Chapter 1

The Arrow-Debreu model

1.1 Exchange economies

In the pure theory of international trade, we consider several countries exchanging goods on international markets at fixed terms of trade. This model is the genesis of the exchange economies that we discuss in this and the next two sections. Here, we shall prove the existence of prices - terms of trade - which clear all markets. Such prices are called equilibrium prices.

The symbol $\mathcal{P}R$ will denote the set of all preferences on $\mathbb{R}^m_+$. We start our discussion with a general definition of exchange economies with a finite dimensional commodity space.

Definition 1.1.1 *An* **exchange economy** $\mathcal{E}$ *is a function from a non-empty set A (called the set of agents or consumers) into $\mathbb{R}^m_+ \times \mathcal{P}$, i.e.,*

$$\mathcal{E} : A \to \mathbb{R}^m_+ \times \mathcal{P}.$$

If $\mathcal{E} : A \to \mathbb{R}^m_+ \times \mathcal{P}$ is an economy, then the value $\mathcal{E}_i = (\omega_i, \succeq_i)$

represents the characteristics of agent i; the element ω_i is called his initial endowment and $\succeq_i$ his preference or taste. If $\mathbf{p}$ is any price vector, then the non-negative real number $\mathbf{p} \cdot \omega_i$ is called *the income* of agent i at prices $\mathbf{p}$ and is denoted by ω_i, i.e., $\omega_i = \mathbf{p} \cdot \omega_i$. When A is a finite set, the vector $\omega = \sum_{i \in A} \omega_i$ is called the **total** (or the *aggregate* or the *social*) *endowment* of the economy.

In this section, we shall study an important class of exchange economies - the neoclassical exchange economies. Their definition is as follows.

Definition 1.1.2 *A* **neoclassical exchange economy** *is exchange economy $\mathcal{E} : A \to \mathbb{R}^m_+ \times \mathcal{P}$ such that:*

1) The set A of agents is finite;

2) Each agent i has a non-zero initial endowment ω_i (i.e., $\omega_i > 0$) and his preference relation $\succeq_i$ is neoclassical; and

3) The total endowment $\omega = \sum_{i \in A} \omega_i$ is strictly positive, i.e. $\omega \gg 0$ holds.

For the rest of our discussion in this section $\mathcal{E}$ will always indicate a neoclassical exchange economy. In this case, each agent i has a neoclassical preference $\succeq_i$, and hence, by the discussion in Section 1.3, each agent i has a demand function $\mathbf{x}_i : \mathrm{Int}(\mathbb{R}^m_+) \to \mathbb{R}^m_+$. The aggregate demand minus the total endowment is known as the excess demand function.

Definition 1.1.3 *If $\mathcal{E}$ is a neoclassical exchange economy, then* **the excess demand function** *for the economy $\mathcal{E}$ is the function $\zeta : \mathrm{Int}(\mathbb{R}^m_+) \to \mathbb{R}^m$ defined by*

$$\zeta(\mathbf{p}) = \sum_{i \in A} \mathbf{x}_i(\mathbf{p}) - \sum_{i \in A} \omega_i = \sum_{i \in A} \mathbf{x}_i(\mathbf{p}) - \omega.$$

In component form the excess demand function will be denoted as

$$\zeta(\cdot) = (\zeta_1(\cdot), \zeta_2(\cdot), \ldots, \zeta_m(\cdot)).$$

The basic properties of the excess demand function are described in the next theorem.

Theorem 1.1.4 *The excess demand function ζ of a neoclassical exchange economy satisfies the following properties.*

1) ζ is homogeneous of degree zero, i.e., $\zeta(\lambda \mathbf{p}) = \zeta(\mathbf{p})$ holds for all $\mathbf{p} \gg 0$ and all $\lambda > 0$.

2) ζ is continuous and bounded from below.

3) ζ satisfies Walras' Law, i.e., $\mathbf{p} \cdot \zeta(\mathbf{p}) = 0$ holds for all $\mathbf{p} \gg 0$.

