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DECAY SOLUTIONS TO RETARDED FRACTIONAL EVOLUTION INCLUSIONS WITH SUPERLINEAR PERTURBATIONS

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Abstract. In this paper, we consider a class of abstract fractional differential inclusion with finite delay in which the multi-valued nonlinearity is possibly superlinear. We analyze some sufficient conditions that ensure the global solvability of problem. Our main result is the existence of a compact set of decay solutions to our problem by estimating the measure of noncompactness and using the fixed point theory for a condensing map. The obtained results will be applied to a concrete polytope fractional differential system.

Key Words and Phrases: Decay solutions, differential inclusion, fixed point, measure of noncompactness; MNC-estimate.
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1. Introduction

We are concerned with the following problem in a Banach space X

$$^{C}D_0^{\alpha}u(t) - Au(t) \in F(t, u_t),\ t \geq 0 \tag{1.1}$$

$$u(s) = \varphi(s),\ s \in [-h, 0], \tag{1.2}$$

where $\alpha \in (0,1)$, and $^{C}D_0^{\alpha}$ stands for the Caputo derivative of order α defined by

$$^{C}D_0^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds.$$

In this model, A is a closed linear operator in X which generates a strongly continuous semigroup $W(\cdot)$, $F : \mathbb{R}^+ \times C([-h,0];X) \to \mathcal{P}(X)$ is a multivalued map. Here u_t stands for the history of the state function, i.e. $u_t(s) = u(t+s)$ for $s \in [-h, 0]$.

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Subsequently, there has been a great deal of research on this field. Without stressing to wide list of references, we quote here some monographs about fractional differential equations in Euclidean spaces and Banach spaces [5, 15, 24].

Differential inclusions (DIs) as appearing in (1.1) arise, for instance, from control theory in which control factor is taken in the form of feedbacks. In such control problems, the presence of delay terms is an inherent feature. Recently, the theory of differential variational inequalities (DVIs) has been an increasingly interesting subject since DVIs come from various realistic problems (see [19]). In dealing with DVIs, an effective method is converting them to DIs. These brief mentions tell us that the study of DIs is able to range over many applications.

Problem $(1.1)-(1.2)$ in case $\alpha = 1$ (with/without retarded terms) has been studied extensively. For a complete reference to DIs in infinite dimensional spaces, we refer the reader to monograph [10]. In addition, there are many contributions for semilinear DIs published in the last few years (see e.g. [1, 6, 7, 9, 11, 16, 18, 20]). Concerning fractional DIs in infinite dimensional spaces, one can find a number of works devoted to the questions of solvability, stability and controllability. References [2, 13, 17, 22, 23, 25] are the notable investigations that are close to the problem under consideration.

An important question raised for problem $(1.1) - (1.2)$ is to study the existence of decay global solutions. Up to now, to prove the existence of decay solution for a semilinear problem, we have to assume that, the nonlinear is sublinear. The main motivation of the present paper is to prove the existence of a compact set of solutions to our problem under the assumption that F possibly superlinear. More precisely, we will show that problem $(1.1) - (1.2)$ has a compact set of decay solution if the semigroup generated by A is compact and exponentially bounded; multivalued nonlinearity F have compact and convex valued is a Caratheodory function and satisfies

$$\|F(t,v)\| = \sup\{\|\xi\| : \xi \in F(t,v)\} \le p(t)G(\|v\|), \forall t > 0, v \in C([-h,0];X),$$

where $p \in L^q_{loc}(\mathbb{R}^+), (q > \frac{1}{\alpha})$ is nonnegative function and $G \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{G(r)}{r}.\sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds < \frac{1}{M}, \tag{1.3}$$

and

$$\lim_{T\to\infty}\sup_{t\ge T}\int_0^{t/2} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds = 0, \tag{1.4}$$

in which, $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, i.e.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}.$$

It easy to get that if G is superlinear, e.g. $G(r) = r^q$ for some $q > 1$, then condition (1.3) is testified obviously. On the other hand, if $p \in L^\infty(\mathbb{R}^+)$ then condition (1.4) is satisfied. Therefore, conditions $(1.3) - (1.4)$ of F assure that the problem contains the case where F can be superlinear.

This paper is organized as follows. In the next section, we give the definition of a solution to problem (1.1)-(1.2) and prove some existence results on the interval $(-h;T]$, for arbitrary $T > 0$ under some different assumptions of the nonlinearity F. Section 3 is devoted to proving the existence of a compact set of decay global solutions. In the last section, we apply the abstract results to a class of polytope differential inclusions.

2. Solvability and dissipativity

Consider the linear problem

$$D_0^\alpha u(t) = Au(t) + f(t), t > 0, \tag{2.1}$$

$$u(0) = u_0. \tag{2.2}$$

where $f \in L^p(0,T;X)$. Let $\{S_\alpha(t), P_\alpha(t)\}_{t\geq 0}$ be the family of operators such that

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} = \int_0^\infty e^{-\lambda t} S_\alpha(t)dt, \tag{2.3}$$

$$(\lambda^{\alpha-1} I - A)^{-1} = \int_0^\infty e^{-\lambda t} t^{\alpha-1} P_\alpha(t)dt. \tag{2.4}$$

By the same arguments as in [12] and [25], we have the following representation of solution for the linear problem (2.1) – (2.2)

$$u(t) = S_\alpha(t)u(0) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds, t > 0. \tag{2.5}$$

Let $\{W(t)\}$ be the C_0-semigroup generated by A. Then we have the formulas for $S_\alpha(t)$ and $P_\alpha(t)$ as follows (see [25])

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta) W(t^\alpha \theta)x d\theta, \tag{2.6}$$

$$P_\alpha(t)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta) W(t^\alpha \theta)x d\theta, x \in X, \tag{2.7}$$

$$\phi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!} \Gamma(n\alpha)\sin(n\pi\alpha).$$

We also give some important properties of the Mittag-Leffler function in the following proposition, as these important properties are useful for the later estimations in this paper.

