p^* + \varepsilon$, for each $n \geq N_\varepsilon$. When $\varepsilon \searrow 0$ we get $p(x) \leq p^*$, proving that p is lower semicontinuous in x. □

Lemma 3.12. *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cp}(Y)$ be u.s.c. on X. Then, the multivalued operator $\overline{co}\, F : X \to P(Y)$ is u.s.c. on X.*

Proof. From Mazur's theorem (see Dugundji [60]) $\overline{co}\, F(x)$ is compact, for all $x \in X$ and hence $\overline{co}\, F$ has compact values. Using Lemma 1.3.7. it is sufficient to prove that $\overline{co}\, F$ is H-u.s.c. on X. Let $x \in X$ be an arbitrary point and $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to x. From

$$\rho(\overline{co}\, F(x_n), \overline{co}\, F(x)) \leq \rho(F(x_n), F(x)), \text{ for all } \ n \in \mathbb{N}^*$$

and using the hypothesis that F is H-u.s.c. on X we get the desired conclusion.□

Lemma 3.13. *Let X, Y be Hausdorff topological spaces and $F : X \to P_{cp}(Y)$ be u.s.c. on X. Then, for each compact subset K of X, $F(K)$ is a compact set in Y.*

A simple fact is:

Lemma 3.15. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ be u.s.c. on X. Then, $\{x \in X | F(x) \neq \emptyset\}$ is a closed subset of X.*

Lemma 3.16. *a) Let X, Y be Hausdorff topological spaces, $F_i : X \to P_{cp}(Y)$ be u.s.c. on X for each $i \in I$ such that $\bigcap_{i\in I} F_i(x) \neq \emptyset$ for each $x \in X$ and $H_j : X \to P_{cp}(Y)$, be u.s.c. for each $j \in \{1, 2, \ldots, n\}$. Then:*

i) $F := \bigcap_{i\in I} F_i$ is u.s.c. on X and has compact values.

ii) $H := \bigcup_{j=1}^{n} H_j$ is u.s.c. on X and has compact values.

b) If Y is a normed spaces and $F_1, F_2 : X \to P_{cp}(Y)$ are u.s.c. then, $T : X \to P_{cp}(Y)$, $T = F_1 + F_2$ is u.s.c. on X.

Another continuity notion for a multifunction is defined as follows:

Definition 3.17. Let $(X,d),(Y,d')$ be metric spaces and $F : X \to P(Y)$. Then F is said to be closed in $x_0 \in X$ if and only if for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and for all $(y_n)_{n\in\mathbb{N}^*} \subset Y$, with $y_n \in F(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} y_n = y_0$ we have $y_0 \in F(x_0)$.

F is closed on X if it is closed in each point $x_0 \in X$.

Remark 3.18. An equivalent definition is the following: $F : X \to P(Y)$ is said to be closed in $x_0 \in X$ if and only if for each $y_0 \notin F(x_0)$ there exist a neighborhood V of x_0 and a neighborhood U of y_0 such that for all $x \in V$ it follows that $F(x) \cap U = \emptyset$.

Lemma 3.19. *Let $(X,d),(Y,d')$ be metric spaces and $F : X \to P(Y)$ closed on X. Then:*

i) $F(x) \in P_{cl}(Y)$, for all $x \in X$

ii) $Graf\,F$ is a closed set with respect to the Pompeiu-Hausdorff topology from $X \times Y$. Moreover, the condition ii) implies that F is closed on X.

Lemma 3.20. *Let X, Y be Hausdorff topological spaces, $F_i : X \to P(Y)$, $i \in I$ be closed on X such that $\bigcap\limits_{i\in I} F_i(x) \neq \emptyset$ for each $x \in X$ and $H_j : X \to P(Y)$, $j \in \{1,\ldots,n\}$ be closed on X. Then:*

i) $F := \bigcap\limits_{i\in I} F_i$ is closed on X.

ii) $H := \bigcup\limits_{j=1}^{n} H_j$ is closed on X.

The relations between upper semi-continuity and closedness are given by the following results:

Lemma 3.21. *Let $(X,d),(Y,d')$ be metric spaces and $F : X \to P_{b,cl}(Y)$ be H-u.s.c. on X. Then F is closed on X.*

Proof. Let $x \in X$ and $((x_n,y_n))_{n\in\mathbb{N}} \subset X \times Y$ such that $(x_n,y_n) \to (x,y)$ as $n \to \infty$ with $y_n \in F(x_n)$, for all $n \in \mathbb{N}$. F is H-u.s.c. in x and hence $\lim\limits_{n\to\infty} \rho(F(x_n),F(x)) = 0$. On the other side, $D(y,F(x)) \leq d(y,y_n) + D(y_n,F(x_n)) + \rho(F(x_n),F(x))$, for all $n \in \mathbb{N}$. If we take $n \to \infty$ it follows that $D(y,F(x)) \leq 0$ and so $y \in \overline{F(x)} = F(x)$. □

For a reverse proposition, we have:

Theorem 3.22. *Let $(X, d), (Y, d')$ be metric spaces, $F_1 : X \to P(Y)$ closed and $F_2 : X \to P_{cp}(Y)$ u.s.c.. Suppose that $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$. Then, the multivalued operator $F = F_1 \cap F_2$ is u.s.c. and it has compact values.*

Corollary 3.23. *Let (X, d) be a metric space, (Y, d') be a compact metric space and $F : X \to P(Y)$ closed on X. Then F is u.s.c. on X and it has compact values.*

Definition 3.24. Let X, Y be topological spaces. A multifunction $F : X \to P(Y)$ is said to be compact if its range $F(X)$ is relatively compact in Y.

Lemma 3.25. *Let X, Y be metric spaces and $F : X \to P_{cp}(Y)$ be a closed and compact multifunction. Then F is u.s.c.*

Lemma 3.26. *Let X, Y be metric spaces and $F : X \to P_{cl}(Y)$ be a closed multifunction. Then for each compact subset K of X its image $F(K)$ is closed in Y.*

Let us consider now the concept of lower semi-continuous multifunction.

Definition 3.27. Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}(\mathcal{Y})$. Then, F is said to be lower semi-continuous (briefly l.s.c.) in $x_0 \in X$ if and only if for each open subset $U \subset Y$ with $F(x_0) \cap U \neq \emptyset$ there exists an open neighborhood V of x_0 such that $F(x) \cap U \neq \emptyset$, for all $x \in V$.

F is l.s.c. on X if it is l.s.c. in each $x_0 \in X$.

Remark 3.28. If $(X, d), (Y, d')$ are metric spaces and $F : X \to P(Y)$, then F is l.s.c. in $x_0 \in X$ if and only if for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$ and for all $y_0 \in F(x_0)$ there exists a sequence $(y_n)_{n\in\mathbb{N}^*} \subset Y$ such that $y_n \in F(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim_{n\to\infty} y_n = y_0$.

Another lower semi-continuity notion is given by:

Definition 3.29. Let (X, d) and (Y, d') be metric spaces and $F : X \to P(Y)$. Then, F is called H-lower semi-continuous (briefly H-l.s.c.) in $x_0 \in X$

if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $F(x_0) \subset V(F(x);\varepsilon)$, for all $x \in B(x_0;\eta)$.

F is H-l.s.c. on X if it is H-l.s.c. in each point $x_0 \in X$.

Remark 3.30. $F : X \to P_{b,cl}(Y)$ is H-l.s.c. in $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\rho_{d'}(F(x_0), F(x)) \leq \varepsilon$, for all $x \in B(x_0;\eta)$.

Lemma 3.31. *Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$ be H-l.s.c. in $x_0 \in X$. Then F is l.s.c. in $x_0 \in X$.*

Regarding the reverse implication we have:

Lemma 3.32. *Let (X, d), (Y, d') be metric spaces and $F : X \to P_{cp}(Y)$ be l.s.c. in $x_0 \in X$. then F is H-l.s.c. in $x_0 \in X$.*

A characterization result for l.s.c. multifunctions is:

Theorem 3.33. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then, the following assertions are equivalent:*

i) F is l.s.c. on X

ii) $F^+(V) := \{x \in X|\ F(x) \subset V\}$ is closed, for each closed set $V \subset Y$.

iii) $F^-(W) := \{x \in X|\ F(x) \cap W \neq \emptyset\}$ is open, for each open set $W \subset Y$.

Lemma 3.34. *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P(Y)$ be l.s.c.. Then, the multivalued operator $\overline{co}\,F$ is l.s.c.*

Lemma 3.35. *Let X, Y, Z be Hausdorff topological spaces. Then:*

i) If $F : X \to P(Y)$ and $G : Y \to P(Z)$ are l.s.c. on X respectively on Y then $G \circ F : X \to P(Z)$ is l.s.c. on X.

ii) If $F_i : X \to P(Y)$, are l.s.c. on X, for each $i \in I$, then $F := \bigcup_{i\in I} F_i$ is l.s.c. on X.

An useful result is:

Lemma 3.36. *Let (X, d), (Y, d') be metric spaces. If $F_1 : X \to P(Y)$ is l.s.c. and $F_2 : X \to P(Y)$ has open graph, such that $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$, then the multivalued operator $F_1 \cap F_2$ is l.s.c..*

Definition 3.37. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is said to be continuous in $x_0 \in X$ if and only if it is l.s.c. and u.s.c. in $x_0 \in X$.

Definition 3.38. Let (X, d), (Y, d') be metric spaces and $F : X \to P(Y)$. Then F is called H-continuous in $x_0 \in X$ (briefly H-c.) if and only if it is H-l.s.c. and H-u.s.c. in $x_0 \in X$.

Remark 3.39. If $(X, d), (Y, d')$ are metric spaces, then $F : X \to P_{b,cl}(Y)$ is H-c. in $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $x \in B(x_0; \eta)$ implies $H_{d'}(F(x), F(x_0)) < \varepsilon$.

Theorem 3.40. *Let (X, d) and (Y, d') be metric spaces. Then $F : X \to P_{cp}(Y)$ is continuous on X if and only if F is H-c. on X.*

The relation between H-continuity and lower semi-continuity is given in:

Lemma 3.41. *Let $(X, d), (Y, d')$ be metric spaces and $F : X \to P_{b,cl}(Y)$ be H-c. on X. Then F is l.s.c. on X.*

Further on, we will present some properties of multivalued Lipschitz-type operators.

Lemma 3.42. *Let (X, d) be a metric space and $F : X \to P_{b,cl}(X)$ be a-Lipschitz. Then:*

a) F is closed on X

b) F is H-l.s.c. on X

c) F is H-u.s.c. on X.

Proof. a)Let $(x_n, y_n)_{n\in\mathbb{N}} \subset X \times X$ such that $(x_n, y_n) \to (x, y)$, when $n \to \infty$ and $y_n \in F(x_n)$, for all $n \in \mathbb{N}$. It follows that $d(y, F(x)) \leq d(y, y_n) + D(y_n, F(x)) \leq d(y, y_n) + H(F(x_n), F(x)) \leq d(y, y_n) + ad(x_n, x)$, for all $n \in \mathbb{N}$. Let us consider $n \to \infty$ and we obtain $D(y, F(x)) \leq 0$, proving that $y \in \overline{F(x)} = F(x)$.

b) Let $x \in X$ such that $x_n \to x$. We have: $\rho(F(x), F(x_n)) \leq H(F(x), F(x_n)) \leq ad(x, x_n) \to 0$. In conclusion, F is H-l.s.c. on X.

c) Using the relation: $\rho(F(x_n), F(x)) \leq H(F(x_n), F(x)) \leq ad(x, x_n) \to 0$, the conclusion follows as before. □

Lemma 3.43. Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be contractive. Then F is u.s.c. on X.

Proof. Let $H \subset Y$ be a closed set. We will prove that $F^{-}(H)$ is closed in X. Let $x \in \overline{F^{-}(H)} \setminus F^{-}(H)$ and $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$, when $n \to \infty$, $x_n \neq x$, for all $n \in \mathbb{N}$ and $x_n \in F^{-}(H)$, for all $n \in \mathbb{N}$. It follows $F(x_n) \cap H \neq \emptyset$, for all $n \in \mathbb{N}$. Let $y_n \in F(x_n) \cap H$, $n \in \mathbb{N}$. Then $D(y_n, F(x)) \leq H(F(x_n), F(x)) < d(x_n, x)$. If $n \to \infty$ we get that $\lim\limits_{n\to\infty} D(y_n, F(x)) = 0$. But $D(y_n, F(x)) = \inf\limits_{y\in F(x)} d(y_n, y) = d(y_n, x'_n)$ (using the compactness of the set $F(x)$). When $n \to \infty$ we have $d(y_n, y'_n) \to 0$, $n \to \infty$. Because $(y'_n)_{n\in\mathbb{N}} \subset F(x)$ we obtain that there exists a subsequence $(y'_{n_k})_{k\in\mathbb{N}}$ which converges to an element $\widetilde{x} \in F(x)$. Then:

$$d(y_{n_k}, \widetilde{x}) \leq d(y_{n_k}, x'_{n_k}) + d(x'_{n_k}, \widetilde{x}) \to 0 \text{ când } k \to \infty$$

Hence, $y_{n_k} \to \widetilde{x} \in F(x)$, as $n \to \infty$. Because, $(y_{n_k})_{k\in\mathbb{N}} \subset H$ and H is closed, we obtain that $\widetilde{x} \in H$. So $F(x) \cap H \neq \emptyset$, which implies $x \in F^{-}(H)$, a contradiction. In conclusion, $\overline{F^{-}(H)} = F^{-}(H)$ and hence $F^{-}(H)$ is closed in X. □

Bibliographical comments.

The notions and results given in this chapter can be found in books and papers on multivalued analysis such as: Aubin-Cellina [14], Aubin-Frankowska [15], Beer [24], Berge [26], Cernea [41], Deimling [58], Hu-Papageorgiou [84], Kamenskii-Obuhovskii-Zecca [93], Kirk-Sims (eds.) [97], Kisielewicz [100], M. Mureşan [124], Petruşel A. [149], I. A. Rus [172], Xu [212], etc.

Part II

Selections, Fixed Points and Strict Fixed Points

Chapter 4

Selection theorems

First, we will consider the basic selection theorems for l.s.c. and u.s.c. multifunctions.

Definition 4.1. Let X, Y be nonempty sets and $F : X \to P(Y)$. Then the single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.

If X is a metric space and $(U_i)_{i \in I}$ is an open covering for X, then a locally Lipschitz partition of unity corresponding to $(U_i)_{i \in I}$ means a family of locally Lipschitz functions $\varphi_i : X \to [0, 1]$ such that:

(i) $supp\varphi_i \subset U_i$, for each $i \in I$

(ii) $(supp\varphi_i)_{i \in I}$ is a closed locally finite covering of X

(iii) $\sum_{i \in I} \varphi_i(x) = 1$, for each $x \in X$.

If $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ are two coverings of a metric space X, then (U_i) is a refinement of (V_i) if for every $i \in I$ there exists $j \in J$ such that $U_i \subset V_j$.

Recall that an open covering $(V_i)_{i \in I}$ of X is said to be locally finite if and only if for each $x \in X$ there exists V an open neighborhood of x such that $card\{i \in I | V_i \cap V \neq \emptyset\}$ is finite. Also recall that in a paracompact space X (in particular in a metric space) each open covering of X has a locally finite open refinement, such that there exists a locally Lipschitz partition of unity subordinated to it.

A very famous result is the so-called Michael' selection theorem. We start by proving the following auxiliary result:

Lemma 4.2. *Let (X,d) be a metric space, Y a Banach space and $F : X \to P_{cv}(Y)$ be l.s.c. on X. Then, for each $\varepsilon > 0$ there exists $f_\varepsilon : X \to Y$ a continuous operator such that for all $x \in X$, we have: $f_\varepsilon(x) \in V^0(F(x);\varepsilon)$.*

Proof. Because F is l.s.c. we associate to each $x \in X$ and to each $y_x \in F(x)$ an open neighborhood U_x of x such that $F(x') \cap B(y_x;\varepsilon) \neq \emptyset$, for all $x' \in U_x$. Since X is a metric space there exists a locally finite refinement $\{U'_x\}_{x\in X}$ of $\{U_x\}_{x\in X}$. Let us recall that $\{\Omega_i\}_{i\in I}$ is a locally finite covering of X if for each $x \in X$ there exists V a neighborhood of x satisfying $\Omega_i \cap V \neq \emptyset$, for all $i = \overline{1,k}$. Moreover, to each locally finite covering it is possible to associate a partition of unity locally Lipschitz, let say $\{\pi_x\}_{x\in X}$, i. e. $\pi_x : X \to [0,1]$ has the following properties: $(supp\pi_x)_{x\in X}$ is a locally finite covering of X, with $supp\pi_x \subset U'_x$ and $\sum_{x\in X} \pi_x(t) = 1$, for each $t \in X$. We define: $f_\varepsilon(t) = \sum\limits_{x\in X} \pi_x(t) y_x$. Then f_ε is continuous, being, locally, a finite sum of continuous operators. Moreover, if $\pi_x(t) > 0$, for $t \in U'_x \subset U_x$ then $y_x \in V^0(F(t),\varepsilon)$ implies that $f_\varepsilon(t) \in V^0(F(t),\varepsilon)$. □

Theorem 4.3. (Michael' selection theorem) *Let (X,d) be a metric space, Y be a Banach space and $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X. Then there exists $f : X \to Y$ a continuous selection of F.*

Proof. Let us define inductively a sequence of continuous operators $u_n : X \to Y$, $n = 1,2,\ldots$ satisfying the following assertions:

i) for all $x \in X$, $D(u_n(x),F(x)) < \frac{1}{2^n}$, for each $n \in \mathbb{N}^*$

ii) for all $x \in X$, $\|u_n(x) - u_{n-1}(x)\| \leq \frac{1}{2^{n-2}}$, for each $n = 2,3,\ldots$

1. Case $n = 1$. The conclusion follows from Lemma 4.2. with $\varepsilon = \frac{1}{2}$.

2. Case $n \Longrightarrow n+1$. Let us suppose that we have defined the operators $u_1,\ldots,u_n$ and we will construct the map u_{n+1} such that i) and ii) hold. For this purpose, we consider the multivalued operator F_{n+1} given by:

$$F_{n+1}(x) = F(x) \cap B\left(u_n(x);\frac{1}{2^n}\right), \text{ for each } x \in X.$$

From i) we obtain that $F_{n+1}(x) \neq \emptyset$, for all $x \in X$. Moreover $F_{n+1}(x)$ is convex, for all $x \in X$. Using Lemma 3.36, we have that F_{n+1} is l.s.c.. From

Lemma 4.2., applied for F_{n+1} we have that there exists a continuous operator $u_{n+1} : X \to Y$ such that: $D(u_{n+1}(x), F_{n+1}(x)) < \frac{1}{2^{n+1}}$, for each $x \in X$. It follows that $D(u_{n+1}(x), F(x)) \leq \frac{1}{2^{n+1}}$. Also, we have:

$$u_{n+1}(x) \in V^0\left(F_{n+1}(x), \frac{1}{2^{n+1}}\right) \text{ which implies } \|u_{n+1}(x) - u_n(x)\| \leq \frac{1}{2^{n-1}}.$$

This completes the induction.

Further on, from ii) we obtain that $(u_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence convergent to a continuous operator $u : X \to Y$. From i) and the fact that $F(x)$ are closed for each $x \in X$, we obtain that $u(x) \in F(x)$, for all $x \in X$. Hence, u is the desired continuous selection and the proof is complete. □

Corollary 4.4. *i) Let (X, d) be a metric space, Y a Banach space and $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X. Let $Z \subset X$ be a nonempty set and $\varphi : Z \to Y$ a continuous selection of $F|_Z$. Then φ admits an extension to a continuous selection of F. In particular, we have that for each $y_0 \in F(x_0)$, with $x_0 \in X$ arbitrary, there exists a continuous selection φ of F such that $\varphi(x_0) = y_0$.*

ii) Let X be a metric space, Y be a Banach space, $F : X \to P_{cl,cv}(Y)$ be l.s.c. on X and $G : X \to P(Y)$ with open graph. If $F(x) \cap G(x) \neq \emptyset$, for all $x \in X$, then $F \cap G$ has a continuous selection.

For u.s.c. multifunctions we have the following approximate selection theorem given by Cellina [14]:

Theorem 4.5. (Cellina's approximate selection theorem) *Let (X, d) be a metric space, Y be a Banach space and $F : X \to P_{cv}(Y)$ be u.s.c. on X. Then for each $\varepsilon > 0$ there exists $f_\varepsilon : X \to Y$ locally Lipschitz such that:*

a) $f_\varepsilon(X) \subset co\, F(X)$,

b) $Graf\, f_\varepsilon \subset V(Graf\, F, \varepsilon)$.

Let us consider now the selection theorem of Browder.

