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ON EXISTENCE RESULTS IN FIXED SET THEORY AND APPLICATIONS TO SELF-SIMILARITY

KHALED BEN AMARA*, AREF JERIBI** AND NAJIB KADDACHI***

*Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax,
Soukra Road Km 3.5 B.P. 1171, 3000, Sfax, Tunisia
E-mail: khaledamaram@gmail.com

**Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax,
Soukra Road Km 3.5 B.P. 1171, 3000, Sfax, Tunisia
E-mail: Aref.Jeribi@fss.rnu.tn

***Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax,
Soukra Road Km 3.5 B.P. 1171, 3000, Sfax, Tunisia
E-mail: najibkadachi@gmail.com

Abstract. In this manuscript, by removing the domain convexity hypothesis, the existence of fixed set results for the sum and the product of $(p+1)$-multi-valued operators $\sum_{i=1}^{p} A \cdot B_i$, acting on Banach algebras satisfying a sequential condition $(\mathcal{P})$ under weak topology is proved. In addition, by using a new definition of the multi-valued operator $\left(\frac{I}{A}\right)$, we obtain new fixed-set theorems for the operators of the form $\left(\frac{I}{A}\right)^{-1} \sum_{i=1}^{p} B_i$ under some suitable conditions on the operators $A, B_1, \dots, B_p$. Applications to self-similarity theory are also given.
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1. Introduction

Many phenomena in a variety of disciplines of applied mathematics may be transformed into fixed point or Fixed set problems. For example, one can see [4, 10, 11, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 34, 35] and the references therein. During last years, many works have made a lot of interesting contributions on the equations of the form

$$x = Ax \cdot Bx + Cx, \qquad x \in \Omega, \tag{1.1}$$

where Ω is a closed and convex subset of a Banach algebra X. Dhage's fixed point theorem for the product of tow operators [16] played a very important role in the analysis and the development of such equations. It was mainly based on the convexity of the bounded domain [13], the celebrate Schauder fixed-point theorem [18], the properties of operators A and B (cf. completely continuous [19, 17], k-set contractive [6, 7], condensing) and the potential tool of the axiomatic measures of non-compactness [30, 31]. In [35], OK noticed that the Krasnosel'skii fixed-point theorem can be reformulated by removing or relaxing the convexity condition of Ω and allowing the fixed point to be a fixed set. So a very natural question is what mild condition on A, B, C and Ω can guarantee that $A \cdot B + C$ has a weakly compact fixed set K in a Banach algebra X, i.e.

$$K = A(K) \cdot B(K) + C(K). \tag{1.2}$$

This idea was developed in [8] where sufficient conditions ensuring the existence of a weakly compact set K satisfying (1.2). In this paper, we establish the existence of a weak compact set K such that

$$K = A(K) \cdot B_1(K) + \cdots + A(K) \cdot B_p(K), \tag{1.3}$$

where A and B_i, $i = 1, \ldots, p$, are $(p+1)$-multi-valued operators, acting on Banach algebras satisfying a certain sequential condition $(\mathcal{P})$ under weak topology. Note that our analysis is based on the De Blasi measure of weak non-compactness which allows us to cover earlier results in the literature. Precisely, we prove that if Ω is a nonempty and weakly closed subset of a Banach algebra X satisfying condition $(\mathcal{P})$ and $A : X \longrightarrow \mathcal{P}_{cl,bd,cv}(X)$ and $B_i : \Omega \longrightarrow \mathcal{P}(X)$ are $(p+1)$ sequentially weakly upper semi-compact multi-valued operators satisfying the following properties:
(i) A and each B_i are $\mathcal{D}$-set-Lipschitzians with $\mathcal{D}$-functions ψ and ψ_i respectively,
(ii) $A(X)$ and $B_i(\Omega)$ are bounded,
(iii) $x \in \sum_{i=1}^{p} Ax \cdot B_i y; \quad y \in \Omega \Rightarrow x \in \Omega.$

Then, there exists a weakly compact subset K in Ω such that

$$K = \left(\frac{I}{A}\right)^{-1} \sum_{i=1}^{p} B_i(K) \tag{1.4}$$

whenever

$$\sum_{i=1}^{p} \|A(\Omega)\| \psi_i(r) + \psi(r)\psi_i(r) + pM\psi(r) < r,$$

for $r > 0$, where $M = \sup_{1 \leq i \leq p} \|B_i(\Omega)\|$.

