Cowles Foundation Paper 107

Reprinted from the Proceedings of the National Academy of Sciences,
Vol. 42, No. 11, pp. 876–878. November, 1956.

*MARKET EQUILIBRIUM**

By Gerard Debreu

Cowles Foundation at Yale University

Communicated by Marston Morse, August 31, 1956

VI

Let there be l commodities in the economy. When the price system is $p \in R^l$, the excess of demand over supply is $z \in R^l$. Generally, p does not uniquely determine z; it determines a set $\zeta(p)$ of which z can be any element. The problem of market equilibrium has the natural formulation: Is there a p compatible with $z = 0$, i.e., is there a p such that $0 \in \zeta(p)$?

In usual contexts, two price systems derived from each other by multiplication by a positive number are equivalent, and all prices do not vanish simultaneously. Thus the domain of p is a cone C with vertex 0, but with 0 excluded.

Since no agent spends more than he receives, the value of total demand does not exceed the value of total supply; hence $p \cdot z \leqq 0$ for every z in $\zeta(p)$. This can also be written $p \cdot \zeta(p) \leqq 0$, i.e., the set $\zeta(p)$ is below (with possibly points in) the hyperplane through 0 orthogonal to p (see Fig. 1).

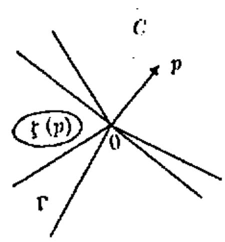

Fig. 1

ERRATUM

Φ, which is a misprint, denotes the empty set.

It is intuitive that, under proper regularity assumptions, there is in C a p, different from 0, for which $\zeta(p)$ intersects Γ, the polar of C (whose definition is recalled in the Appendix). The theorem gives a precise statement of this result. Its interest lies in the fact that, for a wide class of economies, $\Gamma \cap \zeta(p) \neq \Phi$ implies $0 \in \zeta(p)$.[1]

It is convenient to normalize p by restricting it to the unit sphere $S = \{p \in R^l \mid |p| = 1\}$.

THEOREM. *Let C be a closed, convex cone with vertex 0 in R^l, which is not a linear manifold; let Γ be its polar. If the multivalued function ζ from $C \cap S$ to R^l is upper semicontinuous and bounded, and if for every p in $C \cap S$ the set $\zeta(p)$ is nonempty, convex, and satisfies $p \cdot \zeta(p) \leqq 0$, then there is a p in $C \cap S$ such that $\Gamma \cap \zeta(p) \neq \Phi$.*

Proof: Throughout, Z denotes a compact, convex subset of R^l in which ζ takes its values; such a subset exists, since ζ is bounded.

1. The theorem is first proved in the case where Γ has an interior point z^0. It is more convenient here to normalize p by restricting it to the set $P = \{p \in C \mid p \cdot z^0 = -1\}$, which is easily seen to be nonempty, compact, and convex.

Given z in Z, let $\pi(z)$ be the set of maximizers of $p \cdot z$ in P. The set $\pi(z)$ is clearly nonempty and convex, and the multivalued function π from Z to P is easily seen to be upper semicontinuous.

Consider, then, the set $P \times Z$ and the multivalued transformation φ of this set into itself defined by $\varphi(p, z) = \pi(z) \times \zeta(p)$. Since φ satisfies the conditions of the Kakutani[2] fixed-point theorem, there is a pair (p^*, z^*) which belongs to $\varphi(p^*, z^*)$, i.e., $p^* \in \pi(z^*)$ and $z^* \in \zeta(p^*)$.

The first relation implies that $p \cdot z^* \leqq p^* \cdot z^*$ for every p in P; the second implies that $p^* \cdot z^* \leqq 0$; therefore, $p \cdot z^* \leqq 0$ for every p in P; hence $z^* \in \Gamma$. This, with $z^* \in \zeta(p^*)$, proves that $\Gamma \cap \zeta(p^*) \neq \Phi$.

2. In the general case, Γ is considered as the limit of an infinite sequence of cones Γ^m with vertex 0, having *nonempty* interiors. These cones are constrained to be closed, convex, different from R^l, and to contain Γ.