4) If a sequence $\{\mathbf{p}_n\}$ of strictly prices satisfies

$$\mathbf{p}_n = (p_1^n, p_2^n, \ldots, p_m^n) \to \mathbf{p} = (p_1, p_2, \ldots, p_m)$$

and $p_k > 0$ holds for some k, then the sequence $\{\zeta_k(\mathbf{p}_n)\}$ of the k^{th} components of $\{\zeta(\mathbf{p}_n)\}$ is bounded.

5) If $\mathbf{p}_n \gg 0$ holds for each n and $\mathbf{p}_n \to \mathbf{p} \in \partial \mathbb{R}^m_+ \setminus \{0\}$, then there exists at least one k such that $\limsup_{n\to\infty} \zeta_k(\mathbf{p}_n) = \infty$.

Proof. (1) The desired conclusion follows from the fact that $\mathbf{x}_i(\lambda \mathbf{p}) = \mathbf{x}_i(\mathbf{p})$ holds for all $\mathbf{p} \gg 0$, all $\lambda > 0$ and all $i \in A$.

(2) The continuity of the excess demand function follows immediately from Theorem 1.1.8. Since $\mathbf{x}_i(\mathbf{p}) \geq 0$ holds for each i, we see that $\zeta(\mathbf{p}) \geq -\omega$ holds for each $\mathbf{p} \in \mathrm{Int}(\mathbb{R}^m_+)$ and so ζ is bounded from below.

(3) If $\mathbf{p} \gg 0$, then we have

$$\mathbf{p} \cdot \zeta(\mathbf{p}) = \mathbf{p} \cdot \sum_{i\in A} [\mathbf{x}_i(\mathbf{p}) - \omega_i] = \sum_{i\in A} [\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}) - \mathbf{p} \cdot \omega_i] = \sum_{i\in A} 0 = 0.$$

Finally, note that the validity of (4) and (5) can be established easily by invoking Theorem 1.1.9. □

We now define the notion of an equilibrium price vector for a neoclassical exchange economy.

Definition 1.1.5 *A strictly positive price* $\mathbf{p}$ *is said to be an* **equilibrium price** *for a neoclassical exchange economy whenever*

$$\zeta(\mathbf{p}) = 0.$$

Does every neoclassical exchange economy have an equilibrium price ? The celebrated Arrow-Debreu theorem says yes! The rest of the section is devoted to establishing this result.

Since the excess demand function ζ is homogeneous of degree zero (in other words, $\zeta(\lambda\mathbf{p}) = \zeta(\mathbf{p})$ holds for all $\lambda > 0$), we see that a strictly positive price $\mathbf{p}$ is an equilibrium price if and only if $\zeta(\lambda\mathbf{p}) = 0$ holds for all $\lambda > 0$. In other words, if $\mathbf{p}$ is an equilibrium price, then the whole half-ray $\{\lambda\mathbf{p} : \ \lambda > 0\}$ consists of equilibrium prices. This means that the search for equilibrium prices can be confined to sets that contain at least one element from each half-ray. The two most commonly employed normalization of prices are the two sets

$$\Delta = \{\mathbf{p} \in \mathbb{R}^m_+ : \ p_1 + p_2 + \ldots + p_m = 1\}$$

and

$$S_{m-1} = \{\mathbf{p} \in \mathbb{R}^m_+ : \ (p_1)^2 + (p_2)^2 + \ldots + (p_m)^2 = 1\}.$$

Their geometric meaning is shown in Figure ??; notice that each half-ray determined by a positive vector $\mathbf{p}$ intersects both sets. In this chapter, we shall work exclusively with the "simplex" Δ.

Clearly, Δ is a convex and compact subset of $\mathbb{R}^m_+$. The set of all strictly positive prices of Δ will be denoted by $\mathcal{S}$ and is the set

$$\mathcal{S} = \{\mathbf{p} \in \Delta : \ p_i > 0 \text{ for } i = 1, 2, \ldots, m\}.$$

Now we can consider the excess demand function ζ as a function from $\mathcal{S}$ into $\mathbb{R}^m$. According to Theorem 1.1.4, the function $\zeta : \mathcal{S} \to \mathbb{R}^m$ has the following characteristic properties.