The remainder of this paper is organized as follows. Section 2 is devoted to give some standard notations, definitions and auxiliaries results which will be used in our study. In Section 3 and 4, we will establish new Hybrid-type fixed-set theorems for the sum and the product of $(p+1)$-multi-valued operators on Banach algebras satisfying a sequential condition $(\mathcal{P})$ under weak topology. More precisely, in section 3, we establish the existence of a weak compact subset of X that fulfills (1.3). In section 4, on the basis on a new definition of the multi-valued operator $\left(\frac{I}{A}\right)$, we give sufficient conditions ensure the existence of a weak compact subset of X satisfying (1.4). In the last section, we apply our results to the theory of self-similar sets.

2. Preliminaries

In the present section, we give basics concepts that we need during the next sections. In what follows, X will be a Banach space endowed with the norm $\|\cdot\|$ and $\mathcal{B}(X)$, $\mathcal{W}(X)$, respectively, will denote the collection of all nonempty bounded subsets of X and all nonempty weakly compact subsets of X. We denote by $\mathcal{B}_r$ the closed ball in X centered at 0_X, the zero element of X, with radius r. For a subset Ω of X, the symbols $\overline{\Omega}, \overline{\Omega}^w, co(\Omega)$ and $\overline{co}(\Omega)$ are used to denote respectively, the closure, the weak closure, the convex hull and the closed convex hull of Ω.

For the sake convenience let

$$\begin{aligned}
&\mathcal{P}(X) = \big\{S \subset X,\ S \neq \emptyset\big\},\\
&\mathcal{P}_{bd}(X) = \big\{S \in \mathcal{P}(X),\ S \text{ is bounded}\big\},\\
&\mathcal{P}_{cv}(X) = \big\{S \in \mathcal{P}(X),\ S \text{ is convex}\big\},\\
&\mathcal{P}_{cl}(X) = \big\{S \in \mathcal{P}(X),\ S \text{ is closed}\big\},\\
&\mathcal{P}_{cl,bd}(X) = \mathcal{P}_{cl}(X) \cap \mathcal{P}_{bd}(X), \text{ and}\\
&\mathcal{P}_{cl,bd,cv}(X) = \mathcal{P}_{cl,bd}(X) \cap \mathcal{P}_{cv}(X).
\end{aligned}$$

Let Ω be a nonempty subset of a Banach space X and $F : \Omega \to \mathcal{P}(X)$ be a multi-valued mapping. We denote by $Gr(F)$ the graph of F given by

$$Gr(F) = \{(x, y) \in \Omega \times X : y \in F(x)\}.$$

Let $F : X \longrightarrow \mathcal{P}(X)$ be a multi-valued operator. For any $\Omega \in \mathcal{P}(X)$, define

$$F(\Omega) = \cup_{u\in\Omega} F(u) \quad \text{and} \quad F^{-1}(\Omega) = \{u \in X,\ F(u) \cap \Omega \neq \emptyset\}.$$

If $\Omega \in \mathcal{P}_{bd}(X)$, let us put

$$\|\Omega\| = \sup\{\|u\|,\ u \in \Omega\} \quad \text{and} \quad D(u, \Omega) = \inf\{\|u - v\|,\ v \in \Omega\}.$$

Notice that the mapping $d_H : \mathcal{P}_{cl,bd}(X) \times \mathcal{P}_{cl,bd}(X) \longrightarrow \mathbb{R}_+$ given by the formula

$$d_H(\Omega_1, \Omega_2) = \max\left\{\sup_{u\in\Omega_1} D(u, \Omega_2), \sup_{v\in\Omega_2} D(v, \Omega_1)\right\}$$

defines a metric on $\mathcal{P}_{cl,bd}(X)$ [33].