Theorem 4.6. (Browder' selection theorem) *Let X and Y be Hausdorff topological vectorial space and $K \in P_{cp}(X)$. Let $F : K \to P_{cv}(Y)$ be a multivalued operator such that $F^{-1}(y)$ is open, for each $y \in Y$. Then there exists a continuous selection f of F.*

Proof. Because $(F^{-1}(y))_{y\in Y}$ is an open covering of K, there exists a finite refinement of it, denoted by $(F^{-1}(y_i))_{i\in\{1,\ldots,n\}}$. Let $(\alpha_i)_{i\in\{1,\ldots,n\}}$, with $\alpha_i : K \to [0,1]$ be the continuous partition of unity corresponding to this finite covering, i. e. the supports of α_i ($supp\alpha_i := \overline{\{x \in K | \alpha_i(x) \neq 0\}}$ form a locally finite cover of K and $\sum_{i=1}^{n} \alpha_i(x) = 1$. We define $f : K \to Y$ by the following relation: $f(x) = \sum_{i=1}^{n} \alpha_i(x) y_i$. Then f is continuous and for each $x \in K$ with $\alpha_i(x) > 0$ it follows $y_i \in F(x)$. But for each $x \in X$, the set $F(x)$ is convex, and hence we obtain that $f(x) \in F(x)$, for all $x \in X$. □

The concept of locally selectionable multifunction characterize the multivalued operators having "exact" continuous selections. More precisely, we define:

Definition 4.7. Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$. Then F is called locally selectionable at $x_0 \in X$ if for each $y_0 \in F(x_0)$ there exist an open neighborhood V of x_0 and a continuous operator $f : V \to Y$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for all $x \in X$. F is said to be locally selectionable if it is locally selectionable at every $x_0 \in X$.

Remark 4.8. Any locally selectionable multifunction is l.s.c.

Some examples of locally selectionable multifunctions are:

Lemma 4.9. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ such that $F^{-1}(y)$ is open for each $y \in Y$. Then F is locally selectionable.*

We note that a similar result hold for multifunctions with open graph. (It is easy to see that if the graph of F is open then $F^{-1}(y)$ is open for each $y \in X$.)

Lemma 4.10. *Let X, Y be Hausdorff topological spaces and $F, G : X \to P(Y)$ such that $F(x) \cap G(x) \neq \emptyset$, for each $x \in X$. If F is locally selectionable and G has open graph then the multivalued operator $F \cap G$ is locally selectionable.*

A global continuous selection theorem for a locally selectionable multifunction is:

Theorem 4.11. (Aubin-Cellina [14])*Let X be a paracompact space and Y a Hausdorff topological vector space. Then any locally selectionable multifunction $F : X \to P_{cv}(Y)$ has a continuous selection.*

Proof. We associate with each $y \in X$ an element $z \in F(x)$ and a continuous selection $f_y : V \to Y$ such that $f_y(x) \in F(x)$ and $f(y) = z$. Since the space X is paracompact there exists a continuous partition of unity $(a_y)_{y \in X}$ associated with the open covering of X given by $V(y), y \in X$. Denote by $I(x)$ the non-empty finite set of points $y \in X$ having the property that $a_y(x) > 0$. Let us define the operator $f : X \to Y$ by

$$f(x) = \sum_{y \in X} a_y(x) f_y(x) = \sum_{y \in I(x)} a_y(x) f_y(x).$$

Obviously, f is continuous as a finite sum of continuous operators and because $F(x)$ is convex, the convex combination $f(x)$ is also in $F(x)$. $\square$

A very interesting selection result for a continuous multifunction with not necessarily convex values is the following:

Theorem 4.12. (Strother [200]) *Let $F : [0, 1] \to P([0, 1])$ be a continuous multivalued operator. Then there exists a continuous selection of F.*

Proof. Let us define $f : [0, 1] \to [0, 1]$, by $f(x) := inf\{y | y \in F(x)\}$. We will prove that f is a continuous selection of F. Let $x' \in [0, 1]$ be arbitrary and $r > 0$ be a real positive number. Denote by V_{2r} an open interval of length $2r$ with center $f(x')$. Obviously, V_r is also an open set containing $f(x')$. Using the l.s.c. of F there exists an open set U_1 containing x_0 such that $F(x) \cap V_r \neq \emptyset$, for each $x \in U_1$. Hence $x \in U_1$ implies that $inf\{y | y \in F(x)\} = f(x) \geq f(x') - r$. On the other side, consider $V = \{y | y < r + f(x')\}$. The set V is open and it contains $F(x')$. From the u.s.c. of F there exists an open set U_2 containing x' such that $F(x) \subset V$, for each $x \in U_2$. Then for each $x \in U_2$ we have that $f(x) = inf\{y | y \in F(x)\} \leq f(x') + r$.

Let consider now $U := U_1 \cap U_2$. Then for each $x \in U$ we obtain that $|f(x) - f(x')| \leq r$ and therefore $f(x) \in V_{2r}$, proving that f is continuous in x'. $\square$.

Let us consider now the problem of the existence of a Lipschitz selection for a multifunction.

Definition 4.13. Let $F : \mathbb{R}^n \to P_{cp}(\tilde{B}(0;R)))$ be a H-c. multifunction and let $S = \tilde{B}(y^0;b) \subset \mathbb{R}^n$. Let q be any finite collection of points $x_1, x_2, ..., x_{k+1}$ in S such that $\sum_{p=1}^{k} |x_{p+1} - x_p| \leq b$ and Q denote the set of all such collections. Let $V(F,S,q) := \sum_{i=1}^{k} H(F(x_{i+1}), F(x_i))$ and $V(F,S) := sup\{V(F,S,q) | q \in Q\}$. If $V(F,S) < \infty$, then we say that F has bounded variation in S.

Moreover, if $F : [0,T] \to P_{cp}(\tilde{B}(0;R)))$ then, by definition, the variation of F on the subinterval $[t-q,t]$, where $q > 0$, denoted by $V_{t-q}^{t}(F)$ is defined as follows: let R be a partition of $[t-q,t]$ (i.e. $t-q = t_0, t_1 < ... < t_{k+1} = t$) and let $\mathcal{R}$ be the set of all such partitions. Then $V_{t-q}^{t}(F,R) := \sum_{p=1}^{k} H(F(t_{p+1}), F(t_p))$ and $V_{t-q}^{t}(F) := sup\{V_{t-q}^{t}(F,R) | R \in \mathcal{R}\}$.

Theorem 4.14. (Hermes [78], [79]) *Let $T > 0$ and $F : [0,T] \to P_{cp}(\tilde{B}(0;R)))$. Then:*

i) If F is H-c and has bounded variation in $[0,T]$, then F admits a continuous selection.

ii) If F is a-Lipschitz, then there exists an a-Lipschitz selection of F.

Proof. For each positive integer k, consider the points $0, \frac{T}{k}, \frac{2T}{k}, ..., T$. Choose $x_0^k \in F(0), x_1^q \in F(\frac{T}{k})$ such that $|x_0^k - x_1^k| = D(x_0^k, F(\frac{T}{k}))$ and then inductively $x_j^k \in F(\frac{jT}{k})$ such that $|x_{j-1}^k - x_j^k| = D(x_{j-1}^k, F(\frac{jT}{k}))$. Define $f^k : [0,T] \to \mathcal{R}$ be the polygonal arc joining the points $x_j^k, j \in \{0,1,..,k\}$. Then:

i) For each $t \in [0,T]$ and each k there exists an integer $j = j(k)$ such that $|t - \frac{jT}{k}|, \frac{T}{k}$. We can assume, without any loss of generality, that $t \in [\frac{(j-1)T}{k}, \frac{jT}{k}]$. Then $D(f^k(t), F(t)) \leq |f^k(t) - f^k(\frac{jT}{k})| + D(f^k(\frac{jT}{k}), F(t)) \leq H(F(\frac{(j-1)T}{k}), F(\frac{jT}{k})) + H(F(\frac{jT}{k}), F(t))$.

ii) For each t and s from $[0,T]$ and each k, let j,l be integers such that: $|t - \frac{jT}{k}| < \frac{T}{k}$ and $|s - \frac{lT}{k}| < \frac{T}{k}$. We have: $|f^k(t) - f^k(s)| \leq |f^k(t) -$

$f^k(\frac{jT}{k})|+\sum_{r=j}^{l-1}|f^k(\frac{(r+1)T}{k})-f^k(\frac{rT}{k})|+|f^k(\frac{lT}{k})-f^k(s)|\leq H(F(t),F(\frac{jT}{k})|+$
$\sum_{r=j}^{l-1}H(F(\frac{(r+1)T}{k}),F(\frac{rT}{k}))+H(F(s),F(\frac{lT}{k}))$.

Now, we are able to prove a). Let us first remark that the sequence $(f^k)_{k\in bbN^*}$ is equicontinuous. Indeed, for any $\varepsilon > 0$ choose k^* sufficiently large such that if $k \leq k^*$ and $|t_1 - t_2|, \frac{T}{k^*}$ we have $H(t_1), F(t_2)) < \frac{\varepsilon}{3}$. Next, since F is of bounded variation, we obtain that $V_0^t(F)$ is continuous as a function of t on $[0,T]$ and hence uniformly continuous. We can choose $\delta > 0$ such that $V_a^b(F) < \frac{\varepsilon}{3}$, for $|a-b| < \delta$. Since $|\frac{jT}{k} - \frac{lT}{k}| \leq |t-s| + \frac{2T}{k}$ if $k.\frac{4T}{\delta}$ and $|t-s| < \frac{\delta}{2}$, we obtain $V_{\frac{jT}{k}}^{\frac{lT}{k}} < \frac{\varepsilon}{3}$. Then, from ii) we have for $k \geq max(\frac{4T}{j}, k^*)$ and $|t-s| < \delta$ that $|f^k(t) - f^k(s)| < \varepsilon$ and equicontinuity is shown. The sequence (f^k) being bounded, it has an uniformly convergent subsequence converging to $f \in C[0,T]$. let $t \in [0,T]$ and $j(k)$ be an integer such that $|t - \frac{j(k)T}{k}| < \frac{T}{k}$. Using i) and the fact that the images $F(t)$ are closed, we obtain by taking $k \to +\infty$ $f(t) \in F(t)$.

For b), let us assume in ii) that $t < \frac{jT}{k} < ... < \frac{lT}{k} < s$. From the Lipschitz condition, relation ii) becomes: $|f^k(t) - f^k(s)| \leq a[(\frac{jT}{k} - t) + \sum_{p=j}^{l-1}(\frac{(p+1)T}{k} - \frac{pT}{k}) + (s - \frac{lT}{k})] = a|s-t|$. Thus $(f^k)_{k\in bbN^*}$ is equicontinuous, bounded and has a subsequence converging uniformly to $f \in C[0,T]$ and $|f(t) - f(s)| \leq a|t-s|$. From i) we conclude again that $f(t) \in F(t)$, for each $t \in [0,T]$. □

For more general spaces, the Steiner point approach generate a Lipschitz selection as follows:

Theorem 4.15. *Let X be a metric space and $F: X \to P_{cp,cv}(\mathbb{R}^n)$ be a-Lipschitz. Then F admits a b-Lipschitz selection with $b = ak(n)$ and $k(n) = \frac{n!!}{(n-1)!!}$ if n is odd and $k(n) = \frac{n!!}{\pi(n-1)!!}$ if n is even.*

Finally, let us remark that the problem of existence of a Lipschitz selection for a Lipschitz multifunction was settled by Yost (see for example Hu-Papageorgiou [84]) as follows:

Theorem 4.16. (Yost) *Let X be a metric space and Y be a Banach space.*

Then every a-Lipschitz multifunction $F : X \to P_{b,cl,cv}(Y)$ admits a Lipschitz selection if and only if Y is finite dimensional.

A extension of the concept of selection is given by Deguire-Lassonde as follows:

Definition 4.17. Let X be a topological space and $(Y_i)_{i\in I}$ an arbitrary family of topological spaces. The family of continuous operators $\{f_i : X \to Y_i\}_{i\in I}$ is called a selecting family for the family $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ of multifunctions if for each $x \in X$ there exists $i \in I$ such that $f_i(x) \in F_i(x)$.

One easily observe that the notion of selecting family reduces to the concept of continuous selection when I has only one element.

Definition 4.18. Let X be a topological space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $Y_i \in P_{cv}(E_i)$, for all $i \in I$. Then the family $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ of multifunctions is said to be a Ky Fan family if the following are verified:

i) $F_i(x)$ is convex for each $x \in X$ and each $i \in I$.

ii) $F^{-1}(y_i)$ is open for each $y_i \in Y_i$ and each $i \in I$.

iii) for each $x \in X$ there exists $i \in I$ such that $F_i(x) \neq \emptyset$.

In this setting, an important result is:

Theorem 4.19. (Deguire-Lassonde [57]) *Let X be a paracompact space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $X_i \in P_{cv}(E_i)$, for all $i \in I$. Then any Ky Fan family of multivalued operators $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ admits a selecting family $\{f_i : X \to Y_i\}_{i\in I}$.*

Proof. From the definition of the Ky Fan family of multifunctions, we have that the system $(DomF_i(x))_{i\in I}$ is an open covering of X. Using the paracompactness of the space X it follows the existence of a closed refinement $(U_i)_{i\in I}$ such that $U_i \subset Dom(F_i)$, for each $i \in I$. Let us define, for each $i \in I$ the multivalued operator $G_i : X \to Y_i$,by the relation:

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \in U_i \\ Y_i, & \text{if } x \notin Y_i \end{cases}$$

Then, for each $i \in I$, G_i has nonempty and closed values and the sets $F_i^{-1}(y)$ are open for each $y \in Y_i$. From Browder selection theorem, we obtain the existence of a continuous selection $f_i : X \to Y_i$ of F_i, for each $i \in I$. Because for each $x \in X$ there exists $i \in I$ such that $x \in U_i$ implies $f_i(x) \in G_i(x) = F_i(x)$, we obtain that $\{f_i : X \to Y_i | i \in I\}$ is a selecting family for $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$. The proof is complete. □

Using a similar argument (via Michael' selection theorem), we have:

Theorem 4.20. (Deguire-Lassonde [57]) *Let X be a paracompact space, $(E_i)_{i\in I}$ be an arbitrary family of Hausdorff topological vector spaces and $X_i \in P_{cv}(E_i)$, for all $i \in I$. Then any family of l.s.c. multivalued operators $\{F_i : X \to \mathcal{P}(Y_i)\}_{i\in I}$ having the property that for each $x \in X$ there is $i \in I$ with $F_i(x) \neq \emptyset$ admits a selecting family $\{f_i : X \to Y_i\}_{i\in I}$.*

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite nonatomic measure space and E is a Banach space. Let $L^1(\Omega, E)$ be the Banach space of all measurable operators $u : \Omega \to E$ which are Bochner μ-integrable. We call a set $K \subset L^1(\Omega, E)$ decomposable if for all $u, v \in K$ and each $A \in \mathcal{A}$:

$$u\chi_A + v\chi_{\Omega\setminus A} \in K, \qquad (1)$$

where χ_A stands for the characteristic function of the set A.

This notion is, somehow, similar to convexity, but there exist also major differences. For example, the following theorem is a "decomposable" version of the well-known Michael's selection theorem for l.s.c. multifunctions.

Theorem 4.21. (see [30]) *Let (X, d) be a separable metric space, E a separable Banach space and let $F : X \to P_{cl,dec}(L^1(\Omega, E))$ be a l.s.c. multivalued operator. Then F has a continuous selection.*

The purpose of the next part of this section is to prove some "decomposable" versions of Deguire - Lassonde's previous results.

Our first result, concerning the existence of continuous selections for a locally selectionable multivalued operator, is as follows:

Lemma 4.22. *Let (X, d) be a separable metric space, $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite and nonatomic measure space and E be a Banach space. Let $F : X \to P_{dec}(L^1(\Omega, E))$ be a locally selectionable multivalued operator. Then F has a continuous selection.*

Proof. We associate to any $y \in X$ and $z \in F(y)$ an open neighborhood $N(y)$ and a local continuous selection $f_y : N(y) \to L^1(\Omega, E)$, satisfying $f_y(y) =$

z and $f_y(x) \in F(x)$ when $x \in N(y)$. We denote by $\{V_n\}_{n\in\mathbb{N}^*}$ a countable locally finite open refinement of the open covering $\{N(y)|\ y \in X\}$ and by $\{\psi_n\}_{n\in\mathbb{N}^*}$ a continuous partition of unity associated to $\{V_n\}_{n\in\mathbb{N}^*}$.

Then, for each $n \in \mathbb{N}^*$ there exist $y_n \in X$ such that $V_n \subset N(y_n)$ and a continuous operator $f_{y_n} : N(y_n) \to L^1(\Omega, E)$ with $f_{y_n}(y_n) = z_n$, $f_{y_n}(x) \in F(x)$, for all $x \in N(y_n)$. We define $\lambda_0(x) = 0$ and $\lambda_n(x) = \sum\limits_{m\leq n} \psi_m(x)$, $n \in \mathbb{N}^*$. Let $g_{m,n} \in L^1(\Omega, \mathbb{R}_+)$ be the operator defined by $g_{m,n}(t) = \|z_n(t) - z_m(t)\|$, for each $m, n \geq 1$.

We arrange these operators into a sequence $\{g_k\}_{k\in\mathbb{N}^*}$.

Consider the operator $\tau(x) = \sum\limits_{m,n\geq 1} \psi_m(x)\psi_n(x)$. From Lemma 1 in [30], there exists a family $\{\Omega(\tau, \lambda)\}$ of measurable subsets of Ω such that:

(a) $\Omega(\tau, \lambda_1) \subseteq \Omega(\tau, \lambda_2)$, if $\lambda_1 \leq \lambda_2$

(b) $\mu(\Omega(\tau_1, \lambda_1)\Delta\Omega(\tau_2, \lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$

(c) $\int_{\Omega(\tau,\lambda)} g_n d\mu = \lambda \int_\Omega g_n d_\mu$, $\forall\, n \leq \tau_0$ for all $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, and all $\tau_0, \tau_1, \tau_2 \geq 0$.

Define $f_n(x) = f_{y_n}(x)$ and $\chi_n(x) = \chi_{\Omega(\tau(x),\lambda_n(x))\setminus\Omega(\tau(x),\lambda_{n-1}(x))}$ for each $n \in \mathbb{N}^*$.

Let us consider the singlevalued operator $f : X \to L^1(\Omega, E)$, defined by $f(x) = \sum_{n\geq 1} f_n(x)\chi_n(x)$, $x \in X$.Then, f is continuous because the operators τ and λ_n are continuous, the characteristic function of the set $\Omega(\tau, \lambda)$ varies continuously in $L^1(\Omega, E)$ with respect to the parameters τ and λ and because the summation defining f is locally finite. On the other hand, from the properties of the sets $\Omega(\tau, \lambda)$ (see [30]) and because F has decomposable values, it follows that f is a selection of F. $\square$

Next result is a selection theorem for the intersection of two multivalued operators.

Theorem 4.23. *Let (X, d) be a separable metric space, E a separable Banach space, $F : X \to P_{cl,dec}(L^1(\Omega, E))$ be a l.s.c. multivalued operator and $G : X \to P_{dec}(L^1(\Omega, E))$ be with open graph. If $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$ then there exists a continuous selection of $F \cap G$.*

Proof. Let $x_0 \in X$ and for each $y_0 \in F(x_0)$ we define the multifunction

$$F_0(x) = \begin{cases} \{y_0\}, & \text{if } x = x_0 \\ F(x), & \text{if } x \neq x_0. \end{cases}$$

Obviously $F_0 : X \to \mathcal{P}_{cl,dec}(L^1(\Omega, E))$ is l.s.c. From Theorem 4.1. there exists a continuous selection f of F_0, i.e. $f_0(x_0) = y_0$ and $f_0(x) \in F(x)$, for each $x \in X$, $x \neq x_0$. Using Proposition 4, p.81 in [14] it follows that $F \cap G$ is locally selectionable at x_0 and has decomposable values. From Lemma 4.2. the conclusion follows. □

An important result is the following Browder-type selection theorem:

Theorem 4.24. *Let E be a Banach space such that $L^1(\Omega, E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(\Omega, E)$ and let $F : K \to P_{dec}(K)$ be a multivalued operator with open fibres. Then F has a continuous selection.*

Proof. For each $y \in K$, $F^{-1}(y)$ is an open subset of K. Since K is paracompact it follows that the open covering $\{F^{-1}(y)\}_{y \in K}$ admits a locally finite open refinement, let say $K = \bigcup_{j \in J} F^{-1}(y_j)$, with $y_j \in K$. Let $\{\psi_j\}_{j \in J}$ be a continuous partition of unity subordinate to $\{F^{-1}(y_j)\}_{j \in J}$. Using the same construction as in the proof of Lemma 4.2., one can construct a continuous operator $f : K \to K$, $f(x) = \sum_{j \in J} f_j(x)\chi_j(x)$, where $f_j(x) \in F(x)$ for each $x \in K$. This operator is a continuous selection for F. □

Next, we will consider selecting results for multifunctions with decomposable values.

Theorem 4.25. *Let E be a Banach space such that $L^1(\Omega, E)$ is separable. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family of nonempty, decomposable subsets of $L^1(\Omega, E)$ and X a paracompact space. Let us suppose that the family*
$\{F_i : X \to \mathcal{P}_{dec}(K_i) | i \in I\}$ is of Ky Fan-type. Then there exists a selecting family for $\{F_i\}_{i \in I}$.