Proposition 2.1. [15] *Let $g_1(\beta,t) = E_{\alpha,1}(-\beta t^\alpha)$ and $g_2(\beta,t) = t^{\alpha-1}E_{\alpha,\alpha}(-\beta t^\alpha)$, $t > 0$. Then for every $\beta > 0$, we have*

1. *$g_1(\beta,\cdot)$ and $g_2(\beta,\cdot)$ are nonnegative;*
2. *$g_1(\beta,\cdot), g_2(\beta,\cdot) \in L^1_{loc}(\mathbb{R}^+)$;*
3. *$g_1(\beta,\cdot)$ is nonincreasing, $\lim_{t\to\infty} g_1(\beta,t) = 0$ and $g_1(\beta,t) \leq 1, \forall t \geq 0$;*
4. *the relationship of g_1 and g_2 is*

$$\int_0^t g_2(\beta, t-s)ds = \int_0^t g_2(\beta,s)ds = \frac{1-g_1(\beta,t)}{\beta} \leq \frac{1}{\beta}, \forall t \geq 0. \tag{2.8}$$

For a given $\varphi \in C([-h,0],X)$, denote $C_\varphi = \{u \in C([0,T];X) : u(0) = \varphi(0)\}$. For $u \in C_\varphi$, let $u[\varphi] \in C([-h,T];X)$ be defined as follows

$$u[\varphi](t) = \begin{cases} \varphi(t), t \in [-h,0], \\ u(t), t \in (0,T]. \end{cases}$$

Hence, we have

$$u[\varphi]_s(t) = \begin{cases} \varphi(t+s), t+s \in [-h,0], \\ u(t+s), t+s \in [0,T]. \end{cases}$$

For $u[\varphi] \in C([-h,T];X)$, putting

$$\mathcal{P}_F^p(u[\varphi]) = \{f \in L^p(0,T;X) : f(t) \in F(t, u[\varphi]_t), \text{ for a.e. } t \in [0,T]\}.$$

Based on the representation of solutions for the linear problem, we give the following definition.

Definition 2.2. Let $\varphi \in C([-h,0];X)$ be given. A function $u \in C([-h,T];X)$ is said to be an integral solution of problem (1.1)-(1.2) on the interval $[-h,T]$ if and only if $u(t) = \varphi(t)$ for $t \in [-h,0]$, and

$$u(t) = S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds,$$

for any $t \in [0,T]$, where $f \in \mathcal{P}_F^p(u[\varphi])$.

In what follows, we use the notation $\|\cdot\|_\infty$ for the sup norm in the spaces $C([-h,0];X), C([-h,T];X), C([0,T];X)$.

Let $\mathcal{E}$ be a Banach space. We recall the following notion

$$\begin{aligned} \mathcal{P}(\mathcal{E}) &= \{B \subset \mathcal{E} : B \neq \emptyset\}, \\ \mathcal{P}_b(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is bounded}\}, \\ \mathcal{K}(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is compact}\}, \\ \mathcal{K}v(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is convex and compact}\}. \end{aligned}$$

We defined the *solution operator* $\mathcal{F} : C([0,T];X) \to \mathcal{P}(C([0,T];X))$ as follows

$$\mathcal{F}(u)(t) = S_\alpha(t)\varphi(0) + \Big\{ \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds : f \in \mathcal{P}_F^p(u[\varphi]) \Big\}.$$

It is obvious that u is a fixed point of $\mathcal{F}$ iff u is an integral solution of (1.1) – (1.2) on $[-h,T]$.

Concerning problem (1.1) – (1.2), we give the following assumptions:

(**A**) *The C_0-semigroup $\{W(t)\}_{t\geq 0}$ generated by A is compact and exponentially bounded, i.e. there is $M > 1, \beta > 0$ such that*

$$\|W(t)x\| \leq Me^{-\beta t}\|x\|, \forall t \geq 0, \forall x \in X.$$

(**F**) *The multivalued nonlinearity function $F : \mathbb{R}^+ \times C([-h,0];X) \to \mathcal{K}v(X)$ satisfies that $F(\cdot, v)$ admits a strongly measurable selection for each $v \in C([-h,0];X)$ and $F(t,\cdot)$ is u.s.c for each $t \in [0,T]$.*

Under the assumptions (**A**) and (**F**), we will prove the compactness of the solution operator $\mathcal{F}$. Because of the compactness of $S_\alpha(t)$, the remain is prove the compactness of the Cauchy operator $Q : C([0,T];X) \to C([0,T];X)$ defined as follows

$$Q(f)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds. \tag{2.9}$$

Firstly, we give the definition of the measure of noncompactness.