Measure of weak noncompactness notion were introduced by De Blasi in [15], play a decisive role in the theory of fixed point. It is defined on $\mathcal{B}(X)$ by

$$\beta(S) = \inf\{r > 0,\ \text{there exists } K \in \mathcal{W}(X) \text{ such that } S \subseteq K + B_r\}.$$

Let us recall some basic properties of β needed below [1, 5, 15].

Lemma 2.1. *Let S_1 and S_2 be two elements of $\mathcal{B}(X)$. Then the following conditions are satisfied:*

(i_1) $S_1 \subseteq S_2$ *implies* $\beta(S_1) \leq \beta(S_2)$,

(i_2) $\beta(S_1) = 0$ *if and only if* $\overline{S_1}^w \in \mathcal{W}(X)$,

(i_3) $\beta\left(\overline{S_1}^w\right) = \beta(S_1)$,

(i_4) $\beta(S_1 \cup S_2) = \max\{\beta(S_1), \beta(S_2)\}$,

(i_5) $\beta(\lambda S_1) = |\lambda|\beta(S_1)$, *for all* $\lambda \in \mathbb{R}$,

(i_6) $\beta\left(co\left(S_1\right)\right)=\beta\left(S_1\right)$,
(i_7) $\beta\left(S_1+S_2\right)\leq\beta\left(S_1\right)+\beta\left(S_2\right)$, *and*
(i_8) *If* $(S_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of bounded and weakly closed subsets of* X *with* $\lim\limits_{n\to+\infty}\beta\left(S_n\right)=0$, *then* $M_\infty=\cap_{n=1}^{+\infty}S_n$ *is nonempty and* $\overline{S_\infty}^w$ *is a weakly compact subset of* X.

Definition 2.1. Let $F:X\longrightarrow X$ be a single-valued operator. Then, we say that F is weakly sequentially continuous on X if for every sequence $(u_n)_{n\in\mathbb{N}}$ of X with $u_n\rightharpoonup u$, we have $F(u_n)\rightharpoonup F(u)$.

Definition 2.2. A multi-valued operator $F:X\longrightarrow\mathcal{P}(X)$ is called $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function Ψ if

$$d_H(F(u),F(v))\leq\Psi\left(\|u-v\|\right),\quad\text{for all } u,v\in X,$$

for some continuous and nondecreasing function $\Psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that $\Psi(0)=0$. Specifically, if $\Psi(r)<r,\ r>0$, then F is called a nonlinear $\mathcal{D}$-contraction. In particular, if $\Psi(r)=k\,r$, for some constant $0<k<1$, then F is called a contraction mapping.

Definition 2.3. Let S be a nonempty subset of X, and $F:S\longrightarrow\mathcal{P}(X)$ be a multi-valued operator. Then, we call that F has a weakly sequentially closed graph on S, (w.s.c.gr., for short) if for every sequence $(u_n)_{n\in\mathbb{N}}$ of S with $u_n\rightharpoonup u$ in S, and for every sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n\in F(u_n)$, for all $n\in\mathbb{N}$, and $v_n\rightharpoonup v$ in X, we have $v\in F(u)$.

Definition 2.4. Let S be a nonempty subset of X. We say that a multi-valued operator $F:S\longrightarrow\mathcal{P}(X)$ is weakly completely continuous if F has a w.s.c.gr. and, for any bounded subset M of S, the subset $F(M)$ is relatively weakly compact.