Proof. Let $\{U_i\}_{i \in I}$ be the open covering of the paracompact space X given by $U_i = \{x \in X | \ F_i(x) \neq \emptyset\}$ for each $i \in I$. It follows that there exists a locally finite open cover $\{W_i\}_{i \in I}$ such that $\overline{W_i} \subset U_i$ for $i \in I$. Let $V_i = \overline{W_i}$. For each

$i \in I$ let us consider the multivalued operator $G_i : X \to \mathcal{P}(K_i)$, defined by the relation

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \in V_i \\ K_i, & \text{if } x \notin V_i. \end{cases}$$

Then G_i is a multifunction with nonempty and decomposable values having open fibres (indeed, $G_i^{-1}(y) = F_i^{-1}(y) \cup (X \backslash V_i))$, for each $i \in I$.

Using Theorem 4.24. we have that there exists $f_i : X \to K_i$ continuous selection for G_i $(i \in I)$, for each $i \in I$. It follows that for each $x \in X$ there exists $i \in I$ such that $x \in V_i$ and hence $f_i(x) \in G_i(x) = F_i(x)$, proving that $\{f_i\}_{i \in I}$ is a selecting family for $\{F_i\}_{i \in I}$. □

By a similar argument we have:

Theorem 4.26. *Let E be a separable Banach space and X a separable metric space. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family of nonempty, closed, decomposable subsets of $L^1(\Omega, E)$. Let $\{F_i : X \to \mathcal{P}_{cl,dec}(K_i) | i \in I\}$ be a family of l.s.c. multivalued operators such that for each $x \in X$ there is $i \in I$ such that $F_i(x) \neq \emptyset$. Then $\{F_i\}_{i \in I}$ has a selecting family.*

We are now interested for the existence of a Caristi selection for multivalued generalized contractions.

Recall that Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+$ such that:

$$d(x, f(x)) + \varphi(f(x)) \le \varphi(x), \text{ for each } x \in X$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$

An operator $f : X \to X$ satisfying the above relation is called a Caristi type operator.

First result of this type was established by J. Jachymski for a multivalued contraction with closed values.

Theorem 4.27. (J. Jachymski [90]) *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued contraction. Then there exists $f : X \to X$ a Caristi selection (with a Lipschitz map φ) of F.*

An extension for a Reich type multivalued operator is the following:

Theorem 4.28. (A. Petruşel - A. Sîntămărian [153]) *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a Reich type multivalued operator, i. e. there exist* $a, b, c \in \mathbb{R}_+$, *with* $a + b + c < 1$ *and for each* $x, y \in X$

$$H(F(x), F(y)) \leq a \cdot d(x, y) + b \cdot D(x, F(x)) + c \cdot D(y, F(y)).$$

Then there exists $f : X \to X$ *a Caristi selection of* F.

Then, another generalization of Jachymski's result was recently proved by Sîntămărian in [194].

Theorem 4.29. (A. Sîntămărian) *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a generalized multivalued contraction, i. e. for each* $x, y \in X$
$H(F(x), F(y)) \leq a_1 \; d(x, y) + a_2 \; D(x, F(x)) + a_3 \; D(y, F(y)) + a_4 \; D(x, F(y)) + a_5 \; D(y, F(x))$, *where* $a_1 + a_2 + a_3 + 2a_4 \in]0, 1[$.

Then there exists $f : X \to X$ *a Caristi selection of* F.

The following result was proved in A. Petruşel, G. Petruşel [154]:

Theorem 4.30. *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a Ciric type multivalued contraction, i. e. there is* $q \in]0, 1[$ *such that for each* $x, y \in X$
$H(F(x), F(y)) \leq q \cdot \max\{d(x, y), D(x, F(x)), D(y, F(y)), \frac{1}{2}(D(x, F(y)) + D(y, F(x)))\}$.

Then there exists $f : X \to X$ *a Caristi selection of* F.

Proof. Let $\varepsilon := \frac{1-q}{2}$ and $\varphi(x) := \frac{1}{\varepsilon} \cdot D(x, F(x))$.

Then, obviously $\varepsilon + q = \frac{1+q}{2} < 1$ and φ is bounded below by 0.

Since $\frac{1}{\varepsilon+q} > 1$, for each $x \in X$ we can choose $f(x) \in F(x)$ such that

$$d(x, f(x)) \leq \frac{1}{\varepsilon + q} \cdot D(x, F(x)), \text{ for each } x \in X.$$

We have then successively: $D(f(x), F(f(x))) \leq H(F(x), F(f(x))) \leq q \cdot \max\{d(x, f(x)), D(x, F(x)), D(f(x), F(f(x)), \frac{1}{2}(D(x, F(f(x))) + D(f(x), F(x)))\} \leq q \cdot \max\{d(x, f(x)), d(x, f(x)), D(f(x), F(f(x))), \frac{1}{2}D(x, F(f(x)))\}$

$q \cdot \max\{d(x, f(x)), D(f(x), F(f(x))), \frac{1}{2}D(x, F(f(x)))\}$.

1) If $\max\{d(x,f(x)), D(f(x),F(f(x))), \frac{1}{2}D(x,F(f(x)))\} = d(x,f(x))$ then we obtain: $D(f(x),F(f(x))) \le q \cdot d(x,f(x)), x \in X$.

2) If $\max\{d(x,f(x)), D(f(x),F(f(x))), \frac{1}{2}D(x,F(f(x)))\} = D(f(x),F(f(x)))$ then $D(f(x),F(f(x))) \le q \cdot D(f(x),F(f(x))), x \in X$, a contradiction with $q > 1$.

3) If $\max\{d(x,f(x)), D(f(x),F(f(x))), \frac{1}{2}D(x,F(f(x)))\} = \frac{1}{2}D(x,F(f(x)))$ then $D(f(x),F(f(x))) \le \frac{q}{2} \cdot D(x,F(f(x))) \le \frac{q}{2}[d(x,f(x)) + D(f(x),F(f(x)))]$ and hence $D(f(x),F(f(x))) \le \frac{q}{2-q} \cdot d(x,f(x)) \le q \cdot d(x,f(x)), x \in X$.

Hence in all the three cases we have:

$D(f(x),F(f(x))) \le q \cdot d(x,f(x)), x \in X$.

We will prove now that f is a Caristi type operator. Indeed, for each $x \in X$ we have:

$d(x,f(x)) = \frac{1}{\varepsilon} \cdot [(\varepsilon + q) \cdot d(x,f(x)) - q \cdot d(x,f(x))] \le \frac{1}{\varepsilon}[D(x,F(x)) - D(f(x),F(f(x)))] = \varphi(x) - \varphi(f(x))$. □

Remark 4.31. *It is quite obvious that the above theorems includes as particular cases Theorem 4.27 - Theorem 4.29.*

For the case of a multivalued contraction with variable coefficient, Xu proved:

Theorem 4.32. (Xu [213]) *Let (X,d) be a metric space and $F : X \to P_{b,cl}(X)$ be a multivalued operator. Suppose there exists a lower semicontinuous mapping $\alpha : X \to [0,1[$ such that*

$$H(F(x),F(y)) \le \alpha(x) \cdot d(x,y), \text{ for each } x,y \in X.$$

Then there exists $f : X \to X$ a Caristi selection (with a lower semicontinuous map φ) of F.

Another important result is:

Theorem 4.33. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. Suppose there exist the lower semicontinuous mappings $\alpha, \beta, \gamma : X \to \mathbb{R}_+$, with $\alpha(x) + \beta(x) + \gamma(x) < 1$ and for each $x \in X$, such that for each $x,y \in X$ we have:*

$$H(F(x),F(y)) \le \alpha(x) \cdot d(x,y) + \beta(x) \cdot D(x,F(x)) + \gamma(x) \cdot D(y,F(y)).$$

Then there exists $f: X \to X$ a Caristi selection of F.

Proof. Let $\varepsilon(x) := \frac{1-\alpha(x)-\beta(x)}{1-\gamma(x)}$ and $\varphi(x) := \frac{1}{\varepsilon(x)} \cdot D(x, F(x))$.

Then φ is bounded below by 0.

Note that $\frac{\alpha(x)+\beta(x)}{1-\gamma(x)} + \varepsilon(x) = \frac{1}{1-\gamma(x)} > 1$, for each $x \in X$.

Then there is $f(x) \in F(x)$ such that

$$d(x, f(x)) \leq \frac{1}{1-\gamma(x)} \cdot D(x, F(x)), \text{ for each } x \in X.$$

Note that $D(f(x), F(f(x))) \leq H(F(x), F(f(x))) \leq \alpha(x) \cdot d(x, f(x)) + \beta(x) \cdot D(x, F(x)) + \gamma(x) \cdot D(f(x), F(f(x))) \leq \alpha(x) \cdot d(x, f(x)) + \beta(x) \cdot d(x, f(x)) + \gamma(x) \cdot D(f(x), F(f(x)))$.

Hence $D(f(x), F(f(x))) \leq \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} \cdot d(x, f(x)), x \in X$.

It remains to show that f satisfies the Caristi type condition. For each $x \in X$ we have:

$d(x, f(x)) =$
$\frac{1}{\varepsilon(x)} \cdot [(\varepsilon(x) + \frac{\alpha(x)+\beta(x)}{1-\gamma(x)}) \cdot d(x, f(x)) - \frac{\alpha(x)+\beta(x)}{1-\gamma(x)} \cdot d(x, f(x))] \leq$
$\frac{1}{\varepsilon(x)} \cdot [\frac{1}{1-\gamma(x)} \cdot d(x, f(x)) - D(f(x), F(f(x)))] \leq$
$\frac{1}{\varepsilon(x)}[D(x, F(x)) - D(f(x), F(f(x)))] =$
$\varphi(x) - \varphi(f(x))$. $\square$

In a similar way to the above results we have:

Theorem 4.44. *Let (X, d) be a metric space and $F: X \to P_{cl}(X)$. Suppose there exists a lower semicontinuous mapping $q: X \to [0, 1[$ such that for each $x, y \in X$*

$H(F(x), F(y)) \leq q(x) \cdot \max\{d(x, y), D(x, F(x)), D(y, F(y)), \frac{1}{2}(D(x, F(y)) + D(y, F(x)))\}$.

Then there exists $f: X \to X$ a Caristi selection of F.

With respect to the above results, some open questions will be presented now.

I.

Open Problem. Give other examples of generalized multivalued contractions having Caristi type selections.

II.

Let X be a nonempty set and $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X,\ n \in \mathbb{N}\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x,$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is convergent sequence and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write

$$x_n \to x \text{ as } n \to \infty.$$

In what follow we will denote an L-space by $(X, \to)$.

Example 4.45. (L-structures on ordered sets) Let $(X, \leq)$ be an ordered set.

(a) $c_1(X) := \{(x_n)_{n\in\mathbb{N}} | (x_n)_{n\in\mathbb{N}}$ is increasing and there exists $\sup x_n\}$, $Lim(x_n)_{n\in\mathbb{N}} = \sup\{x_n | n \in \mathbb{N}\}$. If $x = \sup\{x_n | n \in \mathbb{N}\}$, $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, then we denote this by $x_n \uparrow x$.

(b) $c_2(X) := \{(x_n)_{n\in\mathbb{N}} | (x_n)_{n\in\mathbb{N}}$ is decreasing and there exists $\inf\{x_n | n \in \mathbb{N}\}$, $Lim(x_n)_{n\in\mathbb{N}} = \inf\{x_n | n \in N\}$. If $(x_n)_{n\in\mathbb{N}}$ is decreasing and $\inf\{x_n | n \in \mathbb{N}\} = x$, then we denote this by $x_n \downarrow x$.

(c) $c(X) := c_1(X) \cup c_2(X)$. If $x = Lim(x_n)_{n\in\mathbb{N}}$, then we denote this by $x_n \stackrel{m}{\to} x$ as $n \to \infty$.

(d) By definition, a sequence $(x_n)_{n\in\mathbb{N}}$ (0)-converges to x if there exist two sequence $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

(i) $a_n \uparrow x$ and $b_n \downarrow x$;

(ii) $a_n \leq x_n \leq b_n,\ n \in \mathbb{N}$.

We denote this convergence by $x_n \stackrel{0}{\to} x$. It is clear that $(X, \uparrow)$, $(X, \downarrow)$, $(X, \stackrel{m}{\to})$, $(X, \stackrel{0}{\to})$ are L-spaces.

Example 4.46. (L-structures on Banach spaces) Let X be a Banach space. We denote by $\to$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \to)$, $(X, \rightharpoonup)$ are L-spaces.

Example 4.47. (L-structures on function spaces) Let X and Y be two metric spaces. Let $\mathbb{M}(X,Y)$ the set of all operators from X to Y. We denote by $\stackrel{p}{\to}$ the point convergence on $\mathbb{M}(X,Y)$, by $\stackrel{unif}{\to}$ the uniform convergence and by $\stackrel{cont}{\to}$ the convergence with continuity (M. Agrisani and M. Clavelli [5]). Then $(\mathbb{M}(X,Y),\stackrel{p}{\to})$, $(\mathbb{M}(X,Y),\stackrel{unif}{\to})$ and $(\mathbb{M}(X,Y),\stackrel{cont}{\to})$ are L-spaces.

Remark 4.48. An L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov' sense: $d(x,y) \in \mathbb{R}^m_+$, in Luxemburg-Jung' sense (see [170], [179]): $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$, $d(x,y) \in K$, K a cone in an ordered Banach space, $d(x,y) \in E$, E an ordered linear space with a notion of linear convergence, etc.), 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details see Fréchet [66], Blumenthal [27] and I. A. Rus [171].

An important abstract concept is:

Definition 4.49. (I. A. Rus-A. Petruşel-Sîntămărian [177], [178]) Let $(X,\to)$ be an L-space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Another important concept is:

Definition 4.50. Let $(X,\to)$ be an L-space. By definition, $f : X \to X$ is called a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

In I. A. Rus [179] the basic theory of Picard and weakly Picard operators is presented. For the multivalued case see Petruşel [150]. For both settings see also [184].

Let $(X,\to)$ be an L-space and $F : X \to P(X)$. It is easy to see that if F admits a weakly Picard selection $f : X \to X$, then F is weakly Picard too.

Open Problem. If F is a weakly Picard mutivalued operator, in which conditions there exists a weakly Picard selection of it ?

Bibliographical comments. Basic continuous selections theorems can be found in many books on multivalued analysis such as: Aubin [13], Aubin-Cellina [14], Aubin-Frankowska [15], Border [28], Deimling [58], Górniewicz [73], Hu-Papageorgiou [84], Kamenskii-Obuhovskii-Zecca [93], Kisielewicz [100], Repovs-Simeonov [166] Tolstonogov [205] and Yuan [217]. A selection theorem for multifunction on $[0, 1]$ was proved in Strother [200], meanwhile results regarding the existence of Lipschitz selections for multifunctions maybe found in Hermes [78] and [79]. The notion of selecting family and the corresponding results were given by Deguire and Lassonde in [56] and [57]. The part concerning decomposability and continuous selection follows the paper A. Petruşel-Moţ [147]. The last part of the section comes from A. Petruşel, G. Petruşel [154].

Chapter 5

Fixed point principles

The aim of this section is to report some basic theorems of the fixed point theory for multifunctions.

Let us recall first some basic notations and concepts.

Definition 5.1. Let X be a metric space. If $F : X \to P(X)$ is a multivalued operator and $x_0 \in X$ is an arbitrary point, then the sequence $(x_n)_{n\in\mathbb{N}}$ is, by definition, the successive approximations sequence of F starting from x_0 if and only if $x_k \in F(x_{k-1})$, for all $k \in \mathbb{N}^*$. Let us remark that in the theory of dynamical systems, the sequence of successive approximations is called the motion of the system F at x_0 or a dynamic process of F starting at x_0. The set $\mathcal{T}(x_0) := \{x_n : x_{n+1} \in F(x_n), n \in \mathbb{N}\}$ is called the trajectory of this motion and the space X is the phase space.

Definition 5.2. Let (X, d) be a generalized metric space and let $F : X \to P_{cl}(X)$ be a multivalued operator. Then F is said to be:

i) a-contraction if and only if $a \in [0, 1[$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2)$, for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \infty$.

ii) (ε, a)-contraction if and only if $\varepsilon > 0$, $a \in [0, 1[$ and $H(F(x_1), F(x_2)) \leq ad(x_1, x_2)$, for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \varepsilon$.

Remark 5.3. Obviously, each multivalued a-contraction is an (ε, a)-contraction.

Theorem 5.4. (Covitz-Nadler [50]) *Let (X, d) be a generalized complete* *metric space. Let $x_0 \in X$ arbitrary and $F : X \to P_{cl}(X)$ be a multivalued (ε, a)-contraction. Then the following alternative holds:*

(1) for each sequence of successive approximations of F starting from x_0 we have $d(x_{i-1}, x_i) \geq \varepsilon$, for all $i \in \mathbb{N}^*$

or

(2) there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

Corollary 5.5. *Let (X, d) be a generalized complete metric space and $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multivalued a-contraction, then the following alternative holds:*

(1) for each sequence of successive approximations of F starting from x_0 we have $d(x_{i-1}, x_i) = \infty$, for all $i \in \mathbb{N}^*$

or

(2) there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

The following result is known in the literature as Nadler theorem (see [125], [50]):

Theorem 5.6. (Nadler [125], Covitz-Nadler [50]) *Let (X, d) be a complete metric space and $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multivalued a-contraction, then there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.*

Definition 5.7. Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. If there exists $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ such that for all $x_1, x_2 \in X$ we have:

$$H(F(x_1), F(x_2)) \leq ad(x, y) + bD(x_1, F(x_1)) + cD(x_2, F(x_2))$$

then F is called a Reich type multivalued operator.

Reich's fixed point theorem (see [165]) is an extension of the Nadler principle:

Theorem 5.8. (Reich [165]) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a Reich type multivalued operator. Then $FixF \neq \emptyset$.*

If the multivalued operator is contractive and the space is compact, then we have the following result:

Theorem 5.9. (Smithson [198]) *Let (X, d) be a compact metric space and $F : X \to P_{cl}(X)$ be a contractive multivalued operator. Then $FixF \neq \emptyset$.*

Another generalization of the Covitz-Nadler principle is:

Theorem 5.10. (Mizoguchi-Takahashi (see [115]) *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ a multifunction such that $H(F(x), F(y)) \leq k(d(x, y))d(x, y)$, for each $x, y \in X$ with $x \neq y$, where $k :]0, \infty[\to [0, 1[$ satisfies $lim_{r \to t^+} k(r) < 1$, for every $t \in [0, \infty[$. Then $FixF \neq \emptyset$.*

For the case of multifunctions from a closed ball of a metric space X into X, Frigon and Granas (see [68]) proved the following extension of Covitz-Nadler principle:

Theorem 5.11. (Frigon and Granas [68]) *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $F : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be an a-contraction such that $D(x_0, F(x_0)) < (1 - a)r$. Then $FixF \neq \emptyset$.*

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$, with $d(x_0, x_1) < (1 - a)r$. Then $H(F(x_0), F(x_1)) \leq a \cdot d(x_0, x_1) < a(1 - a)d(x_0, x_1)$. Then there exists $x_2 \in F(x_1)$ such that $d(x_1, x_2) < a(1-a)r$. Moreover we have $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1 - a)r + a(1 - a)r = (1 - a^2)r$. Thus $x_2 \in \widetilde{B}(x_0; r)$. We can construct inductively a sequence $(x_n)_{n \in \mathbb{N}}$ in $\widetilde{B}(x_0; r)$ having the properties:

(i) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$

(ii) $d(x_n, x_{n+1}) \leq a^n \cdot (1 - a)r$.

From (ii) the sequence is Cauchy, hence it converges to a certain $x^* \in \widetilde{B}(x_0; r)$, while from (i), taking account of the fact that F is closed, we obtain the desired conclusion: $x^* \in F(x^*)$. □

Using the above theorem, Frigon and Granas have proved some continuation results for multifunctions on complete metric spaces.

Definition 5.12. If X, Y are metric spaces and $F_t : X \to P_{cl}(Y)$ is a family of multifunctions depending on a parameter $t \in [0, 1]$ then, by definition, $(F_t)_{t \in [0,1]}$ is said to be a family of k-contractions if:

i) F_t is a k-contraction, for each $t \in [0, 1]$.

ii) $H(F_t(x), F_s(x)) \leq |\phi(t) - \phi(s)|$, for each $t, s \in [0,1]$ and each $x \in X$, where $\phi : [0,1] \to \mathbb{R}$ is a continuous and strictly increasing function.

If (X, d) is a complete metric space and U is an open connected subset of X, then we will denote by $K(\overline{U}, X)$ the set of all k-contractions $F : \overline{U} \to P_{cl}(X)$. Also, denote by $\mathcal{K}_0(\overline{U}, X) = \{F \in \mathcal{K}(\overline{U}, X) | x \notin F(x), \text{ for each } x \in \partial U\}$.

Definition 5.13. $F \in \mathcal{K}_0(\overline{U}, X)$ is called essential if and only if $FixF \neq \emptyset$. Otherwise F is said to be inessential.

Definition 5.14. A family of k-contractions $(F_t)_{t\in[0,1]}$ is called a homotopy of contractions if and only if $F_t \in \mathcal{K}_0(\overline{U}, X)$, for each $t \in [0,1]$. The multifunctions S and T are said to be homotopic if there exists a homotopy of contractions $(F_t)_{t\in[0,1]}$ such that $F_0 = S$ and $F_1 = T$.