Definition 2.3. [10] Let $\mathcal{E}$ be a Banach space. A function $\mu : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^+$ is said to be a measure of noncompactness (MNC) if $\mu(\overline{\text{co}}\, D) = \mu(D)$ for all $D \in \mathcal{P}_b(\mathcal{E})$, here the notation $\overline{\text{co}}$ denote the closure of convex hull of subsets in $\mathcal{E}$. An MNC is called

- nonsingular if $\mu(D \cup \{x\}) = \mu(D)$ for all $D \in \mathcal{P}_b(\mathcal{E})$, $x \in \mathcal{E}$.
- monotone if $\mu(D_1) \le \mu(D_2)$ provided that $D_1 \subset D_2$.
- regular if $\mu(D) = 0$ is equivalent to the relative compactness of D.

The MNC defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ admits a finite } \varepsilon - \text{net}\}$$

is called the Hausdorff measure of noncompactness.

Before proving the compactness property of operator Q, we need following result.

Proposition 2.4. [13] *Let $M \subset C([0,T];X)$ be such that*

(1) $\|f(t)\| \le \nu(t)$ *for a.e.* $t \in [0,T]$ *and for all* $f \in M$*;*
(2) $\chi(M(t)) \le \mu(t)$ *for a.e.* $t \in [0,T]$*,*

where $\nu, \mu \in L^1(0,T)$ are nonnegative functions. Then we have

$$\chi\left(\int_0^t M(s)ds\right) \le 4\int_0^t \chi(M(s))ds, t \in [0,T],$$

here

$$\int_0^t M(s)ds = \left\{\int_0^t f(s)ds : f \in M\right\}.$$

Lemma 2.5. *The Cauchy operator defined by* (2.9) *is compact.*

Proof. Let $D \subset C([0,T];X)$ be a bounded set. We will prove $Q(D)$ is compact. We first testify that $Q(D)(t)$ is compact in X for each $t > 0$. Since Proposition 2.4 and the compactness of P_α, we have

$$\chi(Q(D)(t)) = \chi\left(\int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) D(s) ds\right) = 0.$$

Now we prove that $Q(D)$ is equicontinuous. Let $f \in D, t \in (0,T)$ and $\delta \in (0, T-t]$, then

$$\begin{aligned}
\|Q(f)(t+\delta) - Q(f)(t)\| \le & \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \|P_\alpha(t+\delta-s) f(s)\| ds \\
& + \left\| \int_0^t (t+\delta-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s) ds - \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds \right\| \\
& = I_1(t) + I_2(t).
\end{aligned}$$

We have

$$\begin{aligned}I_1(t) &\leq M\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t+\delta-s)^\alpha)\|f(s)\|ds\\ &\leq M\|f\|\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t+\delta-s)^\alpha)ds\\ &= M\|f\|\int_0^{\delta}\tau^{\alpha-1}E_{\alpha,\alpha}(-\beta\tau^\alpha)d\tau = M\|f\|\frac{1-E_{\alpha,1}(-\beta\delta^\alpha)}{\beta}\to 0 \text{ as } \delta\to 0,\end{aligned}$$

thanks to (2.8).
We also have

$$\begin{aligned}I_2(t) &= \Big\|\int_0^t\tau^{\alpha-1}P_\alpha(\tau)f(t+\delta-\tau)d\tau - \int_0^t\tau^{\alpha-1}P_\alpha(\tau)f(t-\tau)d\tau\Big\|\\ &= \Big\|\int_0^t\tau^{\alpha-1}P_\alpha(\tau)[f(t+\delta-\tau)-f(t-\tau)]d\tau\Big\|\\ &\leq \int_0^t\tau^{\alpha-1}\|P_\alpha(\tau)[f(t+\delta-\tau)-f(t-\tau)]\|d\tau\\ &\leq M\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\beta\tau^\alpha)\|f(t+\delta-\tau)-f(t-\tau)\|d\tau.\end{aligned}$$

Because $t>0$, we imply $1-E_{\alpha,1}(-\beta t^\alpha)>0$. Then since $f\in D\subset C([0,T];X)$, for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\|f(t+\delta-\tau)-f(t-\tau)\| < \frac{\beta\varepsilon}{M[1-E_{\alpha,1}(-\beta t^\alpha)]}.$$

Thus, we imply

$$\begin{aligned}I_2(t) &< M\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\beta\tau^\alpha)\frac{\beta\varepsilon}{M[1-E_{\alpha,1}(-\beta t^\alpha)]}d\tau\\ &< \frac{\beta\varepsilon}{1-E_{\alpha,1}(-\beta t^\alpha)}\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\beta\tau^\alpha)d\tau = \varepsilon.\end{aligned}$$

Hence, $I_2(t)\to 0$ as $\delta\to 0$.

We also have

$$\begin{aligned}\|Q(f)(\delta)-Q(f)(0)\| &\leq \int_0^\delta(\delta-s)^{\alpha-1}\|P_\alpha(\delta-s)f(s)\|ds\\ &\leq M\int_0^\delta(\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(\delta-s)^\alpha)\|f(s)\|ds\\ &\leq M\|f\|\int_0^\delta(\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(\delta-s)^\alpha)ds\\ &= \frac{M\|f\|}{\beta}[1-E_{\alpha,1}(-\beta\delta^\alpha)]\to 0 \text{ as } \delta\to 0,\end{aligned}$$

uniformly in $f\in D$. Hence, $Q(D)$ is equicontinuous. From the Arzela-Ascoli theorem, we imply the compactness of Cauchy operator.