Definition 2.5. Let S be a nonempty subset of X. We say that a multi-valued operator $F:S\longrightarrow\mathcal{P}(X)$ is sequentially weakly upper semicompact in S (s.w.u.sco., for short) if for any sequence $(u_n)_{n\in\mathbb{N}}\subset S$ with $u_n\rightharpoonup u$ in S and for any arbitrary $v_n\in F(u_n)$, the sequence $(v_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence to a point $v\in F(u)$ in S. If F is a single-valued operator, then we say that F is sequentially weakly upper semicompact if for any weakly convergent sequence $(u_n)_{n\in\mathbb{N}}$, the sequence $(F(u_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence.

Remark 2.6. Let us note that every weakly sequentially continuous operator is sequentially weakly upper semicompact. Moreover, if a multi-valued operator F is sequentially weakly upper semicompact, then $F(S)$ is relatively weakly compact for any relatively weakly compact subset S of X.

Definition 2.7. Let $F:X\longrightarrow\mathcal{P}(X)$ be a multi-valued operator. Then, F is called $\mathcal{D}$-set-Lipschitzian (with respect to β) if

$$\beta(F(S))\leq\Psi(\beta(S))\text{ for all bounded sets } S \text{ of } X,\ \text{with } F(S)\in\mathcal{P}_{bd}(X),$$

for some continuous nondecreasing function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\Psi(0) = 0$. The function Ψ is called a $\mathcal{D}$-function of F on X. In the case, when $\Psi(r) = k\,r, k > 0$, the map F is called k-set-Lipschitzian operator, and if $k < 1$, then the multi-valued operator F is called a k-set-contraction operator. Moreover, if $\Psi(r) < r$, then F is called a nonlinear $\mathcal{D}$-set-contraction operator on X.

Remark 2.8. Notice that, if an operator F is sequentially weakly upper semi-compact and $\mathcal{D}$-lipschitzian, then F is $\mathcal{D}$-set-lipschitzian with respect to β, see Lemma 2.4 in [2]. Similarly, if F is a weakly sequentially continuous single-valued operator and $\mathcal{D}$-Lipschitzian, then F is $\mathcal{D}$-set-Lipschitzian with respect to β.

In the paper of A. Ben Amar, S. Chouayekh and A. Jeribi [10], a new class of Banach algebras which includes all Banach algebras satisfying a condition $(\mathcal{P})$ was introduced.
Definition 2.9. [10] A Banach algebra X satisfying the sequential condition $(\mathcal{P})$, if for any sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of X with $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, we have $u_n \cdot v_n \rightharpoonup u \cdot v$.

Remark 2.10. (i) Notice that, every finite dimensional Banach algebra satisfies the sequential condition $(\mathcal{P})$. Moreover, if X is a Banach algebra satisfying $(\mathcal{P})$, then from the Dobrakov's theorem,the Banach algebra $C(K, X)$ is also satisfying the condition $(\mathcal{P})$, where K is a compact Hausdorff space [22].
(ii) In [3], J. Banaś and L. Olszowy proved the equivalence of Banach algebras satisfying $(\mathcal{P})$ and WC-Banach algebras, where the class of WC-Banach algebras is the family of all Banach algebras E such that the product of two arbitrary weakly compact subsets in E is weakly compact.

Definition 2.11. [9] Let X be a Banach algebra and let $F : X \to \mathcal{P}(X)$ be a multi-valued operator. Then, we say that the operator $\frac{I}{F}$ is well defined on $u \in X$ and we write $v \in \left(\frac{I}{F}\right)(u)$ if $u \in v\,F(u)$.

3. Fixed-set results

Throughout the present and the next sections, X will be a Banach algebras satisfying the condition $(\mathcal{P})$. Let $A, B_i : \Omega \longrightarrow \mathcal{P}(X)$, $i = 1, \ldots, p$, are $(p+1)$ multi-valued operators. Under suitable conditions and by removing the convexity hypothesis of the domain Ω, we look for a nonempty weakly compact subset K of Ω such that