The topological transversality theorem read as follows:

Theorem 5.15. (Frigon-Granas [68]) *Let $S, T \in \mathcal{K}_0(\overline{U}, X)$ two homotopic multifunctions. Then S is essential if and only if T is essential.*

The non-linear alternative for multivalued contractions was proved by Frigon and Granas:

Theorem 5.16. (Frigon-Granas [68]) *Let X be a Banach space and $U \in P_{op}(X)$ such that $0 \in U$. If $T : \overline{U} \to P_{cl}(X)$ is a multivalued k-contraction such that $T(\overline{U})$ is bounded, then either:*

i) there exists $x \in \overline{U}$ such that $x \in T(x)$.

or

ii) there exists $y \in \partial U$ and $\lambda \in]0,1[$ such that $y \in \lambda T(y)$.

Let us present now the Leray-Schauder principle for multivalued contractions:

Theorem 5.17. (Frigon-Granas [68]) *Let X be a Banach space and $T : X \to P_{cl}(X)$ such that for each $r > 0$ the multifunction $T|_{\widetilde{B}(0,r)}$ is a k-contraction. Denote by $\mathcal{E}_T := \{x \in X | x \in \lambda T(x), \text{ for some } \lambda \in]0,1[\}$. Then at least one of the following assertions hold:*

i) $\mathcal{E}_T$ is unbounded

ii) $FixT \neq \emptyset$.

Corollary 5.18. *Let X be a Banach space and $T : \overline{U} \to P_{cl}(X)$ be a k-contractions such that for each $x \in \partial U$ at least one of the following assertions hold:*

i) $\|T(x)\| \leq \|x\|$
ii) $\|T(x)\| \leq D(x, T(x))$
iii) $\|T(x)\| \leq (D(x, T(x))^2 + \|x\|^2)^{\frac{1}{2}}$
iv) $\|T(x)\| \leq max(\|x\|, D(x, T(x)))$

Then $FixT \neq \emptyset$

In case F is a nonexpansive (i.e. 1-Lipschitz) multifunction, we have:

Theorem 5.19. (Lim [106]) *Let X be an uniformly convex Banach space $Y \in P_{b,cl,cv}(X)$ and $F : Y \to P_{cp}(Y)$ be nonexpansive. Then $FixF \neq \emptyset$.*

Definition 5.20. Let X be a real Banach space, $Y \in P_{cl}(X)$ and $x \in Y$. We let:

$$T_Y(x) = \left\{ y \in X | \lim_{h \to 0_+} \inf D(x + hy, Y)h^{-1} = 0 \right\}$$

$$\tilde{I}_Y(x) := x + T_Y(x)$$

$$I_Y(x) = \{x + \lambda(y - x) | \ \lambda \geq 0, \ y \in Y\}, \quad \text{for} \quad Y \in P_{cl,cv}(X).$$

The set $I_Y(x)$ is called the inward set at x. Notice that $\tilde{I}_Y(x) = I_Y(x)$ for convex subset Y of X.

Definition 5.21. Let X be a real Banach space, $Y \in P_{cl}(X)$ and the mappings $f : Y \to X$ and $F : Y \to P(X)$. Then:

i) f is called weakly inward if $f(x) \in \tilde{I}_Y(x)$, for each $x \in Y$
ii) F is called weakly inward if $F(x) \subset \tilde{I}_Y(x)$, for each $x \in Y$
iii) F is called inward if $F(x) \cap \tilde{I}_Y(x) \neq \emptyset$, for each $x \in Y$

For weakly inward multivalued contractions we have the following recent result of T. -C. Lim ([105]):

Theorem 5.22. (Lim [105]) *Let X be a Banach space and Y be a nonempty closed subset of X. Assume that $F : Y \to P_{cl}(X)$ is a weakly inward multivalued contraction. Then F has a fixed point in Y.*

Let us consider now some basic topological fixed point principles.

For the beginning, we define the notion of Kakutani-type multifunction:

Definition 5.23. Let X, Y be two vector topological spaces. Then $F : X \to P(Y)$ is said to be a Kakutani-type multifunction if and only if:

i) $F(x) \in P_{cp,cv}(Y)$, for all $x \in X$

ii) F is u.s.c. on X.

Definition 5.24. Let X be a vector topological space and $Y \in P(X)$. Then, by definition, Y has the Kakutani fixed point property (briefly K.f.p.p.) if and only if each Kakutani-type multifunction $F : Y \to P(Y)$ has at least a fixed point in Y.

The most famous topological fixed point result is the Kakutani-Fan theorem (see [92]):

Theorem 5.25. (Kakutani-Fan [92]) *Any compact convex subset K of a Banach space X has the K.f.p.p.*

Corollary 5.26. (Brouwer-Schauder) *Let K be a compact convex subset of a Banach space X and $f : K \to K$ be a continuous operator. Then there exists at least one fixed point for f.*

For the infinite dimensional case we also have the following result (see for example Kirk-Sims [97]) of Bohnenblust-Karlin:

Theorem 5.27. (Bohnenblust-Karlin) *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. The any u.s.c. multifunction $F : Y \to P_{cl,cv}(Y)$ with relatively compact range has at least a fixed point in Y.*

As consequence of the Kakutani-Fan result, Browder and Fan proved:

Theorem 5.28. (Browder-Fan [33]) *Let X be a Hausdorff vector topological space and K be a nonempty compact and convex subset of X. Let $F : K \to P_{cv}(K)$ be a multivalued operator with open fibres. Then $FixF \neq \emptyset$.*

Another generalization of the Kakutani-Fan fixed point principle has been proved by Himmelberg as follows:

Theorem 5.29. (Himmelberg [82]) *Let X be a convex subset of a locally convex Hausdorff topological vector space and Y be a nonempty compact subset* *of* X*. Let* $F : X \to P_{cl,cv}(Y)$ *be an u.s.c. multifunction. Then there exists a point* $\overline{x} \in Y$ *such that* $\overline{x} \in F(\overline{x})$*.*

Recently, X. Wu (see [209]) proved a fixed point theorem for lower semi-continuous multivalued operators in locally convex Hausdorff topological vector spaces. This theorem is the lower semi-continuous version of Himmelberg's fixed point theorem.

Theorem 5.30. (Wu [209]) *Let* X *be a nonempty convex subset of a locally convex Hausdorff topological vector space,* Y *a nonempty compact metrizable subset of* X *and* $F : X \to P_{cl,cv}(Y)$ *a l.s.c. multifunction. Then the exists a point* $\overline{x} \in Y$ *such that* $\overline{x} \in F(\overline{x})$*.*

Bibliographical comments. Basic fixed point theorems for multifunction can be found in several sources, such as: Agarwal-Meehan-O'Regan [1], Border [28], Covitz-Nadler [50], Deimling [58], [59], Espínola-Kirk [62], Espínola-Khamsi [63], Frigon-Granas [68], Hu-Papageorgiou [84], M. Kamenskii-Obuhovskii-Zecca [93], Kirk-Sims [97], I. A. Rus [172], Smithson [198], X. Wu [209], Z. Wu [210], Yuan [217].

Chapter 6

Properties of the fixed point set

The purpose of this section is to present several properties of the fixed point set for some multivalued generalized contractions.

Throughout this section, the symbol $\mathcal{M}$ indicates the family of all metric spaces. Let $X \in \mathcal{M}$.

Recall the following notion from I. A. Rus-Petruşel A.-Sîntămărian (see [177] and [178]).

Definition 6.1. Let (X, d) be a metric space and $T : X \to P(X)$ a multivalued operator. By definition, T is a multivalued weakly Picard (briefly MWP) operator if and only if for all $x \in X$ and all $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of the multivalued operator T.

Let us remark that a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) in the previous definition is, by definition, a sequence of successive approximations of T, starting from (x, y).

We can illustrate this notions by several examples.

Example 6.2. (Nadler [125], Covitz-Nadler [50]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Then T is a MWP operator.

Example 6.3. (Reich [165]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued Reich-type operator. Then T is a MWP operator.

Example 6.4. (I. A. Rus [173]) Let (X, d) be a complete metric space. A multivalued operator $T : X \to P_{cl}(X)$ is said to be a multivalued Rus-type graphic-contraction if Graf(T) is closed and the following condition is satisfied: there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$ such that: $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$

Then T is a MWP operator.

Example 6.5. (Petruşel A. [142]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. The multivalued operator T is called a Frigon-Granas type operator if $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ and satisfies the following assertion:

i) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$ such that:

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in \tilde{B}(x_0; r)$$

If T is a Frigon-Granas type operator such that:

ii) $\delta(x_0, T(x_0)) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1} r$,

then T is a MWP operator.

In 1985, T.-C. Lim (see [103]) proved that if T_1 and T_2 are multivalued contractions on a complete metric space X with a same contraction constant $\alpha < 1$ and if $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$, then the data dependence phenomenon for the fixed point set holds, i.e.

$$H(FixT_1, FixT_2) \leq \eta(1 - a)^{-1}.$$

We will show now that the data dependence problem for the fixed point set for a class of generalized multivalued contractions also has an affirmative answer.

Definition 6.6. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then we define the multivalued operator $T^{\infty} : Graf(T) \to P(FixT)$ by the formula:

$T^{\infty}(x, y) := \{z \in FixT|$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

An important abstract concept in this approach is the following:

Definition 6.7. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is a c-multivalued weakly Picard operator (briefly c-MWP operator) if there is a selection t^{∞} of T^{∞} such that: $d(x, t^{\infty}(x, y)) \leq cd(x, y),$ for all $(x, y) \in Graf(T)$.

Further on we shall present several examples of c-MWP operators.

Example 6.8. A multivalued α-contraction on a complete metric space is a c-MWP operator with $c = (1 - \alpha)^{-1}$.

Example 6.9. A multivalued Reich type operator on a complete metric space is a c-MWP operator with $c = [1 - (\alpha + \beta + \gamma)]^{-1}(1 - \gamma)$.

Example 6.10. A multivalued Rus-type graphic contraction on a complete metric space is a c-MWP operator with $c = (1 - \beta)[1 - (\alpha + \beta)]^{-1}$.

Example 6.11. A multivalued Frigon-Granas type operator $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ satisfying the condition $\delta(x_0, T(x_0)) < [1-(\alpha+\beta+\gamma)](1-\gamma)^{-1}r$ is a c-MWP operator.

An important abstract result of is the following:

Theorem 6.12. *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$. We suppose that:*

i) T_i is a c_i-MWP operator for $i \in \{1, 2\}$

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then $H(FixT_1, FixT_2) \leq \eta \max\{c_1, c_2\}$.

Proof. Let $t_i : X \to X$ be a selection of T_i for $i \in \{1, 2\}$. Let us remark that

$$H(FixF_1, FixT_2) \leq \max\left\{\sup_{x \in FixT_2} d(x, t_1^{\infty}(x, t_1(x))), \sup_{x \in FixT_2} d(x, t_2^{\infty}(x, t_2(x)))\right\}.$$

Let $q > 1$. Then we can choose t_i ($i \in \{1, 2\}$) such that

$$d(x, t_1^\infty(x, t_1(x))) \leq c_1 q H(T_2(x), T_1(x)), \text{ for all } x \in FixT_2$$

and

$$d(x, t_2^\infty(x, t_2(x)) \leq c_2 q H(T_1(x), T_2(x)), \text{ for all } x \in FixT_1.$$

Thus we have $H(FixT_1, FixT_2) \leq q\eta \max\{c_1, c_2\}$. Letting $q \searrow 1$, the proof is complete. □

Remark 6.13. As consequences of this abstract principle, we deduce that the data dependence phenomenon regarding the fixed points set for some generalized multivalued contractions (such as Reich-type operators, Rus-type graphic contractions, Frigon-Granas type operators) holds.

Contrary to the single-valued case, if $T : X \to P_{cl}(X)$ is a multivalued contraction on a complete metric space, then $FixT$ is not necessarily a singleton and hence it is of interest to study the topological properties of it.

Let us recall that a metric space X is called an absolute retract for metric spaces (briefly $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any $Y_0 \in P_{cl}(X)$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension over Y, that is $f : Y \to X$. Obviously, any absolute retract is arcwise connected. In this setting, B. Ricceri (see [167]), stated the following important theorem:

Theorem 6.14. (Ricceri) *Let E be a Banach space and let X be a nonempty, closed, convex subset of E. Suppose $T : X \to P_{cl,cv}(X)$ is a multivalued contraction. Then $FixT$ is an absolute retract for metric spaces.*

We establish the following result on the structure of the fixed point set for a multivalued Reich type operator with convex values.

Theorem 6.15. *Let E be a Banach space, $X \in P_{clc,cv}(E)$ and $T : X \to P_{cl,cv}(X)$ be a l.s.c. multivalued Reich-type operator. Then $FixT \in AR(\mathcal{M})$.*

Proof. Let us remark first that $FixT \in P_{cl}(X)$. (see for example Reich [165]) Let K be a paracompact topological space, $A \in P_{cl}(K)$ and $\psi : A \to FixT$ a continuous mapping. Using Theorem 2 from B. Ricceri [167] (taking $G(t) = X$, for each $t \in K$) it follows the existence of a continuous function

$\varphi_0 : K \to X$ such that $\varphi_0|_A = \psi$. We next consider $q \in]1, (\alpha+\beta+\gamma)^{-1}[$. We claim that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions from K to X with the following properties:

(i) $\varphi_n|_A = \psi$

(ii) $\varphi_n(t) \in T(\varphi_{n-1}(t))$, for all $t \in K$

(iii) $\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq [(\alpha+\beta+\gamma)q]^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$.

To see this, we proceed by induction on n. Clearly, for each $t \in A$ we have that $\psi(t) \in T(\varphi_0(t))$. On the other hand, the multifunction $t \mapsto T(\varphi_0(t))$ is l.s.c. on K with closed, convex values and hence using again Theorem 2 in [167] it follows that there is a continuous function $\varphi_1 : K \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(t) \in T(\varphi_0(t))$, for all $t \in K$. Hence, the conditions (i), (ii), (iii) are true for φ_1. Suppose now we have constructed p continuous functions $\varphi_1, \varphi_2, \ldots, \varphi_p$ from K to X in such a way that (i), (ii), (iii) are true for $n \in \{1, 2, \ldots, p\}$. Using the Reich type contraction condition for T, we have

$$D(\varphi_p(A), T(\varphi_p(t))) \leq H(T(\varphi_{p-1}(t)), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta D(\varphi_{p-1}(t), T(\varphi_{p-1}(t))) + \gamma D(\varphi_p(t), T(\varphi_p(t))) \leq$$

$$\leq \alpha\|\varphi_{p-1}(t) - \varphi_p(t)\| + \beta\|\varphi_{p-1}(t) - \varphi_p(t)\| + \gamma D(\varphi_p(t), T(\varphi_p(t)))$$

so that

$$D(\varphi_p(t), T(\varphi_p(t))) \leq (\alpha+\beta)(1-\gamma)^{-1}\|\varphi_p(t) - \varphi_{p-1}(t)\| \leq$$

$$(\alpha+\beta)(1-\gamma)^{-1}[(\alpha+\beta+\gamma)q]^{p-1}\|\varphi_1(t)-\varphi_0(t)\| < (\alpha+\beta+\gamma)^p q^{p-1}\|\varphi_1(t)-\varphi_0(t)\|$$

$$< [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|.$$

We next define

$$Q_p(t) = \begin{cases} B(\varphi_p(t), [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|), \text{if } t \in K, \varphi_1(t) \neq \varphi_0(t) \\ \{\varphi_p(t)\}, \text{ if } \varphi_1(t) = \varphi_0(t) \end{cases}$$

Obviously $T(\varphi_p(t)) \cap Q_p(t) \neq \emptyset$, for all $t \in K$. We can apply now (taking $G(t) = F(\varphi_p(t))$, $f(t) = \varphi_p(t)$ and the mapping $g(t) = [(\alpha+\beta+\gamma)q]^p\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$). Proposition 3 from Ricceri [167], we obtain that the multifunction $t \mapsto \overline{T(\varphi_p(t)) \cap Q_p(t)}$ is l.s.c. on K with nonempty, closed, convex

values. Because of Theorem 2 in [167], this produces a continuous function $\varphi_{p+1} : K \to X$ such that $\varphi_{p+1}|_t = \psi$ and $\varphi_{p+1}(t) \in \overline{T(\varphi_p(H)) \cap Q_p(t)}$, for all $t \in T$. Thus the existence of the sequence $\{\varphi_n\}$ is established. Consider now the open covering of K defined by the formula: $(\{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\})_{\lambda>0}$. Moreover, because of (iii) and the fact that X is complete, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly on each of the following set $K_\lambda = \{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\}$ $(\lambda > 0)$. Let $\varphi : K \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}}$. Obviously φ is continuous and $\varphi|_A = \psi$. Moreover, a simple computation ensures that : $\varphi(t) \in T(\varphi(t))$ for all $t \in K$ and this completes the proof.
□

Remark 6.16. If $\beta = \gamma = 0$ then the previous theorem coincides with B. Ricceri's result (Theorem 2.4.14. below).

Remark 6.17. Of course, it is also possible to formulate version of Theorem 2.4.16. for multivalued Rus type graphic contraction. It is an open question if such a result holds for a Frigon-Granas type multifunction.

Regarding to the compactness property of the fixed point set of a multivalued contraction mapping, J. Saint Raymond (see [187]) established the following theorem:

Theorem 6.18. (Saint Raymond) *Let T be a multivalued contraction from the complete metric space X to itself. If T takes compact values, the fixed point set $FixT$ is compact too.*

An extension of the previous result is:

Theorem 6.19. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let us suppose that $T : \tilde{B}(x_0; r) \to P_{cp}(X)$ satisfies the following two conditions:*

i) there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$ such that

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta[D(x, T(x)) + D(y, T(y))],$$

for each $x, y \in \tilde{B}(x_0; r)$

ii) $D(x_0, T(x_0)) < [1 - (\alpha + 2\beta)](1 - \gamma)^{-1} r$.

Then the fixed points set $FixT$ is compact.

Proof. From Reich's theorem [165] it follows that $FixT \in P_{cl}(\tilde{B}(x_0;r))$. Assume that $FixT$ is not compact. Because $FixT$ is complete, it cannot be precompact, so there exist $\delta > 0$ and a sequence $(x_i)_{i\in\mathbb{N}} \subset FixT$ such that $d(x_i, x_j) \geq \delta$, for each $i \neq j$. Put $\rho = \inf\{R|$ there exists $a \in \tilde{B}(x_0;r)$ such that $B(a,R)$ contains infinitely many x_i;s$\}$. Obviously $\rho \geq \frac{\delta}{2} > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{1-\alpha-2\beta}{1+\alpha}\rho$ and choose $a \in \tilde{B}(x_0;r)$ such that the set $J = \{i : x_i \in B(a, \rho+\varepsilon)\}$ is infinite.

For each $i \in J$, we have

$$D(x_i, T(a)) \leq H(T(x_i), T(a)) \leq \alpha d(x_i, a) + \beta_i[D(x_i, T(x_i)) + D(a, T(a))] =$$

$$= \alpha d(x_i, a) + \beta D(a, T(a)) < \alpha(\rho+\varepsilon) + \beta d(a, y), \text{ for every } y \in T(a).$$

Then

$$D(x_i, T(a)) < \alpha(\rho+\varepsilon) + \beta[d(a, x_i) + d(x_i, y)] < \alpha(\rho+\varepsilon) + \beta(\rho+\varepsilon) + \beta d(x_i, y),$$

for every $y \in T(a)$. Taking $\inf\limits_{y\in T(a)}$ we get : $D(x_i, T(a)) \leq (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1}$, for each $i \in J$. So, we can choose some $y_i \in T(a)$ such that $d(x_i, y_i) \leq (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1}$, for each $i \in J$. By the compactness of $T(a)$ there exists $b \in T(a)$ such that the following set: $J' = \{i \in J|\ d(y_i, b) < \varepsilon\}$ is infinite. Then, for each $i \in J'$ we get $d(x_i, b) \leq d(x_i, y_i) + d(y_i, b) < (\alpha+\beta)(\rho+\varepsilon)(1-\beta)^{-1} + \varepsilon = (\alpha+\beta)(1-\beta)^{-1}\rho + \varepsilon\left(1 + (\alpha+\beta)(1-\beta)^{-1}\right) < \rho$. This contradicts the definition of ρ, because the set $B(b,R)$ contains infinitely many x_i's $\left(\text{where } R = (\alpha+\beta)\rho(1-\beta)^{-1} + \varepsilon\left(1 + (\alpha+\beta)(1-\beta)^{-1}\right)\right)$. $\square$

Bibliographical comments. The approach of this paragraph follows mainly Petruşel A. [137] and Rus-A. Petruşel-Sîntămărian [178]. Excellent sources on this topic are at least the following titles: Anisiu-Mark [7], Deimling [58], Górniewicz-Marano-Slosarki [74], Górniewicz-Marano [75], Kamenskii-Obuhovskii-Zecca [93], Lim [104], Marano [108], Markin [109], Precup [162], Naselli Ricceri and B. Ricceri [126], B. Ricceri [167], Saint Raymond [187], Schirmer [195], Wang [207], Xu-Beg [214], etc.