In the next theorems, we prove some global existence results for problem $(1.1)-(1.2)$.

Theorem 2.6. *Assume that* $(\mathbf{A})$ *and* $(\mathbf{F})$ *hold. Then there exists* $\delta > 0$ *such that the problem* $(1.1)-(1.2)$ *has at least one integral solution on* $[-h, T]$ *provided* $\|\varphi\|_\infty < \delta$ *and*

(F1) $\|F(t,v)\| = \sup\{\|\xi\| : \xi \in F(t,v)\} \le p(t)G(\|v\|_\infty)$, *for all* $v \in C([-h,0];X)$, *where* $p \in L^q_{loc}(\mathbb{R}^+)$, $q > \frac{1}{\alpha}$; *and* G *is a continuous and nonnegative function such that*

$$\limsup_{r\to 0} \frac{G(r)}{r}. \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds < \frac{1}{M}. \tag{2.10}$$

Proof. Let

$$\ell = \limsup_{r\to 0} \frac{G(r)}{r}, \quad I(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds, \quad m = \sup_{[0,T]} I(t).$$

Then by assumption, one can take $\varepsilon > 0$ such that

$$M(\ell+\varepsilon)m < 1.$$

In addition, there exist $\eta > 0$ such that

$$\frac{G(r)}{r} \le \ell + \varepsilon, \forall r \in (0, 2\eta].$$

Let

$$\delta_0 = \frac{\eta}{M} \inf_{t\in[0,T]} \frac{1 - M(\ell+\varepsilon)I(t)}{E_{\alpha,1}(-\beta t^\alpha) + (\ell+\varepsilon)I(t)}.$$

We have

$$\inf_{t\in[0,T]} [1 - M(\ell+\varepsilon)I(t)] = 1 - M(\ell+\varepsilon) \sup_{t\in[0,T]} I(t) = 1 - M(\ell+\varepsilon)m > 0$$

and

$$\sup_{t\in[0,T]} [ME_{\alpha,1}(-\beta t^\alpha) + M(\ell+\varepsilon)I(t)] \le M + 1$$

(since property of Mittag-Leffler function: $E_{\alpha,1}(-\beta t^\alpha) \le 1,\ \forall t \ge 0$), so we imply $\delta_0 > 0$. Put $\delta = \min\{\delta_0, \eta\}$. If $\varphi \in C([-h,0];X)$ such that $\|\varphi\|_\infty \le \delta$, then

$$\|u[\varphi]_s(\tau)\| \le \|u\|_\infty + \|\varphi\|_\infty \le \eta + \delta \le 2\eta,\ \forall \tau \in [-h, 0].$$

Denote by B_η the closed ball in $C_\varphi([0,T];X)$ centered at origin and with radius η. Considering $\mathcal{F} : \mathsf{B}_\eta \to \mathcal{P}(C_\varphi([0,T];X))$, with each $z(t) \in \mathcal{F}(u)(t)$, we can find $f \in \mathcal{P}^p_F(u)$ such that:

$$z(t) = S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s)f(s)ds.$$

We then have

$$
\begin{aligned}
\|z(t)\| &\leq \|S_\alpha(t)\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)f(s)\|ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}ME_{\alpha,\alpha}(-\beta(t-s)^\alpha)\|f(s)\|ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)\|F(s,u_s)\|ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)G(\|u_s\|_\infty)ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)G(\sup_{\tau\in[-h,0]}\|u[\varphi]_s(\tau)\|)ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)(\ell+\epsilon)\sup_{\tau\in[-h,0]}\|u[\varphi]_s(\tau)\|ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)(\ell+\epsilon)(\eta+\delta)ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\delta \\
&\quad + M(\ell+\varepsilon)(\eta+\delta)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\delta + M(\ell+\varepsilon)(\eta+\delta)I(t).
\end{aligned}
$$

Hence

$$
ME_{\alpha,1}(-\beta t^\alpha)\delta + M(\ell+\varepsilon)(\eta+\delta)I(t) \leq \eta \Leftrightarrow \delta \leq \frac{\eta}{M}.\frac{1-M(\ell+\varepsilon)I(t)}{E_{\alpha,1}(-\beta t^\alpha)+(\ell+\varepsilon)I(t)}.
$$

It is easy to check last inequality because

$$
\delta \leq \delta_0 = \frac{\eta}{M}\inf_{t\in[0,T]}\frac{1-M(\ell+\varepsilon)I(t)}{E_{\alpha,1}(-\beta t^\alpha)+(\ell+\varepsilon)I(t)}.
$$

Thus, we get $\|\mathcal{F}(u)(t)\| \leq \eta$ for all $t \in [0,T]$, and $\mathcal{F}(\mathsf{B}_\eta) \subset \mathsf{B}_\eta$, provided $\|\varphi\|_\infty \leq \delta$.

Consider $\mathcal{F} : \mathsf{B}_\eta \to \mathsf{B}_\eta$. Since the compactness of $\mathcal{F}$, the proof is complete by applying the fixed point theorem for compact multi-valued map.

Theorem 2.6 deals with the case when F is possibly superlinear. In the next theorem, we can relax the smallness condition on initial data, provide that F has a sublinear growth.