Let C^m be the polar of Γ^m; it is contained in C, which is the polar of Γ. Apply, then, the result of paragraph 1 to the pair C^m, Γ^m: there is a pair (p^m, z^m) such that $p^m \in C^m \cap S$, $z^m \in \Gamma^m$, and $z^m \in \zeta(p^m)$.

Since $S \times Z$ is compact, one can extract from the sequence (p^m, z^m) a subsequence converging to (p^*, z^*). Clearly, $p^* \in C \cap S$, $z^* \in \Gamma$, and $z^* \in \zeta(p^*)$ (the last relation by upper semicontinuity of ζ).

Remarks: The central idea of the proof is taken from Arrow-Debreu.[3,4] It consists, given an excess z of demand over supply, in choosing p so as to maximize $p \cdot z$. It has a simple economic interpretation: in order to reduce the excess demand, the weight of the price system is brought to bear on those commodities for which the excess demand is the largest.

Since the convexity assumptions in Kakutani's theorem can be weakened (see Eilenberg-Montgomery[5] and Begle[6]), the assumption that $\zeta(p)$ is convex is inessential.

Gale[7] and Debreu[1] have, independently, stated (and Kuhn[8] has proved in a third way) the theorem in the particular case where C is the set of points in R^l all of whose co-ordinates are non-negative. The underlying economic assumption is that commodities can be freely disposed of. As McKenzie[9] emphasizes, it is very desirable to relax that assumption. The purpose of this note was to give a general market equilibrium theorem with a simple and economically meaningful proof.

Appendix.—Let C be a cone with vertex 0 in R^l; its polar Γ is the set $\{z \in R^l | p \cdot z \leq 0$ for every p in $C\}$. This set is a closed, convex cone with vertex 0. It can also be described as the intersection of the closed half-spaces below the hyperplanes through 0 with normals p in C.

It is immediate that "C^1 contains C^2" implies "Γ^2 contains Γ^1". One can prove that if C is closed, and convex, then C is the polar of Γ, i.e., the relation "is the polar of" becomes symmetric.

Let ψ be a multivalued function from a subset E of R^n to R^n; it is said to be upper semicontinuous if "$x^q \to x^0$, $y^q \in \psi(x^q)$, $y^q \to y^0$" implies "$y^0 \in \psi(x^0)$".

* A technical report of research undertaken by the Cowles Foundation for Research in Economics under contract with the Office of Naval Research. Based on Cowles Foundation Discussion Paper No. 10 (February, 1956). Reproduction in whole or in part is permitted for any purpose of the United States Government.

I thank Shizuo Kakutani for the valuable discussion of this question I had with him, and Lionel McKenzie for his comments on an earlier draft.

[1] G. Debreu, *Value Theory* (forthcoming), chap. 5.

[2] S. Kakutani, "A Generalization of Brouwer's Fixed Point Theorem," *Duke Math. J.*, **8**, 457–459, 1941.

[3] K. J. Arrow and G. Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, **22**, 265–290, 1954, secs. 3.1, 3.2.

[4] G. Debreu, "The Coefficient of Resource Utilization," *Econometrica*, **19**, 273–292, 1951, secs. 11, 12.

[5] S. Eilenberg and D. Montgomery, "Fixed Point Theorems for Multi-valued Transformations," *Am. J. Math.*, **68**, 214–222, 1946.

[6] E. G. Begle, "A Fixed Point Theorem," *Ann. Math.*, **51**, 544–550, 1950.

[7] D. Gale, "The Law of Supply and Demand," *Math. Scand.*, **3**, 155–169, 1955, secs. 2, 3.

[8] H. W. Kuhn, "A Note on 'The Law of Supply and Demand,'" *Math Scand.*, Vol. **4**, 1956.

[9] L. W. McKenzie, "Competitive Equilibrium with Dependent Consumer Preferences," *Second Symposium in Linear Programming* (U.S. National Bureau of Standards, 1955), **1**, 277–294.