Chapter 7

Strict fixed point principles

As we have seen in the **Introduction** of this book, a strict fixed point could be interpreted as an optimal preference of a consumer. Also, strict fixed points appear in optimization problems. The purpose of this chapter is to present several strict fixed point results.

First example is in connection with the so-called δ-Reich type operators. Recall that if (X,d) is a metric space, then $T : X \to P_{b,cl}(X)$ is said to be a δ-Reich operator, if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, with $\alpha+\beta+\gamma < 1$ such that $\delta(T(x), T(y)) \le \alpha d(x,y) + \beta\delta(x, T(x)) + \gamma\delta(y, T(y))$ for each $x, y \in X$.

Theorem 7.1. (Reich [165]) *Let (X,d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a δ-Reich operator.*

Then:

i) $(SF)_T = \{x^*\}$

ii) for each $x \in X$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from x, such that $x_n \to x^*$.

Next example provides a multivalued operator with unique strict fixed point such that there exists a sequence of successive approximations which converges to the unique strict fixed point.

Theorem 7.2. (Corley [49]) *Let (X,d) be a complete metric space and $Y \in P_{cl}(X)$. Let $T : Y \to P(Y)$ be such that:*

i) $y \in T(y)$, for each $y \in Y$

ii) there exist $a \in [0,1[, x_0 \in Y$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from x_0, such that $diamT(x_{n+1}) \leq a \cdot diamT(x_n)$, for $n \in \mathbb{N}$.

Then:

i) $(SF)_T = \{x^*\}$

ii) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from x_0, such that $x_n \to x^*$.

Proof. Obviously $diamT(x_n) \leq a^n \cdot diamT(x_0) \to 0$. Hence, $d(x_m, x_n) \to 0$, as $m, n \to \infty$. So, (x_n) is a Cauchy sequence. Moreover Y is complete and then (x_n) converges to some $\{x^*\} \in Y$. Since $\{x^*\} \in T(x^*)$, by hypothesis and $diamT(x^*) = 0$ we obtain the conclusion $\{x^*\} = T(x^*)$. □

Let remark that if $T : X \to P(X)$ and we define the following sequence of multivalued operators: $T^0(x) = \{x\}$, $T^1(x) = T(T^0(x)) = T(x)$, $T^2(x) = T(T^1(x)) = \bigcup\limits_{y\in T^1(x)} T(y), \ldots, T^n(x) = T(T^{n-1}(x)) = \bigcup\limits_{y\in T^{n-1}(x)} T(y)$, for $x \in X$, then a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in T^n(x)$, $x \in X$ for $n \in \mathbb{N}$ is, by definition, (Tarafdar and Vyborny, see Yuan [217]) a generalized sequence of successive approximations of T starting from $x \in X$. Obviously, each sequence of successive approximations of T starting from arbitrary $x \in X$ is a generalized sequence of successive approximations, but the converse may not be true, since $T^n(x)$ is, in general, bigger than $T(x_{n-1})$, i.e. $T(x_{n-1}) \subset T^n(x)$ but not conversely.

Let (X, d) be a metric space and $Y \in P_{b,cl}(X)$. By definition, $T : Y \to P(Y)$ is caled a multivalued (δ, a)-contraction if and only if there exists a real number $a \in]0, 1[$ such that

$$diam(T(Y)) \leq a \cdot diam(Y), \text{ for each } Y \in P_{b,cl}(X) \cap I(T).$$

Theorem 7.3. (Tarafdar-Vyborny, see Yuan [217]) *Let (X, d) be a complete metric space and $Y \in P_{b,cl}(X)$. Let $T : Y \to P(Y)$ be a multivalued (δ, a)-contraction.*

Then:

i) $(SF)_T = \{x^*\}$

ii) for each $x_0 \in X$, there exists a generalized sequence of successive approximations of T starting from x_0, such that $x_n \to x^*$.

Remark 7.4. X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. Then $(SF)_T \subset F_T \subset \bigcap_{n\in\mathbb{N}} T^n(X)$, where $T^0(X) = X$ and $T^n(X) = T(T^{n-1}(X)) = \bigcup_{y\in T^{n-1}(X)} T(y)$.

Proof. First inclusion is quite obviously. For the second one let $x \in F_T$. Then $x \in T(x) \subset T(X) \subset T^2(X) \subset \cdots \subset T^n(X) \subset \ldots$ Hence $x \in \bigcap_{n\in\mathbb{N}} T^n(X)$. □

Another situation is in connection with the core of a multivalued operator.

Definition 7.5. Let (X, d) be a metric space. Then $T : X \to P(X)$ is called a multivalued Janos operator (briefly MJ operator) if $\bigcap_{n\in\mathbb{N}} T^n(X) = \{x^*\}$.

When T is a singlevalued operator we get the notion of singlevalued Janos operator, introduced by I. A. Rus.

Remark 7.6. If $T : X \to P(X)$ is a MJ operator then $(SF)_T = F_T = \{x^*\}$.

Let X be a Hausdorff topological space. Then $T : X \to P_{cl}(X)$ is said to be a topological contraction if and only if T is u.s.c. on X and for every $A \in P_{cl}(X)$ the following implication holds

$$T(A) = A \text{ implies } A = \{x^*\}.$$

Then we have:

Theorem 7.7. (Tarafdar-Vyborny, see Yuan [217]) *Let X be a compact Hausdorff topological space and $T : X \to P_{cl}(X)$ be a topological contraction.*

Then T is a MJ operator.

Theorem 7.8. *Let (X, d) be a compact metric space and $T : X \to P_{cl}(X)$ be a multivalued (δ, a)-contraction. Then T is a MJ operator.*

Proof. Each multivalued (δ, a)-contraction on a bounded metric space is a topological contraction. □

Theorem 7.9. (I. A. Rus [172]) *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction such that $(SF)_T \neq \emptyset$.*

Then:

i) $(SF)_T = F_T = \{x^*\}$

ii) for each $x \in X$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from x, such that $x_n \to x^*$.

Proof. Let $x^* \in X$ be a strict fixed point for T. Then $T(x^*) = \{x^*\}$. Let $y^* \in T(y^*)$. Then $d(x^*, y^*) \leq \delta(x^*, Ty^*) = H(T(x^*), T(y^*)) \leq ad(x^*, y^*)$. It follows $d(x^*, y^*) = 0$ and hence $y^* = x^*$. Thus $F_T = \{x^*\}$ and then $(SF)_T = \{x^*\}$. Moreover, from Covitz-Nadler fixed point principle, for each $x \in X$ there exists a sequence of successive approximation for T starting from x such that $x_n \to x^*$. □

We will condider now the following problem.

Open Problem. Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued operator. If there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$\delta(T(x), T(y)) \leq a\cdot d(x,y) + b\cdot\delta(x, T(x)) + c\cdot\delta(y, T(y)), \text{ for each } (x,y) \in GrafT,$$

then the problem is to study when $Fix(T) = SFix(T) \neq \emptyset$.

In connection with the above problem we have:

Theorem 7.10. *Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a closed multivalued operator. Suppose that there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$\delta(T(x), T(y)) \leq a\cdot d(x,y) + b\cdot\delta(x, T(x)) + c\cdot\delta(y, T(y)), \text{ for each } (x,y) \in GrafT.$$

Then $Fix(T) = SFix(T) \neq \emptyset$.

Proof. Let $q > 1$ and $x_0 \in X$ be arbitrary. Then there exists $x_1 \in T(x_0)$ such that $\delta(x_0, T(x_0)) \leq q \cdot d(x_0, x_1)$. We have $\delta(x_1, T(x_1)) \leq \delta(T(x_0), T(x_1)) \leq a \cdot d(x_0, x_1) + b \cdot \delta(x_0, T(x_0)) + c \cdot \delta(x_1, T(x_1)) \leq ad(x_0, x_1) + bqd(x_0, x_1) + c \cdot \delta(x_1, T(x_1))$. Hence $\delta(x_1, T(x_1)) \leq \frac{a+bq}{1-c} \cdot d(x_0, x_1)$. By this procedure, we can obtain the sequence $(x_n)_{n\in\mathbb{N}}$ having the property $d(x_n, x_{n+1}) \leq (\frac{a+bq}{1-c})^n \cdot d(x_0, x_1)$, for each $n \in \mathbb{N}$. If we choose $q > \frac{b}{1-a-c}$ then we get that

$\frac{a+bq}{1-c} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, d). Denote by x^* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Since $Graf(T)$ is a closed set in $X \times X$ we obtain the first conclusion $x^* \in T(x^*)$.

Let us establish now the relation $Fix(T) = SFix(T)$. It's enough to prove that $Fix(T) \subset SFix(T)$. For, let $x \in Fix(T)$ be arbitrary. Then, using the hypothesis (with $y = x \in T(x)$) we get successively: $\delta(T(x)) \le (b + c) \cdot \delta(x, T(x)) \le (b + c) \cdot \delta(T(x))$. Suppose, by absurdum, that $cardT(x) > 1$. Then $\delta(T(x)) > 0$ and using the above relation we get that $1 \le b + c$, a contradiction. Hence $\delta(T(x)) = 0$ and so $\{x\} = T(x)$. □

Remark 7.11. Theorem 7.10 is an extension of some results given in S. Reich [165] (see Theorem 7.1) and I.A. Rus [172].

Moreover, the contractive condition on T in the previous theorem can be replaced with a more general one, namely: there exist $a \in [0, 1[$ such that

$$\delta(T(x), T(y)) \le a \cdot \delta(x, T(x)), \text{ for each } (x, y) \in GrafT,$$

since $d(x, y) \le \delta(x, T(x))$ and $\delta(y, T(y)) \le \delta(T(x), T(y))$.

Next, we present a strict fixed point theorem.

Theorem 7.11. *Let (X, d) be a complete metric space, and $T : X \to P_b(X)$ be a set-valued operator. Suppose that there exist $a, b \in \mathbb{R}_+$ with $a+b < 1$ such that for each $x \in X$ there exists $y \in T(x)$ with*

$$\delta(y, T(y)) \le a \cdot d(x, y) + b \cdot \delta(x, T(x)).$$

If the map $f : X \to \mathbb{R}_+$, defined by $f(x) := \delta(x, T(x))$ is lower semicontinuous, then $SFix(T) \ne \emptyset$.

Proof. From the hypothesis we have that for each $x \in X$ there is $y \in T(x)$ such that $\delta(y, T(y)) \le (a + b) \cdot \delta(x, T(x))$. Then, for each $x_0 \in X$ we can construct inductively a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x_0, having the property $\delta(x_n, T(x_n)) \le (a + b)^n \cdot \delta(x_0, T(x_0))$. Hence, we will obtain $d(x_n, x_{n+1}) \le \delta(x_n, T(x_n)) \to 0$, as $n \to +\infty$. As consequence, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Denote by $x^* \in X$ the limit of this sequence.

If we denote $f(x_n) := \delta(x_n, T(x_n))$, then using the lower semicontinuity of

f we can write:

$$0 \leq f(x^*) \leq \liminf_{n \to +\infty} f(x_n) = 0.$$

So, $f(x^*) = 0$ and the conclusion $\{x^*\} = T(x^*)$ follows. □

Remark 7.12. If, instead of the lower semicontinuity of f, we suppose that $Graf(T)$ is closed, then, since $(x_n)_{n\in\mathbb{N}}$ is a sequence of successive approximations for T, we immediately get that $x^* \in T(x^*)$. So, the conclusion of the above result is $Fix(T) \neq \emptyset$. It is an open question if the above fixed point is a strict fixed point for T.

Remark 7.13. In Theorem 7.11. the contractive condition on T can be replaced with a more general one: there exists $a \in [0,1[$ such that for each $x \in X$ there exists $y \in T(x)$ with $\delta(y, T(y)) \leq a \cdot \delta(x, T(x))$, since again $d(x, y) \leq \delta(x, T(x))$.

Bibliographical comments. An important part of this chapter is based on the works [44] and [185]. For other strict fixed point results, see Aubin [16], Aubin- Siegel [17], Corley [49], Czerwik [52], Van Hot [83], Mehta [112], A. Muntean [123], Reich [164], [165], I. A. Rus [172], I. A. Rus- A. Petruşel- G. Petruşel [184], Sîntămărian, [192], S. P. Singh-Watson- Srivastava [196], Yuan [217], etc.

Chapter 8

Multivalued operators of Caristi type

The well-known Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(8.1.) \qquad d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X,$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$ (see Caristi [38]).

There are several extensions and generalizations of this important principle of the nonlinear analysis (see for example Jachymski [90], Ciric [46] etc.).

One of them, asserts that if (X, d) is a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is lower semi-continuous and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\int_0^\infty \frac{ds}{1+h(s)} = \infty$, then each single-valued operator f from X to itself satisfying the condition:

$$(8.2.) \text{ for each } x \in X, \frac{d(x, f(x))}{1 + h(d(x_0, x))} + \varphi(f(x)) \leq \varphi(x),$$

has at least a fixed point. (see Zhong-Zhu-Zhao [220])

For the multivalued case, if F is an operator of the complete metric space

X into the space of all nonempty subsets of X and there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(8.3.)\ \text{ for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x,y) + \varphi(y) \leq \varphi(x),$$

(or equivalently there exists a Caristi type selection of F)

then the multivalued map F has at least a fixed point $x^* \in X$, i. e. $x^* \in F(x^*)$. (see for example [115])

Moreover, if F satisfies the stronger condition:

$$(8.4.)\ \text{ for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x,y) + \varphi(y) \leq \varphi(x)$$

(or equivalently $x \leq_\varphi F(x)$ implies that $Max(X, \leq_\varphi) \subset (SF)_T$, where $Max(X, \leq_\varphi)$ denotes the set of all maximal elements in X with respect to $\leq_\varphi$),

then the multivalued map F has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = F(x^*)$. (see [17])

On the other hand, if F is a multivalued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semi-continuous function such that the following condition holds:

$$(8.5.)\ \text{ for each } x \in X, \inf \{\ d(x,y) + \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x),$$

then F has at least a fixed point.(see [83])

In this framework, let us remark that if we replace condition (8.5.) by a weaker condition (see (8.6.) below), then the conjecture stated by J.-P. Penot in [134] as follows:

Let (X,d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semi-continuous function and F be a multivalued operator of X into the family of all nonempty closed subsets of X satisfying the following condition:

$$(8.6.)\ D(x, F(x)) + \inf \{\ \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x),$$

then F has at least a fixed point.

is false. (see Van Hot [83] for a counterexample).

It is easy to see that (8.4.) $\Rightarrow$ (8.3.) $\Rightarrow$ (8.5.) and (8.5.) $\Rightarrow$ (8.3.) provided that F has nonempty compact values.

The purpose of this section is to present several new results in connection with the above mentioned single-valued and multivalued Caristi type operators in complete metric spaces.

Let (X, d) be a metric space and $F : X \to P(X)$ be a multivalued map.

Definition 8.1. A function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is called:

(i) a weak entropy of F if the condition (8.3) holds.

(ii) an entropy of F if the condition (8.4.) holds.

Moreover, the map $F : X \to P(X)$ is said to be weakly dissipative if there exists a weak entropy of F and it is said to be dissipative if there is an entropy of it.

Let us remark now, that if f is a (single-valued) a-contraction in a complete metric space X, then f satisfies condition (8.1.) with $\varphi(x) = (1-a)^{-1}\ d(x, f(x))$, for each $x \in X$, so that part of the Banach contraction principle which says about the existence of a fixed point can be obtained by Caristi's theorem. For the multivalued case we have the following result:

Theorem 8.2. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be an a-contraction $(0 < a < 1)$. Then:*

(a) F satisfies the condition (8.5.) with $\varphi(x) = (1-a)^{-1}\ D(x, F(x))$, for each $x \in X$.

(b) If, in addition $F(x) \in P_{cp}(X)$, for each $x \in X$, then F is weakly dissipative with a weak entropy given by the formula $\varphi(x) = (1-a)^{-1}\ D(x, F(x))$, for each $x \in X$.

Proof. a) is Corollary 1 in [83] and b) follows immediately from a) and the conditions (8.3.) $\Leftrightarrow$ (8.5.). □

Remark. It is an open question if a multivalued a-contraction $(0 < a < 1)$ is dissipative.

The first main result of this section is:

Theorem 8.3. *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a Reich type multivalued map. Then there exists* $f : X \to X$ *a selection of* F *satisfying the Caristi type condition (8.1.).*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1 - b - c$. We denote by $U_x = \{ y \in F(x) : \varepsilon\, d(x,y) \leq (1-b-c)\, D(x, F(x)) \}$, for each $x \in X$. Obviously, for each $x \in X$, the set U_x is nonempty (otherwise, if $x \in X$ is not a fixed point of F and we suppose that for each $y \in F(x)$ we have $\varepsilon d(x,y) > (1-b-c)\, D(x, F(x))$, then we reach the contradiction $\varepsilon D(x, F(x)) \geq (1-b-c)\, D(x, F(x))$; if $x \in X$ is a fixed point of F, then clearly $U_x \neq \emptyset$).

We can choose a single-valued operator $f : X \to X$ such that $f(x) \in U_x$, i. e. $f(x) \in F(x)$ and $\varepsilon\, d(x, f(x)) \leq (1-b-c)\, D(x, F(x))$, for each $x \in X$.

Then we have successively: $D(f(x), F(f(x))) \leq H(F(x), F(f(x))) \leq a\, d(x, f(x)) + b\, D(x, F(x)) + c\, D(f(x), F(f(x)))$ and hence

$$(1-c)\, D(f(x), F(f(x))) - b\, D(x, F(x)) \leq a\, d(x, f(x)).$$

In view of this we obtain:

$$d(x, f(x)) = (\varepsilon - a)^{-1}\, [\varepsilon\, d(x, f(x)) - a\, d(x, f(x))] \leq$$

$$\leq (\varepsilon - a)^{-1}\, [(1-b-c)\, D(x, F(x)) - (1-c)\, D(f(x), F(f(x))) + b\, D(x, F(x))] =$$

$$= (1-c)/(\varepsilon - a)\, [D(x, F(x)) - D(f(x), F(f(x)))].$$

If we define $\varphi : X \to \mathbb{R}_+$ by $\varphi(x) = (1-c)/(\varepsilon - a)\, D(x, F(x))$, for each $x \in X$, then it is easy to see that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \text{ for each } x \in X.$$

□

Remark 8.4. If the multivalued operator $F : X \to P_{cl}(X)$ is an upper semi-continuous Reich type operator, then φ is a lower semi-continuous entropy of f.

Remark 8.5. If in previous Theorem we take $b = c = 0$, then we obtain a result of Jachymski, see [90]. Moreover, we get that a multivalued a-contraction $(0 \le a < 1)$ is weakly dissipative.

Theorem 8.6. *Let (X, d) be a metric space and $F : X \to P(X)$ be a δ-Reich type operator. Then the multivalued operator F is dissipative.*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1 - b - c$. Let $x \in X$ and $y \in F(x)$. It is not difficult to see that

$$\varepsilon \; d(x, y) \le (1 - b - c) \; \delta(x, F(x)).$$

Using the fact that $y \in F(x)$ and the condition from hypothesis we have

$$\delta(y, F(y)) \le \delta(F(x), F(y)) \le a \; d(x, y) + b \; \delta(x, F(x)) + c \; \delta(y, F(y)).$$

It follows that

$$-a \; d(x, y) \le b \; \delta(x, F(x)) - (1 - c) \; \delta(y, F(y)).$$

So, we have

$$d(x, y) = (\varepsilon - a)^{-1} \; [\varepsilon \; d(x, y) - a \; d(x, y)] \le$$

$$\le (\varepsilon - a)^{-1} \; [(1 - b - c) \; \delta(x, F(x)) + b \; \delta(x, F(x)) - (1 - c) \; \delta(y, F(y))] =$$

$$= (1 - c)/(\varepsilon - a) \; [\delta(x, F(x)) - \delta(y, F(y))].$$

We define $\varphi(x) : X \to \mathbb{R}_+$ as follows: $\varphi(x) = (1 - c)/(\varepsilon - a) \; \delta(x, F(x))$, for each $x \in X$ and we get

$$d(x, y) + \varphi(y) \le \varphi(x), \text{ for each } x \in X \text{ and for all } y \in F(x),$$

i. e. the multivalued operator F is dissipative. □

The following result is an extension of Proposition 1 in Van Hot [83].

Theorem 8.7. *Let (X, d) a complete metric space, $x_0 \in X$ be arbitrarily, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ a lower semi-continuous function and $h : \mathbb{R}_+ \to \mathbb{R}_+$ a* *continuous non-decreasing function such that* $\int_0^\infty \frac{ds}{1+h(s)} = \infty$. *Let* $F : X \to P_{cl}(X)$ *be a multivalued operator such that:*

$$\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\} \leq \varphi(x), \textit{for each } x \in X.$$

Then F *has at least a fixed point.*

Proof. We shall prove that for each $x \in X$ there exists $f(x) \in F(x)$ such that:

$$\frac{d(x,f(x))}{1+h(d(x_0,x))} + 2\varphi(f(x)) \leq 2\varphi(x).$$

If $D(x, F(x)) = 0$ then $x \in F(x)$ and put $x = f(x)$.