Theorem 2.7. *Assume that* (**A**) *and* (**F**) *hold. Moreover,* $F : \mathbb{R}^+ \times C([-h,0];X) \to \mathcal{K}v(X)$ *satisfies*

(**F2**) $\|F(t,v)\| \le p(t)(1+\|v\|_\infty)$, *for all* $t \in [0,T]$ *and* $v \in C([-h,0];X)$, *where* $p \in L^1(0,T)$ *is a nonnegative function and* p *is nondecreasing.*

Then the problem $(1.1)-(1.2)$ *has at least one integral solution on* $[-h,T]$.

Proof. Let $\psi \in C([0,T];\mathbb{R})$ be the unique solution of the integral equation

$$\psi(t) = M\|\varphi\|_\infty + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)(1+\|\varphi\|_\infty+\psi(s))ds$$

and

$$D = \{u \in C_\varphi([0,T];X) : \sup_{\tau\in[0,t]}\|u(\tau)\| \le \psi(t), \forall t \in [0,T]\}.$$

Then D is a closed and convex subset of $C_\varphi([0,T];X)$. Since $\mathcal{F}$ is compact, it suffices to show that $\mathcal{F}(D) \subset D$.

Let $u \in D$, with each $z(t) \in \mathcal{F}(u)(t)$, we can find $f \in \mathcal{P}_F^p(u[\varphi])$ such that:

$$z(t) = S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds.$$

We have

$$\|z(t)\| \le \|S_\alpha(t)\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)f(s)\|ds$$

and from Proposition 2.1, we get

$$\begin{aligned}
\|z(t)\| &\le M\|\varphi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s)\|ds \\
&\le M\|\varphi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)(1+\|u_s\|_\infty)ds \\
&\le M\|\varphi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)(1+\|\varphi\|_\infty+\sup_{[0,s]}\|u(\tau)\|)ds \\
&\le M\|\varphi\|_\infty + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)(1+\|\varphi\|_\infty+\psi(s))ds.
\end{aligned}$$

Since the last integral is nondecreasing in t, we get

$$\begin{aligned}
\sup_{\tau\in[0,t]}\|z(\tau)\| &\le M\|\varphi\|_\infty + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)(1+\|\varphi\|_\infty+\psi(s))ds \\
&= \psi(t),
\end{aligned}$$

which ensures that $\mathcal{F}(u) \in D$. The proof is complete.

The rest of this section is devoted to proving the existence of a bounded absorbing set. To this end, we introduce a Halanay-type inequality in the following lemma.

Lemma 2.8. *Let* v *be a continuous and nonnegative function satisfying*

$$v(t) = \psi(t), \forall t \in [-h,0],\ \ \psi \in C([-h,0];\mathbb{R}^+)$$

and

$$\begin{aligned} v(t) \leq & M E_{\alpha,1}(-\beta t^\alpha) v_0 \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)[a + bv(s) + c \sup_{[-h,s]} v(\tau)] ds, \ t \geq 0, \end{aligned}$$

for $M, \beta, a, c > 0, b \geq 0$ *such that* $b + c < \beta$. *Then*

$$v(t) \leq \frac{\beta - b}{\beta - b - c}\Big[M v_0 + a \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(\beta - b)(t-s)^\alpha) ds\Big] + \sup_{[-h,0]} \psi(s)$$

and

$$\limsup_{t\to\infty} v(t) = \frac{M(\beta - b)v_0 + a}{\beta - b - c} + \sup_{[-h,0]} \psi(s).$$

Proof. This lemma is a result of Proposition 3 [14] with

$$s(t, \mu) = E_{\alpha,1}(-\mu t^\alpha), \ r(t, \mu) = t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^\alpha).$$

We are now in a position to show the dissipativity of our problem.

Theorem 2.9. *Let the hypotheses of Theorem 2.7 hold for all* $T > 0$ *and* $p \in L^\infty(0, T)$, *moreover* $\|p\|_\infty = \text{esssup}_{t>0} p(t) < \gamma < \beta/M$. *Then there exists a bounded absorbing set for the solutions of* (1.1) − (1.2) *with arbitrary initial data.*

Proof. Let u be a solution of (1.1) − (1.2). Then

$$\begin{aligned} \|u(t)\| &\leq \|S_\alpha(t)\varphi(0)\| + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s) f(s)\| ds \\ &\leq M E_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1} M E_{\alpha,\alpha}(-\beta(t-s)^\alpha)\|f(s)\| ds \\ &\leq M E_{\alpha,1}(-\beta t^\alpha)\|u(0)\| \\ &\quad + M \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha) p(s)(1 + \|u_s\|_\infty) ds \\ &\leq M E_{\alpha,1}(-\beta t^\alpha)\|u(0)\| \\ &\quad + M \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha) p(s)(1 + \sup_{[-h,s]} \|u(\tau)\|) ds \\ &\leq M E_{\alpha,1}(-\beta t^\alpha)\|u(0)\| \\ &\quad + M \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha) \gamma(1 + \sup_{[-h,s]} \|u(\tau)\|) ds \\ &\leq M E_{\alpha,1}(-\beta t^\alpha)\|u(0)\| \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)(M\gamma + M\gamma \sup_{[-h,s]} \|u(\tau)\|) ds. \end{aligned}$$

Applying Lemma 2.8 with $a = M\gamma, b = 0, c = M\gamma$, we have:

$$\limsup_{t\to\infty} \|u(t)\| \leq \frac{M\beta||u(0)|| + M\gamma}{\beta - M\gamma} + \sup_{[-h,0]} \|\varphi(s)\|.$$

This implies that the ball $B(0,R) \subset X$ with

$$R = \frac{M\beta||\varphi(0)|| + M\gamma}{\beta - M\gamma} + \sup_{[-h,0]} \|\varphi(s)\| + 1$$

is an absorbing set for the solutions of (1.1) – (1.2) with arbitrary initial data.