$$A(K) \cdot B_1(K) + A(K) \cdot B_2(K) + \cdots + A(K) \cdot B_p(K) = K,$$

or

$$\left(\frac{I}{A}\right)(K) = \sum_{i=1}^{p} B_i(K).$$

Theorem 3.1. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A, B_i : \Omega \longrightarrow \mathcal{P}(X)$, $i = 1, \ldots, p$, are s.w.u.sco. and satisfy:*
(i) A and each B_i are $\mathcal{D}$-set-Lipschitzians with $\mathcal{D}$-functions ψ and ψ_i respectively,
(ii) $A(\Omega)$ and each $B_i(\Omega)$ are bounded, and
(iii) $\sum_{i=1}^{p} A(\Omega) \cdot B_i(\Omega) \subset \Omega$.

Then, there exists a weakly compact subset K in X such that $K = \sum_{i=1}^{p} A(K) \cdot B_i(K)$ whenever

$$\sum_{i=1}^{p} \left[\|A(\Omega)\| + \psi(r)\right] \psi_i(r) + p\,\delta\psi(r) < r,$$

$r > 0$, *where* $\delta = \sup_{1 \leq i \leq p} \|B_i(\Omega)\|$.

Proof. Let s_0 be a fixed element in Ω and let $\mathcal{M}$ be the set of all weakly closed subset M of Ω for which $s_0 \in M$ and $\sum_{i=1}^{p} A(M) \cdot B_i(M) \subset M$. Obviously, $\mathcal{M}$ is non-empty since $\Omega \in \mathcal{M}$. Take

$$M_0 = \bigcap_{M \in \mathcal{M}} M.$$

As M_0 is weakly closed, $s_0 \in M_0$ and $\sum_{i=1}^{p} A(M_0) \cdot B_i(M_0) \subset M_0$, we have $M_0 \in \mathcal{M}$. Let

$$L = \overline{\sum_{i=1}^{p} A(M_0) \cdot B_i(M_0) \cup \{s_0\}}^{w}.$$

Note that L is a weakly closed subset of M_0 and satisfies $s_0 \in L$ and

$$\sum_{i=1}^{p} A(L) \cdot B_i(L) \subset \sum_{i=1}^{p} A(M_0) \cdot B_i(M_0) \subset L.$$

This discussion implies that $L = M_0 \in \mathcal{M}$. Notice that the subset L is weakly compact. To achieves this, we arguing by a contradiction argument and we suppose that $\beta(L) > 0$. From the subadditivity property of β, one has

$$\begin{aligned}
\beta(L) &= \beta\left(\sum_{i=1}^{p} A(L) \cdot B_i(L)\right) \\
&\leq \sum_{i=1}^{p} \beta\left(A(L) \cdot B_i(L)\right) \\
&\leq \sum_{i=1}^{p} \|A(L)\|\beta\left(B_i(L)\right) + \|B_i(L)\|\beta\left(A(L)\right) + \beta\left(A(L)\right)\beta\left(B_i(L)\right) \\
&\leq \sum_{i=1}^{p} \|A(\Omega)\|\psi_i\left(\beta(L)\right) + \|B_i(\Omega)\|\psi\left(\beta(L)\right) + \psi\left(\beta\left(L\right)\right)\psi_i\left(\beta\left(L\right)\right) \\
&< \beta(L).
\end{aligned}$$

This contradiction implies that L is weakly compact. Let $\mathcal{M}_L$ be the set of all weakly compact subsets M of L for which $M \subset L$ and $\sum_{i=1}^{p} A(M) \cdot B_i(M) \subset M$. Obviously, the subset $\mathcal{M}_L$ is non-empty since $L \in \mathcal{M}_L$. Using the weak compactness of L and the fact that any chain in $(\mathcal{M}_L, \supseteq)$ has the finite intersection property, we deduce that the intersection of all members of any chain in $(\mathcal{M}_L, \supseteq)$ is non-empty, and hence any chain in $(\mathcal{M}_L, \supseteq)$ has a lower bound in $\mathcal{M}_L$. Thus, we can apply the Zorn's lemma to deduce that $(\mathcal{M}_L, \supseteq)$ has a minimal element K. From the definition of $\mathcal{M}_L$, K is a weakly compact subset of L and satisfies $\sum_{i=1}^{p} A(K)\cdot B_i(K) \subset K$. Define now the operator $F : \mathcal{M}_L \longrightarrow \mathcal{P}(X)$ by the formula $F(M) = \sum_{i=1}^{p} A(M)\cdot B_i(M)$. Noting that A and B_i are sequentially weakly upper semi-compact, we have from the condition $(\mathcal{P})$ and Remark 2.6, that $\overline{F(K)}^w$ is weakly compact. Using the inclusion $F(K) \subset K$ one has