If $D(x, F(x)) > 0$ then we get successively:

$$\frac{D(x,F(x)}{1+h(d(x_0,x)} + \inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + 2\varphi(y) : y \in F(x)\}$$

$$\leq 2\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\} \leq 2\varphi(x), \text{for each } x \in X.$$

It follows that:

$$\inf\{\frac{d(x,y)}{1+h(d(x_0,x))} + 2\varphi(y) : y \in F(x)\} < 2\varphi(x)$$

and hence there exists $f(x) \in F(x)$ such that:

$$\frac{d(x,f(x))}{1+h(d(x_0,x))} + 2\varphi(f(x)) \leq 2\varphi(x).$$

If we define $\psi(t) = 2\varphi(t)$ we get that f satisfies the hypothesis of Lemma 1.2. in [220] and hence there exists $x^* \in X$ such that $x^* = f(x^*) \in F(x^*)$. □

In what follows we shall discuss the data dependence of the fixed points set of multivalued operators which satisfy the Caristi type condition (8.3) and the data dependence of the strict fixed points set of multivalued operators which satisfy the Caristi type condition (8.4).

Theorem 8.8. *Let* (X,d) *be a complete metric space and* $F_1, F_2 : X \to P(X)$ *be two multivalued operators. We suppose that:*

(i) there exist two lower semi-continuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ *such that for all* $x \in X$*, there exists* $y \in F_i(x)$ *so that*

$$d(x, y) \leq \varphi_i(x) - \varphi_i(y), \ i \in \{1, 2\};$$

(ii) there exists $c_i \in \,]0, +\infty[$ *such that*

$$\varphi_i(x) \leq c_i \ d(x, y), \text{ for each } x \in X \text{ and for all } y \in F_i(x), \ i \in \{1, 2\};$$

(iii) there exists $\eta > 0$ *such that*

$$H(F_1(x), F_2(x)) \leq \eta, \text{ for all } x \in X.$$

Then

$$H(Fix(F_1), Fix(F_2)) \leq \eta \ \max \{ \ c_1, \ c_2 \ \}.$$

Proof. From the condition (i) we have that $Fix(F_i) \neq \emptyset$, $i \in \{1, 2\}$. Let $\varepsilon \in \,]0, 1[$ and $x_0 \in Fix(F_1)$. It follows, from Ekeland variational principle (see for example [59]), that there exists $x^* \in X$ such that

$$\varepsilon \ d(x_0, x^*) \leq \varphi_2(x_0) - \varphi_2(x^*)$$

and

$$\varphi_2(x^*) - \varphi_2(x) < \varepsilon \ d(x, x^*), \text{ for each } x \in X \setminus \{x^*\}.$$

For $x^* \in X$, there exists $y \in F_2(x^*)$ so that

$$d(x^*, y) \leq \varphi_2(x^*) - \varphi_2(y).$$

If we suppose that $y \neq x^*$, then we reach the contradiction

$$d(x^*, y) \leq \varphi_2(x^*) - \varphi_2(y) < \varepsilon \ d(y, x^*).$$

So $y = x^*$ and therefore $x^* \in F_2(x^*)$, i. e. $x^* \in Fix(F_2)$.

Let $q \in \mathbb{R}, \ q > 1$. Then, there exists $x_1 \in F_2(x_0)$ such that

$$d(x_0, x_1) \leq q \ H(F_1(x_0), F_2(x_0)).$$

Taking into account the conditions (ii) and (iii) we are able to write $\varepsilon\ d(x_0, x^*) \leq \varphi_2(x_0) - \varphi_2(x^*) = \varphi_2(x_0) \leq c_2\ d(x_0, x_1) \leq c_2\ q\ H(F_1(x_0), F_2(x_0)) \leq c_2\ q\ \eta$. Hence

$$d(x_0, x^*) \leq \eta\ c_2\ q\ /\ \varepsilon.$$

Analogously, for all $y_0 \in Fix(F_2)$, there exists $y^* \in Fix(F_1)$ such that

$$d(y_0, y^*) \leq \eta\ c_1\ q\ /\ \varepsilon.$$

Using the last two inequalities, we obtain

$$H(Fix(F_1), Fix(F_2)) \leq \eta\ q\ \varepsilon^{-1}\ \max\{\ c_1,\ c_2\ \}.$$

From this, letting $q \searrow 1$ and $\varepsilon \nearrow 1$, the conclusion follows. □

Remark 8.9. In the condition (ii) of the previous Theorem it is sufficient to ask that $\varphi_i(x) = 0$, for all $x \in Fix(F_i)$ and the existence of $c_i \in\ ]0, +\infty[$ such that

$$\varphi_i(x) \leq c_i\ d(x, y),$$

for each $x \in Fix(F_j)$ and for all $y \in F_i(x)$, $i, j \in \{1, 2\}$, $i \neq j$.

Theorem 8.10. *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) there exists $\varphi : X \to \mathbb{R}_+$ a lower semi-continuous function such that

$$d(x, y) \leq \varphi(x) - \varphi(y),\ \textit{for each } x \in X \textit{ and for all } y \in F(x);$$

(ii)there exists $c \in\ ]0, +\infty[$, such that

$$\varphi(x) \leq c\ d(x, y),\ \textit{for each } x \in X \textit{ and for all } y \in F(x).$$

Then $Fix(F) = SFix(F) \neq \emptyset$.

Proof. From the condition (i) we have that $SFix(F) \neq \emptyset$. Let $x^* \in Fix(F)$ and $y \in F(x^*)$. It follows that

$$d(x^*, y) \leq \varphi(x^*) - \varphi(y) = -\varphi(y) \leq 0.$$

Hence $d(x^*, y) = 0$ and therefore $y = x^*$. So $F(x^*) = \{x^*\}$, i. e. $x^* \in SFix(F)$ and thus we are able to write that $Fix(F) \subseteq SFix(F)$. □

Remark 8.11. In condition (ii) of the previous Theorem it is sufficient to impose that $\varphi(x) = 0$, for all $x \in Fix(F)$.

Example 8.12. Let $F : [0,1] \to P([0,1])$, $F(x) = [x/3, x/2]$, for each $x \in [0,1]$ and $\varphi : X \to \mathbb{R}_+$, $\varphi(x) = kx$, for each $x \in [0,1]$, where $k \in \mathbb{R},\ k \geq 1$. It is not difficult to see that $|x - y| \leq \varphi(x) - \varphi(y)$, for each $x \in [0,1]$ and for all $y \in F(x)$ and there exists $c = 2k > 0$ such that $\varphi(x) \leq c\ |x - y|$ for each $x \in [0,1]$ and for all $y \in F(x)$. From Theorem 8.10 we have $Fix(F) = SFix(F) \neq \emptyset$.

Theorem 8.13. *Let (X, d) be a complete metric space and $F_1, F_2 : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) there exist two lower semi-continuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ such that

$$d(x, y) \leq \varphi_i(x) - \varphi_i(y), \text{ for each } x \in X \text{ and for all } y \in F_i(x),\ i \in \{1, 2\};$$

(ii) there exists $c_i \in\ ]0, +\infty[$ such that

$$\varphi_i(x) \leq c_i\ d(x, y), \text{ for each } x \in X \text{ and for all } y \in F_i(x),\ i \in \{1, 2\};$$

(iii) there exists $\eta > 0$ such that

$$H(F_1(x), F_2(x)) \leq \eta, \text{ for all } x \in X.$$

Then

$$H(Fix(F_1), Fix(F_2)) = H(SFix(F_1), SFix(F_2)) \leq \eta\ \max\ \{\ c_1,\ c_2\ \}.$$

Example 8.14. Let $F_1, F_2 : [0,1] \to P([0,1])$, $F_1(x) = [x/3, x/2]$, for each $x \in [0,1]$ and $F_2(x) = [(x+1)/2, (x+2)/3]$, for each $x \in [0,1]$. Let $\varphi_1, \varphi_2 : [0,1] \to \mathbb{R}_+$, $\varphi_1(x) = x$, for each $x \in [0,1]$ and $\varphi_2(x) = 1-x$, for each $x \in [0,1]$.

By an easy calculation we get that $|x - y| \leq \varphi_i(x) - \varphi_i(y)$, for each $x \in [0,1]$ and for all $y \in F_i(x)$, $i \in \{1,2\}$ and there exist $c_1 = 2$ and $c_2 = 2$ such that $\varphi_i(x) \leq c_i\,|x-y|$, for each $x \in [0,1]$ and for all $y \in F_i(x)$, $i \in \{1,2\}$. Also, there exists $\eta = 2/3 > 0$ so that $H(F_1(x), F_2(x)) \leq \eta$, for all $x \in [0,1]$. Then, from our Theorem we have $H(Fix(F_1), Fix(F_2)) = H(SFix(F_1), SFix(F_2)) \leq 4/3$.

Bibliographical comments. For the results of this section and more details see Petruşel A.-Sîntămărian [153]. Also, the works of Aubin-Siegel [17], Bae-Cho-Yeom [21], Caristi [38], Ciric [46], Van Hot [83], Mizoguchi-Takahashi [115], Penot [135], Zhong-Zhu-Zhao [220] are important for the topic of single-valued and multivalued Caristi operators.

Chapter 9

Coincidence points and Nash equilibrium

The aim of this chapter is to establish some coincidence results for multivalued operators. Also, in the second part of the chapter, the technique of crossed cartesian product of multivalued operators is used for existence resuls for a Nash equilibrium point of a noncooperative game.

S. Sessa and G. Mehta (see [191]) established some general coincidence theorems for upper semi-continuous multifunctions using Himmelberg's fixed point principle.

The first aim of this section is to prove some coincidence theorems for lower semi-continuous multifunctions on locally convex Hausdorff topological vector spaces using, instead of Himmelberg's result, the new fixed point principle of X. Wu We will then show a lower semi-continuous version of the well-known Browder's coincidence theorem. An application to game theory is also considered.

Theorem 9.1. *Let X be a nonempty convex and paracompact subset of a locally convex Hausdorff topological vector space E, D a nonempty set of a topological vector space Y. If $S : D \to P(X)$ and $T : X \to P(D)$ are such that:*

(a) S is l.s.c.

(b) $S(y) \in P_{cl,cv}(X)$

(c) $Q(x) = co\, T(x)$ *is a subset of* D

(d) $S(D) \subset C$, *where* C *is a compact and metrizable subset of* X

(e) for each $x \in X$ *there exists* $y \in D$ *such that* $x \in int\, Q^{-1}(y)$.

Then there exist $\overline{x} \in X$ *and* $\overline{y} \in D$ *such that* $\overline{x} \in S(\overline{y})$ *and* $\overline{y} \in Q(\overline{x})$.

Proof. We denote by $U(y) = int\, Q^{-1}(y)$, for each $y \in D$. Then the family $(U(y))_{y \in D}$ is an open covering of the paracompact space X (see (e)). Then, from the definition of paracompactness we obtain that there exists $(U(y_i))_{i \in I}$ an open locally finite covering of X and $\{f_{y_i} | i \in I\}$ a partition of unity by continuous nonnegative real functions defined on X subordinate to the covering $(U(y_i))_{i \in I}$. We can define a continuous operator $f : X \to D$ by $f(x) = \sum_{i \in I} f_{y_i}(x) y_i$ for each $x \in X$. If $f_{y_i}(x) \neq 0$ then $x \in supp f_{y_i} \subset U(y_i) \subset Q^{-1}(y_i)$, that is $y_i \in Q(x)$. Since $Q(x)$ is convex for each $x \in X$ by (c) and $f(x)$ is a convex combination of elements from $Q(x)$, it follows that $f(x) \in Q(x)$, for each $x \in X$. We consider now the multivalued operator $W : X \to \mathcal{P}(X)$ by $W(x) = S(f(x))$, for each $x \in X$. Then W is l.s.c. since f is continuous and S is l.s.c. Moreover by (b) W has nonempty, closed, convex values and $W(X) \subset S(D) \subset C$. Since C is compact and metrizable, then using Wu's fixed point theorem we get that there exists $\overline{x} \in C$ such that $\overline{x} \in W(\overline{x})$. It follow that $\overline{x} \in S(f(\overline{x}))$ and hence $\overline{y} = f(\overline{x}) \in Q(\overline{x})$, proving the conclusion of this theorem. □

If $E = Y$ and $T(x)$ is convex for each $x \in X$ then we get the following coincidence result, similar to Sessa's coincidence theorem for u.s.c. multifunctions (see [190]).

Corollary 9.2. *Let* X *be a nonempty convex and paracompact subset of a locally convex Hausdorff topological vector space* E, D *a nonempty set of* E *and* $S : D \to P(X)$, $T : X \to P(D)$ *two multivalued operators satisfying the following assertions:*

a) S *is l.s.c.*

b) $S(y) \in P_{cl,cv}(X)$

c) $T(x) \in P_{cv}(D)$

d) $S(D) \subset C$, *where* C *is a nonempty compact, metrizable subset of the* *space X*

e) for each $x \in X$ *there exists* $y \in D$ *such that* $x \in int\, T^{-1}(y)$.

Then there exist $\overline{x} \in X$ *and* $\overline{y} \in D$ *such that* $\overline{x} \in S(\overline{y})$ *and* $\overline{y} \in T(\overline{x})$.

Remark 9.3. Condition (e) from previous Corollary appears in Tarafdar [202] and it generalize the well-known Browder's condition:

(f) for each $y \in D$ the set $T^{-1}(y)$ is open in X.

Using condition (f) instead of (e) we deduce the following result:

Theorem 9.4. *Let* X *be a nonempty convex compact and metrizable subset of a locally convex Hausdorff topological vector space* E, D *a nonempty set of a topological vector space* Y, *and* $S : D \to P(X)$, $T : X \to P(D)$ *two multivalued operators satisfying:*

a) S *is l.s.c.*

b) $S(y) \in P_{cl,cv}(X)$, *for each* $y \in D$

c) $T(x) \in P_{cv}(D)$, *for each* $x \in X$

d) $T^{-1}(y)$ *is open in* X, *for each* $y \in D$.

Then there exist $\overline{x} \in X$ *and* $\overline{y} \in D$ *such that* $\overline{x} \in S(\overline{y})$ *and* $\overline{y} \in T(\overline{x})$.

As consequence of the previous result we get:

Theorem 9.5. *Let* X *be a nonempty convex compact and metrizable subset of a locally convex Hausdorff topological vector space* E, D *a nonempty subset of a topological vector space* Y *and* $S, T : D \to P(X)$ *be multifunctions such that:*

a) S *is l.s.c.*

b) $S(y) \in P_{cl,cv}(X)$ *for each* $y \in D$

c) $T^{-1}(x)$ *is a nonempty convex subset of* D *for each* $x \in X$

d) $T(y)$ *is open in* X *for each* $y \in D$.

Then there exists $\overline{y} \in D$ *such that* $S(y) \cap T(\overline{y}) \neq \emptyset$.

Proof. Let us define the multifunction $\widetilde{T} : X \to P(D)$ by $\widetilde{T}(x) = T^{-1}(x)$, for each $x \in D$. Then S and $\widetilde{T}$ satisfy all the hypothesis of the previous theorem and hence there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that $\overline{x} \in S(\overline{y})$ and $\overline{y} \in \widetilde{T}(\overline{x})$. From the definition of $\widetilde{T}$ we obtain $\overline{y} \in T^{-1}(\overline{x})$ and so $\overline{x} \in S(\overline{y}) \cap T(\overline{y})$. □

An important tool for nonlinear problems solved by fixed point techniques is:

Theorem 9.6. (**Marano** [108]) *Let X, Y be nonempty, closed and convex subsets of the Banach spaces E_1, respectively E_2. If $F_1 : Y \to P_{cl,cv}(X)$ and $F_2 : X \to P_{cl}(Y)$ are a_1, respectively a_2 Lipschitz multifunctions and $a_1 a_2 \in]0, 1[$, then the fixed point set of the multivalued operator $T : X \times Y \multimap X \times Y$, defined by $T(x, y) := F_1(y) \times F_2(x)$, for each $(x, y) \in X \times Y$ is a nonempty absolute retract.*

Let us first recall some notions of the game theory. Let us remark first that the current status of the theory of games as a mathematical theory is due to John von Neumann who, between 1928 and 1941, proposed a general framework, with a view to applications in social sciences, within which conflicts and cooperation of players may be taken into account. His fundamental work, published in 1944 in cooperation with O. Morgenstern, Theory of Games and Economic Behavior is the skeletal structure of this topic even today.

Denote by X_i the set of all strategies of the i player, where $i \in \{1, 2, ..., n\}$. Then, $X := \prod_{i=1}^{n} X_i$ is the set of all strategy (or decision) vectors. Each $x = (x_1, x_2, ..., x_n) \in X$ induces an outcome, or a strategy or a decision for each player.

Players preferences are described using the preference multifunction $\widetilde{U}_i : X \multimap X$, defined by $\widetilde{U}_i(x) := \{y \in X | y \text{ is preferred to x } \}$.

We also define, the good reply multifunction.

Denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$, where $X_{-i} := \prod_{k=1, k \neq i}^{n} X_k$. and $x|y_i := (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \in X$.
Then, by definition, y_i is a good reply for the player i with respect to the strategy vector x if $x|y_i \in \widetilde{U}_i(x)$.

In this setting, the good reply multifunction for the player i is $U_i : X_{-i} \multimap X_i$ defined by

$$U_i(x_{-i}) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x|u_i), \text{ for each } u_i \in X_i\}.$$

A game in strategic form or an abstract economy is, by definition the pair

$(X_i, U_i)_{i\in\{1,2,...,n\}}$.

For example, if we consider $p_i : X \to \mathbb{R}$, for $i \in \{1, 2, ..., n\}$, the pay-off function (respectively the loss function) of the i player, then the good reply multifunction can be expressed by:

$$U_i(x_{-i}) := \{y_i \in X_i | p_i(x|y_i) \geq p_i(x|u_i), \text{ for each } u_i \in X_i\} \text{ (respectively } \leq).$$

By definition, $x^* \in X$ is a Nash equilibrium point for an abstract economy if $x_i^* \in U_i(x_{-i}^*)$, for $i \in \{1, 2, ..., n\}$.

Let us remark that the good reply multifunction can be also defined on X, as follows $U_i : X \multimap X_i$ given by:

$$U_i(x) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x)\}.$$

In this setting $x^* \in X$ is a Nash equilibrium point if $U_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$.

In order to define the next concept, we will take into consideration the above definition for the good reply multifunction.

Another important factor in game theory is the constraint (feasibility) multifunction.

We denote by $F_i : X \multimap X_i$, the constraint multifunction for the i player, where $i \in \{1, 2, ..., n\}$. Then define

$$F := \prod_{i=1}^{n} F_i : X \multimap X, \text{ by } F(x) := \prod_{i=1}^{n} F_i(x)$$

.

Obviously, the feasible strategy vectors are the fixed points of F, i. e. $x \in F(x)$.

By definition, a generalized game or a generalized abstract economy is a strategic game (or an abstract economy), which also includes the constraint multifunction F_i, i.e. $(X_i, U_i, F_i)_{i\in\{1,2,...,n\}}$.

A Nash equilibrium point for an generalized abstract economy is a strategy vector $x^* \in X$ such that $x^* \in F(x^*)$ and $U_i(x^*) \cap F_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$.

Let us consider now a 2-person game (or an abstract economy with neighborhood effects) given by $(X_1, U_1), (X_2, U_2)$, where X_1, X_2 denote the set of strategies of the player 1, respectively player 2, and $U_1 : X_2 \multimap X_1$, $U_2 : X_1 \multimap X_2$ are the good reply multifunctions for each player.

By definition, (x_1^*, x_2^*) is a Nash equilibrium point (or a consistent bistrategy) if $x_1^* \in U_1(x_2^*)$ and $x_2^* \in U_2(x_1^*)$.

We note that the problem of finding consistent bistrategies or Nash equilibrium points for a 2-person game is in fact a fixed point problem. Indeed, let us define the multivalued operator T as the crossed cartesian product of the multivalued operators U_1 and U_2, i. e.

$$T(x_1, x_2) := U_1(x_2) \times U_2(x_1), \text{ for each } (x_1, x_2) \in X_1 \times X_2.$$

Then $(x_1^*, x_2^*) \in X_1 \times X_2$ is a Nash equilibrium if and only if $(x_1^*, x_2^*) \in T(x_1^*, x_2^*)$.

It is important to remark that, to our knowledge, the technique of crossed cartesian product of multivalued operators was used for the first time in Debreu [55] in order to prove the existence of a market equilibrium. For more details on this subject see the recent paper of R. Espínola, G. López, A. Petruşel [65].

The following result is an easy consequence of a fixed point theorem.

Theorem 9.7. *Suppose that the behaviors of the players are described by two continuous singlevalued operators u_1 and u_2 and that the strategy sets X_1 and X_2 are convex compact subsets of finite-dimensional vector spaces. Then there exists at least one Nash equilibrium point for the 2-person game.*

Proof. The conclusion follows by an immediate application of Brouwer-Schauder fixed point theorem. □

By applying a fixed point theorem of X. Wu [209] (see Theorem 5.30) we can get the following existence result for a Nash equilibrium point.