Theorem 1.1.6 *If $\zeta(\cdot) = (\zeta_1(\cdot), \zeta_2(\cdot), \ldots, \zeta_m(\cdot))$ is the excess demand function for a neoclassical exchange economy, then*

1. ζ is continuous and bounded from below on $\mathcal{S}$;

2. ζ satisfies Walras' Law, i.e., $\mathbf{p} \cdot \zeta(\mathbf{p}) = 0$ holds for each $\mathbf{p} \in \mathcal{S}$;

3. $\{\mathbf{p}_n\} \subseteq \mathcal{S}$, $\mathbf{p}_n \to \mathbf{p} = (p_1, \ldots, p_m)$ and $p_k > 0$ imply that the sequence $\{\zeta_k(\mathbf{p}_n)\}$ of the k^{th} components of $\{\zeta(\mathbf{p}_n)\}$ is bounded; and

4. $\mathbf{p}_n \to \mathbf{p} \in \partial\mathcal{S}$ with $\{\mathbf{p}_n\} \subseteq \mathcal{S}$ imply $\lim_{n\to\infty} ||\zeta(\mathbf{p}_n)||_1 = \infty$.

To establish that every neoclassical exchange economy has an equilibrium price, we shall invoke a fixed point theorem due to S. Kakutani. For convenience, we recall a few things about correspondences. A *correspondence* (or a *multivalued function*) betweeb two sets X and Y is any function $\phi : X \to 2^Y$, i.e., the value $\phi(x)$ is a subset of Y for each x. As usual, 2^Y denotes the set of all subsets of Y. The *graph* of a correspondence $\phi : X \to 2^Y$ is the subset of $X \times Y$ defined by

$$G_\phi = \{(x, y) \in X \times Y : \ x \in X \text{ and } y \in \phi(x)\}.$$

If X and Y are topological spaces, then a correspondence $\phi : X \to 2^Y$ is said to have a *closed graph* whenever its graph G_ϕ is a closed subset of $X \times Y$. A point $x \in X$ is said to be a *fixed point* for a correspondence

$\phi : X \to 2^X$ whenever $x \in \phi(x)$ holds. The fixed point theorem of S. Kakutani can be stated now as follows.

Theorem 1.1.7 (Kakutani) *Let C be a non-empty, compact and convex subset of some $\mathbb{R}^m$. If $\phi : C \to 2^C$ is a non-empty and convex-valued correspondence with closed graph, then ϕ has a fixed point, i.e., there exists some $x \in C$ with $x \in \phi(x)$.*

We are now ready to establish a general result that will guarantee the existence of equilibrium prices for every neoclassical exchange economy.

Theorem 1.1.8 *For a function $\zeta(\cdot) = (\zeta_1(\cdot), \zeta_2(\cdot), \ldots, \zeta_m(\cdot))$ from $\mathcal{S}$ into $\mathbb{R}^m$ assume that:*

1) ζ is a continuous and bounded from below;

2) ζ satisfies Walras' Law, i.e., $\mathbf{p} \cdot \zeta(\mathbf{p}) = 0$ holds for each $\mathbf{p} \in \mathcal{S}$;

3) $\{\mathbf{p}_n\} \subseteq \mathcal{S}$, $\mathbf{p}_n \to \mathbf{p} = (p_1, \ldots, p_m)$ and $p_i > 0$ imply that the sequence $\{\zeta_i(\mathbf{p}_n)\}$ of the i^{th} components of $\{\zeta(\mathbf{p}_n)\}$ is bounded; and

4) $\mathbf{p}_n \to \mathbf{p} \in \partial\mathcal{S}$ with $\{\mathbf{p}_n\} \subseteq \mathcal{S}$ imply $\lim_{n\to\infty} ||\zeta(\mathbf{p}_n)||_1 = \infty$.

Then, there exists at least one vector $\mathbf{p} \in \mathcal{S}$ satisfying $\zeta(\mathbf{p}) = 0$.

Proof. Let $\zeta : \mathcal{S} \to \mathbb{R}^m$ be a function satisfying the four properties of the theorem. As usual, ζ will be written in component form as $\zeta(\cdot) = (\zeta_1(\cdot), \zeta_2(\cdot), \ldots, \zeta_m(\cdot))$.