3. Existence of decay integral solutions

Our goal in this section is to prove the existence of a compact set of decay solution to the problem (1.1) – (1.2) under the assumption that the multivalued nonlinearity is possibly superlinear. In this section, we assume that the condition (**A**) and (**F**) are satisfied. Moreover,

(**F3**) *$F : \mathbb{R}^+ \times C([-h,0];X) \to \mathcal{K}v(X)$ is a continuous mapping such that*

$$\|F(t,v)\| = \sup\{\|\xi\| : \xi \in F(t,v)\} \le p(t)G(\|v\|_\infty),\ \forall t > 0, v \in C([-h,0];X),$$

where $p \in L^q_{loc}(\mathbb{R}^+)$ is nonnegative function and $G \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{G(r)}{r}.\sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds < \frac{1}{M}, \tag{3.1}$$

and

$$\lim_{T\to\infty}\sup_{t\ge T}\int_0^{t/2} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds = 0. \tag{3.2}$$

In order to study the stability of solutions to problem (1.1) – (1.2), we consider the function space

$$BC_0 = \{u \in C([-h,+\infty);X) : \lim_{t\to\infty} u(t) = 0\}$$

with the norm

$$\|u\|_\infty = \sup_{t\ge -h}\|u(t)\|.$$

Then BC_0 is a Banach space.

Given $\varphi \in C([-h,0];X)$, let

$$BC_0^\varphi = \{u \in BC_0 : u(\cdot,0) = \varphi(\cdot,0)\}.$$

Then BC_0^φ with the supremum norm $\|\cdot\|$ is a closed subspace of BC_0.

To study the existence of decay solutions to (1.1) – (1.2), we make use of the fixed point theory for condensing maps.

Definition 3.1. [10] Let E be a Banach space and $D \subseteq E$. A multimap $\Phi : D \to \mathcal{K}(E)$ is said to be condensing relative to a MNC μ (or μ-condensing) if for every $\Omega \subseteq D$ that is not relatively compact we have, respectively

$$\mu(\Phi(\Omega)) \le \mu(\Omega).$$

We recall a fixed point principle for condensing multi-valued maps, which is the main tool for our purpose.

Theorem 3.2. [10] *Let $\mathcal{M}$ be a bounded convex closed subset of E and $\Phi : \mathcal{M} \to \mathcal{K}v(\mathcal{M})$ be a closed and μ-condensing. Then Fix $\Phi = \{x \in \mathcal{M} : x \in \Phi(x)\}$ is nonempty.*

Let $\pi_T, T > 0$ be the truncated function on BC_0, i.e., for $D \subset BC_0, \pi_T(D)$ is the restriction of D on the interval $[-h, T]$. Then one can see that the MNC χ_∞ and d_∞ in BC_0, defined by

$$\chi_\infty(D) = \sup_{T>0} \chi_T(\pi_T(D));$$
$$d_\infty(D) = \lim_{T\to\infty} \sup_{t\geq T} \sup_{x\in D} \|x(t)\|,$$

satisfies all properties given in Definition 2.3, except regularity. The following MNC defined in [3],

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D),$$

possesses all properties stated in Definition 2.3. In addition, if $\chi^*(D) = 0$ then D is relatively compact in BC_0.

Lemma 3.3. *Let* (**A**), (**F**) *and* (**F3**) *hold and* $\|\varphi\|_\infty < \delta > 0$.

Then $d_\infty(\mathcal{F}(D)) < d_\infty(D)$ *for all bounded set* $D \in BC_0^\varphi$.

Proof. Let

$$\ell = \lim_{r\to 0}\sup \frac{G(r)}{r}, \quad m = \sup_{t\geq 0} I(t), \quad I(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds.$$

Then by assumption, one can take $\varepsilon > 0$ such that

$$M(\ell + \varepsilon)m < 1. \tag{3.3}$$

In addition, there exist $\eta > 0$ such that

$$\frac{G(r)}{r} \leq \ell + \varepsilon, \forall r \in (0, 2\eta].$$

We show that $d_\infty(\mathcal{F}(D)) \leq M(\ell + \epsilon)md_\infty(D)$ for all bounded set $D \subset BC_0^\varphi$.
Let $v \in \mathcal{F}(D)$ and $u \in D$ be such that $v \in \mathcal{F}(u)$. We have

$$\begin{aligned}
\|v(t)\| &\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty \\
&\quad + M\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)G(\|u_s\|_\infty)ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty \\
&\quad + M(\ell+\epsilon)\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)\|u_s\|_\infty ds \\
&\leq ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty \\
&\quad + M(\ell+\epsilon)\int_0^{\frac{t}{2}} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s) \sup_{\tau\in[-h,0]} \|u[\varphi]_s(\tau)\|ds \\
&\quad + M(\ell+\epsilon)\int_{t/2}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s) \sup_{\tau\in[-h,0]} \|u[\varphi]_s(\tau)\|ds.
\end{aligned}$$