Theorem 9.8. (A. Muntean-A. Petruşel [122]) *Let X_1 be a nonempty paracompact and convex subset of a locally convex Hausdorff topological vector space E_1 and X_2 a nonempty subset of a Hausdorff topological vector space E_2. Let $U_1 : X_2 \to P_{cl,cv}(X_1)$ be lower semi-continuous and $U_2 : X_1 \to P(X_2)$*

defined by $U_2(x) := coU(x)$, *for each* $x \in X_1$, *where* $U : X_1 \to P(X_2)$. *If there exists a compact metrizable subset* C *of* X_1 *such that* $U_1(X_1) \subset C$ *then, there exists at least a Nash equilibrium point for the 2-person game* $\{(X_1, U_1), (X_2, U_2)\}$.

The following theorem is, not only an existence result for the Nash equilibrium points of an 2-person game, but also produces a topological property of the Nash equilibrium point set:

Theorem 9.9. *Let* $(X_1, U_1), (X_2, U_2)$ *be a 2-person game. Suppose that:*

(i) X_1, X_2 *are nonempty, closed and convex subsets of the Banach spaces* E_1, *respectively* E_2.

(ii) U_i *is an* a_i*-Lipschitz multifunction with nonempty, closed and convex values, for* $i \in \{1, 2\}$.

(iii) $a_1 a_2 \in]0, 1[$.

Then the set of all Nash equilibrium points is nonempty and arcwise connected.

Proof. Let us remark that the Nash equilibrium point set is equal with the fixed point set of the multivalued operator $T(x_1, x_2) := U_1(x_2) \times U_2(x_1)$, for each $(x_1, x_2) \in X_1 \times X_2$. The conclusion follows by Theorem 9.6. □

The traditional way of modeling game theory is to assume that each player classifies the bistrategies using an utility function. This function has several names, for example: evaluation function, criterion function, gain function, loss function, cost function. The terminology is only a matter of taste. Such a function may be associated with a partial order $\geq$ called the partial order of preferences, as follows:

$$(x_1, x_2) \in X_1 \times X_2 \text{ is preferred to } (y_1, y_2) \in X_1 \times X_2$$

if and only if

$$f(x_1, x_2) \leq f(y_1, y_2), \text{ (for loss or cost functions) },$$

respectively

$$(x_1, x_2) \in X_1 \times X_2 \text{ is preferred to } (y_1, y_2) \in X_1 \times X_2$$

if and only if

$$f(x_1, x_2) \geq f(y_1, y_2), \text{ (for utility or gain functions) }.$$

Let us consider that the two players choose separately their strategies using their loss functions f_1 and f_2. Suppose that $f_1, f_2 : X_1 \times X_2 \to \mathbb{R}$. We set $f(x_1, x_2) := (f_1(x_1, x_2), f_2(x_1, x_2)) \in \mathbb{R}^2$.

A two person game in normal form is defined by a function f from $X_1 \times X_2$ to $\mathbb{R}^2$, also called the biloss operator.

If the first player (P1) know the strategy $x_2 \in X_2$ of the second player (P2), then he may be tempted to choose a strategy $x_1^* \in X_1$, which minimizes his loss $x_1 \to f(x_1, x_2)$. In other words, he may choose a strategy in the set

$$U_1(x_2) := \{x_1^* \in X_1 | f_1(x_1^*, x_2) = \inf_{x_1 \in X_1} f_1(x_1, x_2)\}.$$

This enables us to a decision rule $U_1 : X_2 \multimap X_1$ for (P1). Similarly, we can define a decision rule U_2 for (P2), by the formula:

$$U_2(x_1) := \{x_2^* \in X_2 | f_2(x_1, x_2^*) = \inf_{x_2 \in X_2} f_2(x_1, x_2)\}.$$

The decision rules U_1, U_2 associates with the loss functions f_1, f_2 are called the canonical decision rules. A consistent pair of bistrategies (x_1^*, x_2^*) based on the canonical decision rules is called a noncooperative equilibrium or a Nash equilibrium of the game. Thus, a pair (x_1^*, x_2^*) is a noncooperative equilibrium if and only if

$$f_1(x_1^*, x_2^*) = \inf_{x_1 \in X_1} f_1(x_1, x_2^*) \text{ and } f_2(x_1^*, x_2^*) = \inf_{x_2 \in X_2} f_1(x_1^*, x_2).$$

So, a noncooperative equilibrium is a situation in which each player optimizes his own criterion, assuming that his partner's choice is known and hence fixed. Such a case is also called a situation with individual stability.

If we assume that the players communicate, exchange information and cooperate, then there it may exist strategy pairs (x_1, x_2) satisfying

$$f_1(x_1, x_2) < f_1(x_1^*, x_2^*) \text{ and } f_2(x_1, x_2) < f_2(x_1^*, x_2^*),$$

where the two players have losses strictly less than in the case of noncooperative equilibrium. This is situation with collective stability, since the players can each find better strategies for themselves. So, a strategy pair (x_1^*, x_2^*) is said to be a Pareto optimum if there are no other strategy pairs $(x_1, x_2) \in X_1 \times X_2$ such that $f_1(x_1, x_2) < f_1(x_1^*, x_2^*)$ and $f_2(x_1, x_2) < f_2(x_1^*, x_2^*)$.

There exists noncooperative Nash equilibria which are Pareto optimal, but there are only few such examples and no general theorems are known.

This approach can easily be extended to n-person games. Denote by X_i the set of all strategies of the i player, where $i \in \{1, 2, ..., n\}$. Then, $X := \prod X_i$ is the set of all strategy (or decision) vectors. Each $x = (x_1, x_2, ..., x_n) \in X$ induces an outcome, or a strategy or a decision for each player. It is called a multistrategy.

Denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$, where $X_{-i} := \prod_{k=1, k\neq i}^{n} X_k$. and $x|y_i := (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \in X$.
Then, by definition, y_i is a good reply for the player i with respect to the strategy vector x if $x|y_i \in \widetilde{U}_i(x)$.

In this setting, the good reply multifunction (or the decision rule multifunction) for the player i is $U_i : X_{-i} \multimap X_i$ defined by:

$$U_i(x_{-i}) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x)\}.$$

A multistrategy $x \in X$ is said to be a consistent strategy if for each $i \in \{1, \cdots, n\}$ we have $x_i \in U_i(x_{-i})$.

We shall suppose now that the decision rule multifunction of the players are determined again by loss operators.

Then, a game in normal form is a n-person game in which the behavior of each player is defined by a loss function $f_i : X \to \mathbb{R}$, with $i \in \{1, 2, \cdots, n\}$.

Such a game can may be summarized by the multiloss operator $f : X \to \mathbb{R}^n$, given by $f(x) = (f_1(x), \cdots f_n(x))$. The associated decision rules are the multivalued operators

$$U_i(x_{-i}) := \{x_i \in X_i | f_i(x_i, x_{-i}) = \inf_{y_i \in X_i} f_i(y_i, x_{-i})\},$$

where (u, x_{-i}) denotes the vector $(x_1, \cdots, x_{i-1}, u, x_{i+1}, \cdots, x_n)$.

By definition, $x^* \in X$ is a Nash equilibrium point or a noncooperative equilibrium for an abstract economy if $x_i^* \in U_i(x_{-i}^*)$, for $i \in \{1, 2, ..., n\}$.

This definition leads to the following characterization. We introduce the function $g : X \times X \to \mathbb{R}$, defined by

$$g(x, y) = \sum_{i=1}^{n} (f_i(x_i, x_{-i}) - f_i(y_i, x_{-i})).$$

Lemma 9.10. *The following assertions are equivalent:*

i) $x^* \in X$ *is a noncooperative Nash equilibrium*

ii) for each $i \in \{1, 2, \cdots, n\}$ *and each* $y_i \in X_i$ *we have* $f_i(x_i^*, x_{-i}^*) - f_i(y_i, x_{-i}^*) \leq 0$

iii) for each $y \in X$ *we have* $g(x^*, y) \leq 0$.

Another auxiliary result is:

Theorem 9.11. (Fan's inequality) *Let* X *be a compact convex subset of a Hilbert space and let* $g : X \times X \to \mathbb{R}$ *satisfying:*

i) $x \to g(x, y)$ *is lower semicontinuous for each* $y \in X$

ii) $y \to g(x, y)$ *is concave for each* $x \in X$.

Then there exists $x^* \in X$ *such that* $\sup_{y \in X} g(x^*, y) \leq \sup_{y \in X} g(y, y)$.

An existence result for a noncooperative Nash equilibrium is the following:

Theorem 9.12. (Nash) *We suppose that:*

1) the sets X_i *are convex and compact, for each* $i \in \{1, 2, ..., n\}$.

2) the operators f_i *are continuous for each* $i \in \{1, 2, ..., n\}$ *and the functions* $y_i \to f_i(y_i, x_{-i})$ *are convex.*

Then there exists at least one noncooperative Nash equilibrium.

Proof. The proof follows from the previous Lemma and Fan's inequality. We introduce $X := \prod_{i=1}^{n} X_i$ and $g(x, y) = \prod_{i=1}^{n} (f_i(x_i, x_{-i}) - f_i(y_i, x_{-i}))$. The set X is compact and convex while the operator g is continuous in first variable and concave in the second one. From Fan's inequality we have that there exists $x^* \in X$ such that $\sup_{y \in X} g(x^*, y) \leq \sup_{y \in X} g(y, y) = 0$, since $g(y, y) = 0$, for each y. Now the final conclusion follows from Lemma 9.10. □

Bibliographical comments. The results given here extent to the l.s.c. multifunctions case some results from Sessa-Mehta (see [191]). Mainly, this section follow the paper A. Muntean- Petruşel A. [122], the book of Aubin [16] and the monograph of Yuan [217].

For other results and interesting applications see: Ansari-Idzik-Yao [9], Buică [35], Dugundji-Granas [61], Petruşel A. [144], [145], O'Regan [130], Rus [169], [172].

Part III

K^2M Operators

Chapter 10

Basic concepts for K^2M operators

Since the K^2M operators technique is an important tool in mathematical economics, we start this section by presenting the concept of K^2M operator.

Let X a vector space over $\mathbb{R}$. A subset A of X is called a linear subspace if for all $x, y \in A$ $x + y \in A$ and for all $x \in X$ and each $\lambda \in \mathbb{R}$ we have that $\lambda \cdot x \in A$. If A is a nonempty subset of X, then *spanA* is, by definition, the intersection of all subspaces which contains A, i. e. the smallest linear subspace containing A. We have the following characterization of the span.

$$spanA = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \text{ with } x_i \in A, \lambda_i \in \mathbb{R}, n \in \mathbb{N}\}.$$

Also, a k-dimensional flat (or a k-dimensional linear variety) in X is a subset L of X with $dimL = k$ such that for each $x, y \in L$, with $x \neq y$, the whole line joining x and y is included in L, i. e. $(1-\lambda) \cdot x + \lambda \cdot y \in L$, for each $\lambda \in \mathbb{R}$.

Definition 10.1 A subset A of a vector space X is said to be finitely closed if its intersection with any finite-dimensional flat $L \subset X$ is closed in the Euclidean topology of L.

Obviously if X is a vector topological space then any closed subset of X is finitely closed.

Definition 10.2. A family $\{A_i | \ i \in I\}$ of sets is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

Definition 10.3. Let X be a vector space and Y a nonempty subset of X. The multifunction $G : Y \to P(X)$ is called a Knaster-Kuratowski-Mazurkiewicz operator (briefly K^2M operator) if and only if

$$co\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

for each finite subset $\{x_1, \dots, x_n\} \subset Y$.

The main property of K^2M operators is given in:

Theorem 10.4. (K^2M principle) *Let X be a vector space, Y a nonempty subset of X and $G : Y \to P(X)$ a K^2M operator such that $G(x)$ is finitely closed, for each $x \in Y$. Then the family $\{G(x)|\ x \in Y\}$ of sets has the finite intersection property.*

Proof. We argue by contradiction: assume that there exist $\{x_1, \dots, x_n\} \subset X$ such that $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. Denote by L the finite dimensional flat spanned by $\{x_1, \dots, x_n\}$, i.e. $L = span\{x_1, \cdots, x_n\}$. Let us denote by d the Euclidean metric in L and by $C := co\{x_1, \dots, x_n\} \subset L$.

Because $L \cap G(x_i)$ is closed in L, for all $i \in \{1, 2, \dots, n\}$ we have that:

$$D_d(x, L \cap G(x_i)) = 0 \ \Leftrightarrow \ x \in L \cap G(x_i), \text{ for all } i = \overline{1, n}.$$

Since $\bigcap_{i=1}^{n} [L \cap G(x_i)] = \emptyset$ it follows that the map $\lambda : C \to \mathbb{R}$ given by

$$\lambda(c) = \sum_{i=1}^{n} D_d(c, L \cap G(x_i)) \neq 0, \text{ for each } c \in C.$$

Hence we can define the continuous map $f : C \to C$ by the formula

$$f(c) = \frac{1}{\lambda(c)} \sum_{i=1}^{n} D_d(c, L \cap G(x_i)) x_i.$$

By Brouwer's fixed point theorem there is a fixed point $c_0 \in C$ of f, i. e. $f(c_0) = c_0$. Let

$$I = \{i|\ D_{d_E}(c_0, L \cap G(x_i)) \neq 0\}.$$

Then for $i \in$ we have $c_0 \notin L \cap G(x_i)$ which implies

$$c_0 \notin \bigcup_{i \in I} G(x_i).$$

On the other side:

$$c_0 = f(c_0) \in co\{x_i |\ i \in I\} \subset \bigcup_{i \in I} G(x_i)$$

(last inclusion follows from the K^2M assumption of G). This is a contradiction. □

As an immediate consequence we obtain the following theorem:

Corollary 10.5. (Ky Fan) *Let X be a vector topological space, Y a nonempty subset of X and $G : Y \to P_{cl}(X)$ a K^2M operator. If at least one of the sets $G(x)$, $x \in Y$ is compact, then*

$$\bigcap_{x \in Y} G(x) \neq \emptyset.$$

We observe that same conclusion can be reached in another way, by involving an auxiliary family of sets and a suitable topology on X.

Corollary 10.6. (Ky Fan) *Let X be a vector space, Y a nonempty subset of X and $G : Y \to P(X)$ a K^2M operator. Assume that there is a multivalued operator $T : Y \to P(X)$ such that $G(x) \subset T(x)$ for each $x \in X$ and*

$$\bigcap_{x \in Y} T(x) = \bigcap_{x \in Y} G(x).$$

If there is some topology on X such that each $T(x)$ is compact, then

$$\bigcap_{x \in Y} G(x) \neq \emptyset.$$

Chapter 11

Ky Fan fixed point theorem

One of the simplest application of K^2M principle is the well-known fixed point theorem of Ky Fan. We start this section with the following auxiliary result.

Lemma 11.1. (Ky Fan) *Let X be a normed space, Y a compact convex subset of X and $f : Y \to X$ be a continuous operator. Then there exists at least one $y_0 \in Y$ such that*

$$\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|.$$

Proof. Define $G : Y \to \mathcal{P}(X)$ by

$$G(x) = \{y \in Y | \|y - f(y)\| \leq \|x - f(y)\|\}.$$

Because f is continuous the sets $G(x)$ are closed in Y and therefore compact. We verify that G is a K^2M operator. For, let $y \in co\{x_1, \ldots, x_n\} \subset Y$. If $y \notin \bigcup_{i=1}^{n} G(x_i)$ then $\|y - f(y)\| > \|x_i - f(y)\|$ for $i \in \{1, 2, \cdots, n\}$. This shows that all the points x_i lie in an open ball of radius $\|y - f(y)\|$ centered at $f(y)$. Therefore, the convex hull of it is also there and in particular y. Thus $\|y - f(y)\| > \|y - f(y)\|$, which is a contradiction. By the compactness of $G(x)$ we find a point y_0 such that $y_0 \in \bigcap_{x \in Y} G(x)$ and hence $\|y_0 - f(y_0)\| \leq \|x - f(y_0)\|$, for all $x \in Y$. This clearly implies $\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|$ and the proof is complete. □

Theorem 11.2. (Ky Fan) *Let Y be a compact convex subset of a normed space X. Let $f : Y \to X$ be a continuous operator such that for each $x \in Y$ with $x \neq f(x)$, the line segment $[x, f(x)]$ contains at least two points of Y. Then f has at least a fixed point.*

Proof. By the previous Lemma, we obtain an element $y_0 \in Y$ with $\|y_0 - f(y_0)\| = \inf_{x \in Y} \|x - f(y_0)\|$. We will show that y_0 is a fixed point of f. The segment $[y_0, f(y_0)]$ must contain a point of Y other than y_0, let say x. Then $x = ty_0 + (1-t)f(y_0)$, with some $t \in]0, 1[$. Then $\|y_0 - f(y_0)\| \leq t\|y_0 - f(y_0)\|$ and since $t < 1$, we must have $\|y_0 - f(y_0)\| = 0$. □

Chapter 12

Game theory

The following general coincidence result follows from the K^2M principle:

Theorem 12.1. (Ky Fan) *Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. Let $A, B : X \to \mathcal{P}(Y)$ two multivalued operators satisfying the following assumptions:*

i) $A(x) \in \mathcal{P}_{op}(Y)$ and $B(x) \in P_{cv}(Y)$, for each $x \in X$

ii) $A^{-1}(y) \in P_{cv}(X)$ and $B^{-1}(y) \in \mathcal{P}_{op}(X)$, for each $y \in Y$.

Then there exists an element $x_0 \in X$ such that $A(x_0) \bigcap B(x_0) \neq \emptyset$, i. e. $C(A, B) \neq \emptyset$.

Proof. Let $Z = X \times Y$ and $G : X \times Y \to \mathcal{P}(E \times F)$ be given by

$$G(x, y) = Z - (B^{-1}(y) \times A(x)).$$

Because $G(x, y) \in P_{cl}(X \times Y)$ and $X \times Y$ is compact we get that $G(x, y) \in P_{cp}(X \times Y)$.

It is easy to observe that:

$$Z = \cup\{B^{-1}(y) \times A(x) |\ (x, y) \in Z\} \qquad (12.1).$$

Indeed, let $(x_0, y_0) \in Z$ be arbitrarily. Choose an $(x, y) \in A^{-1}(y_0) \times B(x_0) \neq \emptyset$ which is equivalent with $(x_0, y_0) \in B^{-1}(y) \times A(x)$. Thus from (12.1) we have:

$$\bigcap_{z \in Z} G(z) = \emptyset.$$

From the first Corollary of K^2M principle G cannot be a K^2M operator. Hence there exist $z_1, z_2, \ldots, z_n \in Z$ such that

$$co\{z_1, \ldots, z_n\} \not\subset \bigcup_{i=1}^{n} G(z_i),$$

which means that there is a $w \in co\{z_1, \ldots, z_n\}$,

$$w = \sum_{i=1}^{n} \lambda_i z_i$$

with

$$w \notin \bigcup_{i=1}^{n} G(z_i).$$

Because Z is convex and $z_i \in Z,$ for each $i = \overline{1,n}$ we obtain that $w \in Z$. Hence:

$$w \in Z - \bigcup_{i=1}^{n} G(z_i) = \bigcap_{i=1}^{n} (B^{-1}(y_i) \times A(x_i)).$$

How

$$w = \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i \right)$$

it follows that

$$\sum_{i=1}^{n} \lambda_i x_i \in B^{-1}(y_i)$$

and

$$\sum_{i=1}^{n} \lambda_i y_i \in A(x_i), \text{ for each } i = \overline{1,n}.$$

Successively we have:

$$y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i \right) \text{ and } x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i \right), \text{ for each } i = \overline{1,n} \Rightarrow$$

$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i \right) \text{ and } \sum_{i=1}^{n} \lambda_i x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i \right) \Rightarrow$$

$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i \right) \text{ and } \sum_{i=1}^{n} \lambda_i y_i \in A\left(\sum_{i=1}^{n} \lambda_i x_i \right).$$

Writing $x_0 = \sum_{i=1}^{n} \lambda_i x_i$ we got that $A(x_0) \cap B(x_0) \neq \emptyset$ and hence $C(A, B) \neq \emptyset$. $\square$

We give now an immediate application to game theory, by establishing a general version of the von Neumann min-max principle due to Sion.

Recall that a functional $f; X \to \mathbb{R}$ on a topological space is called lower (respectively upper) semicontinuous if $\{x \in X | f(x) > r\}$ (respectively $\{x \in X | f(x) < r\}$) is open for each $r \in \mathbb{R}$. Also, if X is a convex set of a vector space, then f is quasi-concave (respectively quasi-convex) if $\{x \in X | f(x) > r\}$ (respectively $\{x \in X | f(x) < r\}$) is convex for each $r \in \mathbb{R}$.

Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. By definition, a point $(x^*, y^*) \in X \times Y$ is called a saddle point for f if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \text{ for each } (x, y) \in X \times Y.$$

The above condition is equivalent with

$$\max_{x \in X} f(x, y^*) = f(x^*, y^*) = \min_{y \in Y} f(x^*, y).$$

Moreover, in this case $(x^*, y^*) \in X \times Y$ is a saddle point for f if and only if

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

If P and Q are two players having X and respectively Y their the strategies set, then for $x \in X$ and $y \in Y$ the value $f(x, y)$ represents the gain of P and so, the lost of Q. If $(x^*, y^*) \in X \times Y$ is a saddle point for f then $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$, for each $(x, y) \in X \times Y$. Hence, if Q choose the strategy y^*, then the gain of P is at most $f(x^*, y^*)$ and the maximum will be attained if P has the strategy x^*. Also, if P choose the strategy x^*, the the lost of Q is at least $f(x^*, y^*)$ and the minimum will be obtained if Q has the strategy y^*. In this way, $(x^*, y^*) \in X \times Y$ assures the optimal balance between the interests of the two players.