For each $\mathbf{p} \in \mathcal{S}$, we define a subset $\Lambda(\mathbf{p})$ of $\{1, 2, \ldots, m\}$ by

$$\Lambda(\mathbf{p}) = \{k \in \{1, 2, \ldots, m\} : \ \zeta_k(\mathbf{p}) = \max\{(\zeta_i(\mathbf{p}) : \ i = 1, 2, \ldots, m)\}.$$

That is, when $\mathbf{p} \in \mathcal{S}$, the set $\Lambda(\mathbf{p})$ consists of all those commodities which have the greatest excess demand. Clearly, $\Lambda(\mathbf{p}) \neq \emptyset$. For $\mathbf{p} \in$

$\Delta \setminus \mathcal{S} = \partial\mathcal{S}$, let

$$\Lambda(\mathbf{p}) = \{k \in \{1, 2, \ldots, m\} : \ p_k = 0\}.$$

Clearly, $\Lambda(\mathbf{p}) \neq \emptyset$ holds in this case too.

Now we define a correspondence $\phi : \Delta \to 2^\Delta$ by the formula

$$\phi(\mathbf{p}) = \{\mathbf{q} \in \Delta : \ q_k = 0 \text{ for all } k \notin \Lambda(\mathbf{p})\}.$$

Since $\Lambda(\mathbf{p}) \neq \emptyset$, it easily follows that $\phi(\mathbf{p}) \neq \emptyset$ for all $\mathbf{p} \in \Delta$. Moreover, note that $\phi(\mathbf{p})$ is a convex and compact subset of Δ - in fact, $\phi(\mathbf{p})$ is a *face* of Δ. In addition, note that if $\Lambda(\mathbf{p}) = \{1, 2, \ldots, m\}$, then $\phi(\mathbf{p}) = \Delta$.

Thus, we have defined a correspondence $\phi : \Delta \to 2^\Delta$ which is nonempty, compact, and convex-valued. We claim that ϕ has also a closed graph.

To establish that ϕ has a closed graph, assume that $\mathbf{p}_n \to \mathbf{p}$ in Δ, $\pi_n \to \pi$ in Δ and $\pi_n \in \phi(\mathbf{p}_n)$ for each n. we have to show that $\pi \in \phi(\mathbf{p})$. We distinguish two cases.

Case I. $\mathbf{p} \in \mathcal{S}$.

In this case, we can assume that $\mathbf{p}_n \gg 0$ holds for each n. Now let $k \notin \Lambda(\mathbf{p})$. This means that $\zeta_k(\mathbf{p}) < \max\{\zeta_i(\mathbf{p}) : \ i = 1, 2, \ldots, m\}$. Since ζ is continuous at $\mathbf{p}$, there exists some N such that

$$\zeta_k(\mathbf{p}_n) < \max\{\zeta_i(\mathbf{p}_n) : \ i = 1, 2, \ldots, m\}$$

holds for all $n \geq N$, and therefore $k \notin \Lambda(\mathbf{p}_n)$ holds for all $n \geq N$. Now from the relation $\pi_n = (\pi_1^n, \pi_2^n, \ldots, \pi_m^n) \in \phi(\mathbf{p}_n)$, we see that $\pi_k^n = 0$ for all $n \geq N$. In view of $\pi_n \to \pi$, we have $\lim_{n\to\infty} \pi_k^n = \pi_k$, and so $\pi_k = 0$. In other words, $\pi_k = 0$ holds for all $k \notin \Lambda(\mathbf{p})$, and so $\pi \in \phi(\mathbf{p})$.

Case II. $\mathbf{p} \in \Delta \setminus \mathcal{S} = \partial\mathcal{S}$.

Without loss of generality, we can suppose that $\mathbf{p} = (0, 0, \ldots, 0, p_{r+1}, \ldots, p_m)$, where $1 \le r < m$ and $p_i > 0$ holds for each $i = r+1, r+2, \ldots, m$. In this case we distinguish two subcases.

Case IIa. There exists a subsequence of $\{\mathbf{p}_n\}$ (which we can assume it to be $\{\mathbf{p}_n\}$ itself) lying in $\mathcal{S}$.