We get $\sup_{\tau\in[-h,0]} \|u[\varphi]_s(\tau)\| \le 2\eta$ by the similar arguments as in the proof of Theorem 2.6. So we imply

$$\|v(t)\| \le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty + 2M(\ell+\epsilon)\eta\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds$$
$$+ M\Big[\sup_{s\ge t/2}\sup_{\tau\in[-h,0]} \|u[\varphi]_s(\tau)\|\Big](\ell+\epsilon)\int_{t/2}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds.$$

Noting that, for a given $T>0$, we choose $T_1 = 2(T+h)$ and for $t \ge T_1$ we have

$$\|v(t)\| \le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty + 2M(\ell+\epsilon)\eta\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds$$
$$+ M\Big[\sup_{s\ge T}\|u[\varphi](s)\|\Big](\ell+\epsilon)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds$$
$$\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty + 2M(\ell+\epsilon)\eta\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds$$
$$+ M\Big[\sup_{u\in D}\sup_{s\ge T}\|u[\varphi](s)\|\Big](\ell+\epsilon)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds.$$

Combining with $E_{\alpha,1}(-\beta t^\alpha)$ is nonincreasing when $t \ge 0$, we get

$$\sup_{t\ge T_1}\|v(t)\| \le ME_{\alpha,1}(-\beta T_1^\alpha)\|\varphi\|_\infty + M\Big[\sup_{u\in D}\sup_{s\ge T}\|u[\varphi](s)\|\Big](\ell+\epsilon)\sup_{t\ge T_1} I(t)$$
$$+ 2M(\ell+\epsilon)\eta\sup_{t\ge T_1}\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds.$$

Since $v \in \mathcal{F}(D)$ is taken arbitrarily, we get

$$\sup_{v\in\mathcal{F}(D)}\sup_{t\ge T_1}\|v(t)\| \le ME_{\alpha,1}(-\beta T_1^\alpha)\|\varphi\|_\infty + M\Big[\sup_{u\in D}\sup_{s\ge T}\|u[\varphi](s)\|\Big](\ell+\epsilon)m$$
$$+ 2M(\ell+\epsilon)\eta\sup_{t\ge T_1}\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds.$$

Let $T\to\infty$ then $T_1\to\infty$ and we obtain

$$d_\infty(\mathcal{F}(D)) \le M(\ell+\epsilon)md_\infty(D) < d_\infty(D),$$

thanks to (3.2), (3.3) and the fact that $E_{\alpha,1}(-\beta T_1^\alpha)\to 0$ as $T_1\to\infty$.

The proof is complete.

Theorem 3.4. *If hypothesis of Lemma 3.3 hold, then the problem* (1.1) – (1.2) *has a compact set of decay solutions.*

Proof. We show that there exists $\eta > 0$ such that $\mathcal{F}(\mathsf{B}_\eta) \subset \mathsf{B}_\eta$. Assume to the contrary that, for each $n \in \mathbb{N}$, there is $u^n \in BC_0^\varphi$ such that $\|u^n\| \le n$ but $\|\mathcal{F}(u^n)\| > n$. Then

$$\begin{aligned}
\|\mathcal{F}(u^n)(t)\| &\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)G(\|u_s^n\|_\infty)ds \\
&\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty \\
&\quad + M\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)(\ell+\varepsilon)\|u_s^n\|_\infty ds \\
&\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi\|_\infty \\
&\quad + M(\ell+\varepsilon)(\|\varphi\|_\infty + n)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds \\
&\le M\|\varphi\|_\infty + M(\ell+\varepsilon)(\|\varphi\|_\infty + n)m.
\end{aligned}$$

So we imply

$$1 < \frac{1}{n}\|\mathcal{F}(u^n)\| \le \frac{M}{n}\Big[1+(\ell+\varepsilon)m\Big]\|\varphi\|_\infty + M(\ell+\varepsilon)m.$$

Passing to the limit as $n \to \infty$, we have a contradiction.

Considering $\mathcal{F} : \mathsf{B}_\eta \to \mathsf{B}_\eta$, we show that $\mathcal{F}$ is χ^*-condensing. If $D \subset \mathsf{B}_\eta$ then $\chi_T(D) = 0 \Rightarrow \chi_\infty(D) = 0$. Using the result of Lemma 3.3, we have

$$\begin{aligned}
\chi^*(\mathcal{F}(D)) &= \chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D)) = d_\infty(\mathcal{F}(D)) \le M(\ell+\varepsilon)md_\infty(D) \\
&\le M(\ell+\varepsilon)m[d_\infty(D) + \chi_\infty(D)] = M(\ell+\varepsilon)m\chi^*(D) < \chi^*(D).
\end{aligned}$$

Thus $\mathcal{F}$ is χ^*-condensing and it admits a fixed point, according to Theorem 3.2. Denote by $\mathcal{D}$ the set of fixed points of $\mathcal{F}$ in B_η. Then $\mathcal{D}$ is closed and $\mathcal{D} \subset \mathcal{F}(\mathcal{D})$. Hence,

$$\chi^*(\mathcal{D}) \le \chi^*(\mathcal{F}(\mathcal{D})) \le M(\ell+\varepsilon)m\chi^*(\mathcal{D}),$$

which ensures that $\chi^*(\mathcal{D}) = 0$ and $\mathcal{D}$ is a compact set. The proof is complete.

4. Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following polytope fractional differential system:

$$\partial_t^\alpha u(t,x) = \Delta u(t,x) + f(t,x),\ x \in \Omega, t > 0, \tag{4.1}$$

$$f(t,x) = \eta\tilde{f}_1(t,u(t-h,x)) + (1-\eta)\tilde{f}_2(t,u(t-h,x)), \eta \in [0,1] \tag{4.2}$$

$$u(t,x) = 0,\ x \in \partial\Omega, t > 0, \tag{4.3}$$

$$u(s,x) = \varphi(x,s), x \in \Omega, s \in [-h,0], \tag{4.4}$$

where $\tilde{f}_i : [0,T] \times \mathbb{R} \to \mathbb{R}, i = 1,2$, are continuous functions.

Let

$$X = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

endowed with the norm $\|v\| = \sup_{x\in\overline{\Omega}} |v(x)|$.

Let $A = \Delta$ with $D(A) = \{v \in C_0(\overline{\Omega}) \cap H_0^1(\Omega) : \Delta v \in C_0(\overline{\Omega})\}$, and

$$\mathcal{C}_h = C([-h, 0]; C_0(\overline{\Omega})).$$

Then it is known that A is the generator of a compact semigroup on X (see [4], Theorem 2.3).

Let λ_1 be the first eigenvalue of Δ on $H_0^1(\Omega)$, that is,

$$\lambda_1 = \sup\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in H_0^1(\Omega), u \neq 0\right\}.$$

Following Theorem 4.2.2 of [8], we have

$$\|S(t)\| \leq Me^{-\lambda_1 t},\ M = \exp\left(\frac{\lambda_1 |\Omega|^{2/n}}{4\pi}\right)$$

where $|\Omega|$ is the volume of Ω. Hence (**A**) is satisfied with $\beta = \lambda_1$ and M as above.

Assume the following on functions $\tilde{f}_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, i \in \{1; 2\}$

(i) $\tilde{f}_i(\cdot, z)$ is measurable for each $z \in \mathbb{R}$; $\tilde{f}_i(t, \cdot)$ is continuous for a.e. $t \in [0, T]$;

(ii) $|\tilde{f}_i(t, z)| \leq p(t)|z|^\gamma$, $\forall (t, z) \in [0, T] \times \mathbb{R}$, where $p \in L^\infty(\mathbb{R}^+)$ and $\gamma > 1$.

Let $f_i : [0, T] \times \mathcal{C}_h \to X$ be the functions defined by

$$f_i(t, v)(x) = \tilde{f}_i(t, v(-h, x)), i \in \{1; 2\},$$

and $F(t, v) = \overline{\text{co}}\{f_1(t, v), f_2(t, v)\}$. Then $F : \mathbb{R}^+ \times \mathcal{C}_h \to \mathcal{P}(X)$ is a multimap with closed convex values. It is easy to check that for each (t, v), $F(t, v)$ is a bounded set in the finite dimensional space spanned by $\{f_1, f_2\}$, and so F has compact values. Now, we show that $F(t, \cdot)$ is u.s.c. Let $\{v_k\} \subset \mathcal{C}_h$ converge to v. Then by the continuity of $\tilde{f}_i$, we get $f_i(t, v_k) \to f_i(t, v)$ in X. For $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f_i(t, v_k) \in f_i(t, v) + \epsilon B_X, \forall k \geq n, i \in \{1; 2\}$, where B_X is the unit ball in X centered at origin. This implies $F(t, v_k) \subset F(t, v) + \epsilon B_X, \forall k \geq n$, and since F has compact values, we have upper-semicontinuity of $F(t, \cdot)$. Hence (**F**) is satisfied.

Let $z \in F(t, v)$, we have

$$\begin{aligned}
|z(x)| &\leq \eta|\tilde{f}_1(t, v(-h, x))| + (1 - \eta)|\tilde{f}_2(t, v(-h, x))| \\
&\leq \eta p(t)|v(-h, x)|^\gamma + (1 - \eta)p(t)|v(-h, x)|^\gamma \\
&\leq p(t)|v(-h, x)|^\gamma.
\end{aligned}$$

Therefore, $\|z\| \leq p(t)\|v(-h, \cdot)\|^\gamma \leq p(t)\|v\|^\gamma$. And thus, $\|F(t, v)\| \leq p(t)\|v\|^\gamma$. This mean that $G(\|v\|) = \|v\|^\gamma$ and condition (3.1) is satisfied.

Now we check condition (3.2). Let $p \in L^\infty(\mathbb{R}^+)$ and $\|p\| = \text{esssup}_{t\geq 0}|p(t)|$, then we have

$$
\begin{aligned}
&\sup_{t\geq T}\int_0^{t/2}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)p(s)ds\\
\leq{}& \|p\|\sup_{t\geq T}\int_0^{t/2}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)ds\\
={}& \|p\|\sup_{t\geq T}\int_{t/2}^{t}s^{\alpha-1}E_{\alpha,\alpha}(-\beta s^\alpha)ds\\
\leq{}& \|p\|\int_{T/2}^{\infty}s^{\alpha-1}E_{\alpha,\alpha}(-\beta s^\alpha)ds\to 0 \text{ as } T\to\infty,
\end{aligned}
$$

thanks to the fact that $s^{\alpha-1}E_{\alpha,\alpha}(-\beta s^\alpha)\in L^1(\mathbb{R}^+)$ is followed from (2.8). Now we get that (3.2) is fulfilled and we obtain the existence of a decay solution to problem (4.1)-(4.4).

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