The following result was proved by John von Neumann in 1927 for the case of $\mathbb{R}^n$. We present here the version based on Sion's proof.

Theorem 12.2. (Min-max principle) *Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. Let $f : X \times Y \to \mathbb{R}$ satisfying:*

i) $y \to f(x,y)$ is lower semicontinuous and quasi-convex for each $x \in X$

ii) $x \to f(x,y)$ is upper semicontinuous and quasi-concave for each fixed $y \in Y$.

Then

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

Proof. Because of upper semicontinuity, $\max_{x\in X} f(x,y)$ exists for each $y \in Y$ and it is a lower semicontinuous function of y, so $\min_{y\in Y}\max_{x\in X} f(x,y)$ exists. Similarly, $\max_{x\in X}\min_{y\in Y} f(x,y)$ exists too. Since $f(x,y) \le \max_{x\in X} f(x,y)$ we have:

$$\min_{y\in Y} f(x,y) \le \min_{y\in Y}\max_{x\in X} f(x,y),$$

and therefore

$$\max_{x\in X}\min_{y\in Y} f(x,y) \le \min_{y\in Y}\max_{x\in X} f(x,y).$$

We shall prove now that the strict inequality cannot hold. For, assume it did. Then there exists some real r with:

$$\max_{x\in X}\min_{y\in Y} f(x,y) < r < \min_{y\in Y}\max_{x\in X} f(x,y).$$

Define $A, B : X \to \mathcal{P}(Y)$ by:

$$A(x) = \{y \in Y | f(x,y) > r\} \text{ and } B(x) = \{y \in Y | f(x,y) < r\}.$$

These multivalued operators would satisfy the coincidence result of Ky Fan. Indeed, $A(x)$ is open by the lower semicontinuity of $y \to f(x,y)$, each $B(x)$ is convex by the quasi-convexity of $y \to f(x,y)$ and it is nonempty because $\max_{x\in X}\min_{y\in Y} f(x,y) < r$. Since $A^{-1}(y) = \{x \in X | f(x,y) > r\}$ and $B^{-1}(y) = \{x \in X | f(x,y) < r\}$, we find in the same way that each $A^{-1}(y)$ is nonempty and convex and each $B^{-1}(y)$ is open. Then, by Ky Fan coincidence result there is $(x_0, y_0) \in X \times Y$ with $y_0 \in A(x_0) \bigcap B(x_0)$, which gives the contradiction $r < f(x_0, y_0) < r$. The proof is complete. $\square$.

Chapter 13

Variational inequalities

An application of the K^2M principle to the theory of variational inequalities will be presented.

Let $(H, (\cdot, \cdot))$ be a Hilbert space and X be any subset of H. We recall that an operator $f : X \to H$ is monotone decreasing on X if $(f(x)-f(y), x-y) \leq 0$, for all $x, y \in X$. We say that $f : X \to H$ is hemi-continuous if $f|_{L \cap X}$ is continuous for each one-dimensional flat $L \subset H$.

Theorem 13.1. (Hartman-Stampacchia) *Let H be a Hilbert space, X a closed bounded convex subset of H and $f : X \to H$ monotone decreasing and hemi-continuous. Then there exists an element $y_0 \in X$ such that $(f(y_0), y_0 - x) \geq 0$, for all $x \in X$.*

Proof. For each $x \in X$, let $G(x) = \{y \in X | (f(y), y - x) \geq 0\}$. We will prove that

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

We will establish first that G is a K^2M operator. Indeed, let $y_0 \in co\{x_1, \ldots, x_n\}$. Suppose, by contradiction, that $y_0 \notin \bigcap_{i=1}^{n} G(x_i)$. Then we have $(f(y_0), y_0 - x_i) < 0$, for each $i \in \{1, \cdots, n\}$. Since all the x_i would lie in the half-space $\{x \in H | (f(y_0), y_0) < f(y_0), x)\}$, so also would $co\{x_1, \ldots, x_n\}$ and therefore, since $y_0 \in co\{x_1, \ldots, x_n\}$ we have got the contradiction $(f(y_0), y_0) < (f(y_0), y_0)$. Thus G is a K^2M operator.

Consider now the multivalued operator $T : X \to \mathcal{P}(H)$ given by:

$$T(x) = \{y \in X | (f(x), y - x) \geq 0\}.$$

We show that T satisfies the requirements of the second Corollary of K^2M principle.

(i) $G(x) \subset T(x)$, for all $x \in X$. For, let $y \in G(x)$. Then $(f(y), y-x) \geq 0$. By the monotonicity of f we have that $(f(y) - f(x), y - x) \leq 0$ and so $0 \leq (f(y), y - x) \leq (f(x), y - x)$. It follows $y \in T(x)$.

(ii) $\bigcap_{x \in X} T(x) = \bigcap_{x \in X} G(x)$. For, it is enough to show

$$\bigcap_{x \in X} T(x) \subset \bigcap_{x \in X} G(x).$$

Assume $y_0 \in \bigcap_{x \in X} T(x)$. Choose any $x \in X$ and let $z_t = tx + (1 - t)y_0 = y_0 - t(y_0 - x)$. Because X is convex, we have that $z_t \in X$, for each $0 \leq t \leq 1$. Since $y_0 \in T(z_t)$, for each $t \in [0, 1]$, we find that $(f(z_t), y_0 - z_t) \geq 0$ for all $t \in [0, 1]$. This means that $t(f(z_t), y_0 - x) \geq 0$, for all $t \in [0, 1]$ and in particular, that $(f(z_t), y_0 - x) \geq 0$, for $t \in]0, 1]$. Let $t \to 0$. From the continuity of f on the ray joining y_0 and x, we obtain that $f(z_t) \to f(y_0)$ and therefore we have $(f(y_0, y_0 - x) \geq 0$. Thus $y_0 \in G(x)$, for each $x \in X$ and the second assumption is proved.

(iii) We now equip H with the weak topology. Then X, as a closed bounded convex set in a Hilbert space, is weakly compact. Therefore each $T(x)$, being the intersection of the closed half-space $\{y \in H | (f(x), y) \geq (f(x), x)\}$ with X is, for the same reason also weakly compact.

All the requirements of the second Corollary of K^2M principle are satisfied and hence $\bigcap_{x \in X} G(x) \neq \emptyset$. The proof is complete. $\square$

Chapter 14

Stability results for the K^2M point set

Let X be a bounded complete convex subset of a normed space E and denote by $\mathcal{M}$ be the collection of all K^2M multifunctions $G : X \to P_{cp}(X)$. For each $G_1, G_2 \in \mathcal{M}$ define $\vartheta(G_1, G_2) := \sup_{x\in X} H(G_1(x), G_2(x))$. Clearly, $(\mathcal{M}, \vartheta)$ is a metric space. For each $G \in \mathcal{M}$, we have that there exists at least an element $x^* \in X$ such that $x \in \bigcap_{x\in X} G(x)$. We shall call such a point x^* a K^2M point of G and denote by $K^2M(G)$ the set of all K^2M points of G.

Lemma 14.1. *$(\mathcal{M}, \vartheta)$ is a complete metric space.*

Proof. Let $(G_n)_{n\in\mathbb{N}^*}$ be a Cauchy sequence in $\mathcal{M}$. Then, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}^*$ such that $\vartheta(G_n, G_m) < \epsilon$, for any $n, m \geq n_0$. It follows that, for each $x \in X$ the sequence $(G_n(x))_{n\in\mathbb{N}^*}$ is Cauchy in $P_{cp}(X)$. Since X is complete, there exists $G : X \to P_{cp}(X)$ such that $H(G_n(x), G(x)) \to 0$, as $n \to +\infty$, for each $x \in X$. Moreover, we get that $\sup_{x\in X} H(G_n(x), G(x)) \to 0$, as $n \to +\infty$.

Suppose, by absurdum, that G were not a K^2M multifunction. Then, there exist $\{x_1, x_2, \cdots, x_m\} \in X$ and $x' \in co\{x_1, x_2, \cdots, x_m\}$ such that $x' \notin \bigcup_{i=1}^{m} G(x_i)$. Since $G(x_i)$ is compact for each $i \in \{1, 2, \cdots, m\}$ there exists $\epsilon_0 > 0$

such that $x' \notin \bigcup_{i=1}^{m}[V^0(G(x_i), \epsilon_0)]$. Since $\sup_{x \in X} H(G_n(x), G(x)) \to 0$ we can find an $n_1 \in \mathbb{N}^*$ such that for any $n \geq n_1$ we have $\bigcup_{i=1}^{m} G_n(x_i) \subset \bigcup_{i=1}^{m}[V^0(G(x_i), \epsilon_0)]$. Thus $x' \notin \bigcup_{i=1}^{m} G_n(x_i)$, for any $n \geq n_1$, which contradicts the assumption that G_n is a K^2M multifunction, for all $n \in \mathbb{N}^*$. Hence G is a K^2M operator and the proof is complete. □

Theorem 14.2. *Let $F : \mathcal{M} \to P(X)$ be a multivalued operator defined by the relation $F(G) := K^2M(G)$. Then F is an u.s.c. multifunction.*

Proof. For any $G \in \mathcal{M}$, for any sequence $(x_n)_{n \in \mathbb{N}^*}$ in $K^2M(G)$ with $x_n \to x^*$ we have that $x_n \in G(x)$, for each $x \in X$. Since $G(x)$ is compact, then $x^* \in G(x)$, for each $x \in X$ and so $x \in \bigcap_{x \in X} G(x)$, $x^* \in K^2M(G)$. Hence $K^2M(G)$ is closed and because $K^2M(G)$ is a subset of the compact set $G(x)$, $x \in X$ we obtain that $K^2M(G)$ is compact too.

Suppose that F were not u.s.c. at $G \in \mathcal{M}$. Then there exists $\epsilon_0 > 0$ and a sequence $(G_n)_{n \in \mathbb{N}^*}$ in $\mathcal{M}$ with $G_n \to G$, such that for each $n \in \mathbb{N}^*$ there exists $x_n \in K^2M(G_n)$ and $x_n \notin V^0(K^2M(G), \epsilon_0)$. Since $x_n \in K^2M(G_n)$ we get $x_n \in \bigcap_{x \in X} G_n(x)$.
For any $x \in X$, since $G_n(x) \to G(x)$, as $n \to +\infty$ (and all these sets are compact) we have that $\bigcap_{i=1}^{+\infty} G_n(x) \bigcup G(x)$ is compact and taking into account that $x_n \to x^*$ we obtain that $x^* \in G(x)$. Thus $x^* \in \bigcap_{x \in X} G(x)$ and $x^* \in K^2M(G) \subset V^0(K^2M(G), \epsilon_0)$, which contradicts the assumption that $x_n \to x^*$ and $x_n \notin K^2M(G) \subset V^0(K^2M(G), \epsilon_0)$, for each $n \in \mathbb{N}^*$. □

The following definition is important in the sequel.

Definition 14.3. Let $G \in \mathcal{M}$. Then $x \in K^2M(G)$ is said to be essential if for any $\epsilon > 0$ there exists $\delta > 0$ such that for each $G' \in \mathcal{M}$ with $\vartheta(G, G') < \delta$, there exists $x' \in K^2M(G')$ with $d(x, x') < \epsilon$.

Theorem 14.4. *$F : \mathcal{M} \to P_{cp}(X)$ is l.s.c. at $G \in \mathcal{M}$ if and only if G is* *essential.*

Proof. From the lower semicontinuity of F at $G \in \mathcal{M}$ we obtain that for each $\epsilon > 0$ there is $\delta > 0$ such that $K^2M(G) \subset V^0(K^2M(G'), \epsilon)$, for each $G' \in \mathcal{M}$, with $\vartheta(G, G') < \delta$. For each $x \in K^2M(G)$, there exists $x' \in K^2M(G')$ with $d(x, x') < \epsilon$ so x is essential and G is essential.

For the reverse implication, suppose that G is essential. If F were not l.s.c. at G, then there exists $\epsilon_0 > 0$ and a sequence $(G_n)_{n\in\mathbb{N}^*}$ in $\mathcal{M}$ with $G_n \to G$, such that for each $n \in \mathbb{N}^*$ there is $x_n \in K^2M(G_n)$ and $x_n \notin V^0(K^2M(G_n), \epsilon_0)$. Since $K^2M(G)$ is compact, we may assume that $x_n \to x \in K^2M(G)$. Since x is essential, $G_n \to G$ and $x_n \to x$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\epsilon_0}{2}$ and $x \in V^0(K^2M(G_n), \frac{\epsilon_0}{2})$, for all $n \geq N$. Hence $x_n \in V^0(K^2M(G_n), \epsilon_0)$, for all $n \in \mathbb{N}^*$, a contradiction. In conclusion, F must be l.s.c. at G. □

Bibliographical comments. We refer to Dugundji-Granas [61], Yuan [217], Border [28], Y. Q. Chen, Y. J. Cho, J. K. Kim, B. S. Lee [43] and Yu-Xiang [216], for more details and other results on this topic.

Part IV

Other Techniques in Mathematical Economics

Chapter 15

Maximal elements

The following theorems give sufficient conditions for a multivalued operator on a compact set to have a maximal element. They also allow us to extend the classical results of equilibrium theory to cover consumers whose preferences may not be representable by utility functions. The problem faced by a consumer is to choose a consumption pattern given his income and prevailing prices. In a market economy, a consumer must purchase his consumption vector at the market prices. The set of all admissible commodity vectors that he can afford at prices p, given an income M (or M_i) is called the budget set and will be denoted by A (or A_i). The budget set can be represented as:

$$A = \{x \in X | p \cdot x \leq M\}.$$

Of course, the budget set can be also empty. An important feature of the budget set is that it is positively homogeneous of degree zero in prices and income. That is, it remains unchanged if the price vector and income are multiplied by the same positive number. If $X = \mathbb{R}^m_+$ and $p > 0$ then the budget set is compact. If some prices are allowed to be zero, then the budget set is no longer compact.

Let us denote by $U(x)$ the set of all consumption vectors which the consumer strictly prefer to x, i. e.

$$U(x) = \{y \in A | y \text{ is strictly preferred to } x\}.$$

Obviously, $U : A \multimap A$ and it is called the preference multifunction or the multivalued operator of preferences. A vector $x^* \in A$ is an optimal preference for a given consumer if and only if $U(x^*) = \emptyset$. Such elements x^* are also called U-maximal or simply maximal. The set of all maximal vectors in the budget set is called the consumer's demand set.

Remark 15.1. Let us remark that if a binary relation U on a set Y is given as follows: it associates to each $x \in Y$ a set $U(x) \subset Y$, which may be interpreted as the set of those elements in Y that are "better" or "larger" than x, then we obtain in fact a multivalued operator $U : Y \multimap Y$, defined by $U(x) = \{y \in Y | y \text{ is better than } x\}$.

Theorem 15.2. (Sommenschein) *Let $Y \subset \mathbb{R}^m_+$ be compact and convex and let $U : Y \multimap Y$ a multivalued operator such that:*

i) $x \notin co\ U(x)$, for all $x \in Y$

ii) If $y \in U^{-1}(x)$ then there exists some $z \in Y$ (possibly $z = y$) such that $y \in int\ U^{-1}(z)$.

Then the U-maximal set is nonempty and compact.

Proof. We have that

$$\{x \in Y | U(x) = \emptyset\} = \bigcap_{x \in Y} (Y - U^{-1}(x)).$$

By hypothesis (ii) we have that

$$\bigcap_{x \in Y} (Y - U^{-1}(x)) = \bigcap_{z \in Y} (Y - \text{int}\ U^{-1}(z)).$$

This latter intersection is compact. Define a multivalued operator by

$$F(x) = Y - \text{int}\ U^{-1}(x), \text{ for each } x \in Y.$$

Each $F(x)$ is compact. If $y \in \text{co}\ \{x_i | i \in \{1, \cdots, n\}\}$ then $y \in \cup_{i=1}^n F(x^i)$. Indeed, if we suppose that $y \notin \cup_{i=1}^n F(x^i)$ then $y \in U^{-1}(x^i)$, for all i, and so $x^i \in U(y)$, for all i. But then $y \in \text{co}\ \{x_i | i \in \{1, \cdots, n\}\} \subset \text{co}\ U(y)$, which violates (i). It then follows from the K^2M corollary that $\bigcap_{x \in y} F(x) \neq \emptyset$. □

Remark 15.3. Arrow applied Sonnenschein result to the problem of existence of equilibrium in a political model.

Corollary 15.4. (Ky Fan lemma-Alternate statement) *Let* $Y \subset \mathbb{R}^m_+$ *be compact and let* $U : Y \multimap Y$ *a multivalued operator such that:*

i) $x \notin U(x)$*, for all* $x \in Y$

ii) $U(x)$ *is convex, for each* $x \in Y$

iii) $Graf U$ *is open in* $Y \times Y$.

Then the U*-maximal set is nonempty and compact.*

Chapter 16

Walras type price equilibrium

Recall that a price p is a free disposal equilibrium price if $f(p) \leq 0$, where f denotes the singlevalued excess demand operator.

Theorem 16.1. (Hartman-Stampacchia) *Let Y a compact and convex subset of $\mathbb{R}^m_+$ and let $f : Y \to \mathbb{R}^m_+$ be continuous. Then there exists an element $p^* \in Y$ such that*

$$p^* \cdot f(p^*) \geq p \cdot f(p^*), \text{ for all } p \in Y.$$

Furthermore the set of all such p^* is compact.

Proof. Define a binary relation U on Y by: $q \in U(p)$ if and only if $q \cdot f(p) > p \cdot f(p)$. Obviously we got a multivalued operator

$$U(p) := \{q \in Y | q \cdot f(p) > p \cdot f(p)\}, \text{ for each } p \in Y.$$

Since f is continuous U has open graph. Also $U(p)$ is convex and $p \notin U(p)$, for each $p \in Y$. Thus by Ky Fan lemma (alternative statement) there is a $p^* \in Y$ such that $U(p^*) = \emptyset$., i. e. for each $p \in Y$ it is not true that $p \cdot f(p^*) > p^* \cdot f(p^*)$. Thus for all $p \in Y$ we have $p^* \cdot f(p^*) \geq p \cdot f(p^*)$. Conversely, any such p^* is U-maximal, so the U-maximal set is compact by the same lemma. □

Theorem 16.2. *Let Y be a compact convex set in $\mathbb{R}^{m+1}_+$ and let $f : Y \to \mathbb{R}^{m+1}_+$ be continuous and satisfy $p \cdot f(p) \leq 0$, for all p.*

Then the set $\{p \in Y | f(p) \leq 0\}$ of free disposal equilibrium prices is nonempty and compact.

Proof. Compactness is immediate. From Hartman-Stampacchia theorem and Walras' law there is an element $p^* \in Y$ such that

$$p \cdot f(p^*) \leq p^* \cdot f(p^*) \leq 0, \text{ for all } p \in Y.$$

Thus $f(p^*) \leq 0$. $\square$

Chapter 17

The excess demand multifunction

If we denote by E the excess demand multifunction, then p is an equilibrium price if $0 \in E(p)$ and it is called a free disposal equilibrium price if there exists an element $z \in E(p)$ such that $z \leq 0$.

An auxiliary result is:

Lemma 17.1. Let $C \subset \mathbb{R}^m$ be a closed convex and let $K \subset \mathbb{R}^m$ be compact convex.

Then $K \cap C^* \neq \emptyset$ if and only if for each $p \in C$ there exists $z \in K$ such that $p \cdot z \leq 0$.

The following theorem is fundamental with respect to the existence of a market equilibrium of an economy and generalizes a similar result for a singlevalued excess demand operator.

Theorem 17.2. (Gale-Debreu-Nikaido) *Let $E : \Delta \to P_{cp,cv}(\mathbb{R}^m_+)$ be an u. s. c. multivalued operator such that for each $p \in \Delta$ we have $p \cdot z \leq 0$, for all $z \in E(p)$. Put $N = -\mathbb{R}^{n+1}_+$.*

Then the set $\{p \in \Delta | N \cap E(p) \neq \emptyset\}$ of free disposal equilibrium prices is nonempty and compact.

Proof. For each $p \in \Delta$ set

$$U(p) = \{q | q \cdot z > 0, \text{ for all } z \in E(p)\}.$$

Then $U(p)$ is convex for each p and $p \notin U(p)$. Also $U(p)$ is open for each p. Indeed, if $q \in U^{-1}(p)$, we have $p \cdot z > 0$ for all $z \in E(q)$. Then, since E is upper semicontinuous $E^+(\{x | p \cdot x > 0\})$ is a neighborhood of q in $U^{-1}(p)$.

Now p is U-maximal if and only if

$$\text{for each } q \in \Delta \text{ there is a } z \in E(p) \text{ such that } q \cdot z \leq 0.$$

Using an auxiliary result (see lemma below), it follows that p is U-maximal if and only if $E(p) \cap N \neq \emptyset$. Thus by Sonnenschein theorem the set $\{p | E(p) \cap N \neq \emptyset\}$ is nonempty and compact. □

Bibliographical comments. For other results and more connections with multivalued analysis theory see the nice book of Border [28].

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