In this case, note that $\Lambda(\mathbf{p}) = \{1, 2, \ldots, r\}$, and so

$$\phi(\mathbf{p}) = \{\mathbf{q} \in \Delta : \ q_i = 0 \text{ for } i = r+1, r+2, \ldots, m\}.$$

Now from our hypothesis, it follows that the sequence $\{\zeta_i(\mathbf{p}_n)\}$ is bounded for each $i = r+1, \ldots, m$ and that $\lim_{n\to\infty} ||\zeta(\mathbf{p}_n)||_1 = \infty$. Therefore, since ζ is bounded from below, there exists some n_0 such that $\Lambda(\mathbf{p}_n) \subseteq \{1, 2, \ldots, r\}$ holds for each $n \ge n_0$. The latter and $\pi_n \in \phi(\mathbf{p}_n)$ imply $\pi_n \in \phi(\mathbf{p})$ for all $n \ge n_0$. Consequently $\pi = \lim_{n\to\infty} \pi_n \in \phi(\mathbf{p})$.

Case IIb. No subsequence of $\{\mathbf{p}_n\}$ lies in $\mathcal{S}$.

In this case, we can assume $\{\mathbf{p}_n\} \subseteq \partial\mathcal{S}$ and $\mathbf{p} = (0, \ldots, 0, p_{r+1}, \ldots, p_m)$. Since $\lim_{n\to\infty} p_i^n = p_i$ holds for each $i = 1, \ldots, m$, we infer that there exists some N such that $\Lambda(\mathbf{p}_n) \subseteq \{1, \ldots, r\}$ holds for all $n \ge N$. From $\pi_n \in \phi(\mathbf{p}_n)$, it follows that $\pi_i^n = 0$ for all $n \ge N$ and all $i = r+1, r+2, \ldots, m$. This (in view of $\pi_n \to \pi$) implies that $\pi_i = 0$ for $i = r+1, \ldots, m$ and so $\pi \in \phi(\mathbf{p})$.

Thus, we have established that the correspondence ϕ has a closed graph. Now, by Kakutani's fixed point theorem (Theorem 1.1.7), ϕ has a fixed point, say $\mathbf{p}$, i.e., $\mathbf{p} \in \phi(\mathbf{p})$. We claim that $\mathbf{p}$ is an equilibrium price.

To see this, note first that $\mathbf{p} \notin \partial\mathcal{S}$. Indeed, if $\mathbf{p} \in \partial\mathcal{S}$, then we have $p_k = 0$ for each $\mathbf{p} = 0 \notin \Delta$, a contradiction. Thus, $\mathbf{p} \in \mathcal{S}$, i.e., $\mathbf{p} \gg 0$.

Next, put $M = \max\{\zeta_i(\mathbf{p}) : \ i = 1, 2, \ldots, m\}$, and note that $p_i > 0$

for all $i = 1, \ldots, m$ and $\mathbf{p} \in \phi(\mathbf{p})$ imply that $\Lambda(\mathbf{p}) = \{1, 2, \ldots, m\}$. This means that $\zeta_i(\mathbf{p}) = M$ holds for each i. On the other hand, using Walras' Law, we see that

$$M = \left(\sum_{i=1}^{m} p_i\right) M = \sum_{i=1}^{m} p_i M = \sum_{i=1}^{m} p_i \zeta_i(\mathbf{p}) = \mathbf{p} \cdot \zeta(\mathbf{p}) = 0,$$

and this implies that $\zeta(\mathbf{p}) = 0$. The proof of the theorem is now complete. □

A special form of Arrow-Debreu theorem can be stated as follows.

Theorem 1.1.9 (Arrow-Debreu) *Every neoclassical exchange economy has an equilibrium price, i.e., there exists at least one price* $\mathbf{p} \gg 0$ *satisfying* $\zeta(\mathbf{p}) = 0$.

Proof. The conclusion follows immediately by observing that (by Theorem 1.1.6) any excess demand function satisfies the hypothesis of Theorem 1.1.8. □

Is should be emphasized that the proof of the preceding result is non constructive. It guarantees the existence of equilibrium prices but it does not provide any method of computing them. A constructive proof of the existence was first given by H. E. Scarf. As a matter of fact, it is very difficult to predict where the equilibrium prices lie on the simplex even in very simple cases. The next example illustrates this point.

Example 1.1.10 *Consider an economy having* $\mathbb{R}^2$ *as commodity space and three agents - i.e.,* $A = \{1, 2, 3\}$ *- with the following characteristics:*

Agent 1: Initial endowment $\omega_1 = (1, 2)$ *and utility function* $u_1(x, y) = xy$.

Agent 2: Initial endowment $\omega_2 = (1,1)$ *and utility function* $u_2(x,y) = x^2y$.

Agent 3: Initial endowment $\omega_3 = (2,3)$ *and utility function* $u_3(x,y) = xy^2$.

Note that the preferences represented by the above utility functions are all neoclassical - and all are only strictly monotone on $\mathrm{Int}(\mathbb{R}^2_+)$. *The total endowment is the vector* $\omega = \omega_1 + \omega_2 + \omega_3 = (4,6)$.

Next, we shall determine the demand functions $\mathbf{x}_1(\cdot)$, $\mathbf{x}_2(\cdot)$ *and* $\mathbf{x}_3(\cdot)$. *To this end, let* $\mathbf{p} = (p_1, p_2) \gg 0$ *be fixed.*

The first agent maximizes the utility function $u_1(x,y) = xy$ *subject to the budget constraint* $p_1x + p_2y = p_2 + 2p_2$. *Employing Lagrange multipliers, we see that at the maximizing point we must have* $\nabla u = (y,x) = \lambda\mathbf{p}$. *This leads us to the system of equations*

$$y = \lambda p_1, \quad x = \lambda p_2 \quad \textit{and} \quad p_1x + p_2y = p_1 + 2p_2.$$

Solving the above system, we obtain

$$\mathbf{x}_1(\mathbf{p}) = \left(\frac{p_1 + 2p_2}{2p_1}, \frac{p_1 + 2p_2}{2p_2}\right).$$

The second agent maximizes the utility function $u_2(x,y) = x^2y$ *subject to* $p_1x + p_2y = p_1 + p_2$. *Using Lagrange multipliers again, we obtain the system*

$$2xy = \lambda p_1, \quad x^2 = \lambda p_2 \quad \textit{and} \quad p_1x + p_2y = p_1 + p_2.$$

Solving the system, we obtain

$$\mathbf{x}_2(\mathbf{p}) = \left(\frac{2p_1 + 2p_2}{3p_1}, \frac{p_1 + p_2}{3p_2}\right).$$

Finally, for the third agent we have the system

$$y^2 = \lambda p_1, \quad 2xy = \lambda p_2 \quad \textit{and} \quad p_1 x + p_2 = 2p_1 + 3p_2,$$

from which we get

$$\mathbf{x}_3(\mathbf{p}) = \left(\frac{2p_1 + 3p_2}{3p_1}, \frac{4p_1 + 6p_2}{3p_2} \right).$$

Therefore,

$$\mathbf{x}_1(\mathbf{p}) + \mathbf{x}_2(\mathbf{p}) + \mathbf{x}_3(\mathbf{p}) = \left(\frac{11p_1 + 16p_2}{6p_1}, \frac{13p_1 + 20p_2}{6p_2} \right)$$

and so

$$\begin{aligned} \zeta(\mathbf{p}) &= \left(\frac{11p_1 + 16p_2}{6p_1}, \frac{13p_1 + 20p_2}{6p_2} \right) - (4, 6) \\ &= \left(-\frac{13p_1 - 16p_2}{6p_1}, \dots \frac{13p_1 - 16p_2}{6p_2} \right). \end{aligned}$$

Clearly, $\zeta(\mathbf{p}) = 0$ *holds if and only if* $13p_1 - 16p_2 = 0$*. Taking into account that* $p_1 + p_2 = 1$*, we infer that an equilibrium price is*

$$\mathbf{p}_{eq} = \left(\frac{16}{29}, \frac{13}{29} \right) \approx (0.45, 0.55).$$

The equilibrium half-ray is "close" to the bisector line $p_2 = p_1$*.*