$$\sum_{i=1}^{p} A\left(\overline{F(K)}^w\right)\cdot B_i\left(\overline{F(K)}^w\right) \subset \sum_{i=1}^{p} A(K)\cdot B_i(K) \subset \overline{F(K)}^w \subset K.$$

Since K is a minimal element in $\mathcal{M}_L$ one easily sees that $\overline{F(K)}^w = K$. Now, we shall show that $F(K)$ is weakly closed. To see this, let $x \in \overline{F(K)}^w$. It follows from the Eberlein-Šmulian's theorem, see Corollary 2.7 in [2], and the weak compactness of $\overline{F(K)}^w$, that there is a sequence $\{x_n,\ n \in \mathbb{N}\}$ in $F(K)$ converging to some element x. So, there exist a sequence $\{\xi_n,\ n \in \mathbb{N}\}$ with $\xi_n \in A(\theta_n)$, $\forall n \in \mathbb{N}$, and a sequence $\{\rho_n^i,\ n \in \mathbb{N}\}$ with $\rho_n^i \in B_i(\sigma_n)$, $\forall n \in \mathbb{N}$, $i = 1, \dots, p$, for some sequences $\{\theta_n,\ n \in \mathbb{N}\}$ and $\{\sigma_n,\ n \in \mathbb{N}\}$ in K such that

$$x_n = \sum_{i=1}^{p} \xi_n \cdot \rho_n^i.$$

According to the weak compactness of K, by applying the well-known result of Eberlein-Šmulian again we obtain $\theta_{n_k} \rightharpoonup \theta$ and $\sigma_{n_k} \rightharpoonup \sigma$, where $(\theta_{n_k})_k$ and $(\sigma_{n_k})_k$ are two subsequences of $\{\theta_n,\ n \in \mathbb{N}\}$ and $\{\sigma_n,\ n \in \mathbb{N}\}$ respectively. Since A and B_i are s.w.u.sco, for all $i = 1, \dots, p$, it is not difficult to verify the existence of two renamed subsequences $(\xi_{n_k})_k$ and $(\rho_{n_k})_k$ such that

$$\xi_{n_k} \rightharpoonup \xi \in A(\theta) \quad \text{and} \quad \rho_{n_k}^i \rightharpoonup \rho^i \in B_i(\sigma),$$

which implies by the fact that X satisfying $(\mathcal{P})$, that

$$x_{n_k} = \sum_{i=1}^{p} \xi_{n_k}\cdot \rho_{n_k}^i \rightharpoonup \sum_{i=1}^{p} \xi\cdot\rho^i \in \sum_{i=1}^{p} A(\theta)\cdot B_i(\sigma) \subset \sum_{i=1}^{p} A(K)\cdot B_i(K).$$

Thus, $x = \sum_{i=1}^{p} \xi\cdot\rho^i \in F(K)$ and the claim is proved. Accordingly, we result that

$$K = \overline{F(K)}^w = F(K).$$

In the particular case, when A and B_i are single-valued operators, for all $1 \le i \le p$, Theorem 3.1 is verified if we suppose that A as well as each B_i is weakly sequentially continuous.

Corollary 3.2. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A, B_i : X \longrightarrow X$, $i = 1, \dots, p$, satisfy:*
(i) A and each B_i are $\mathcal{D}$-Lipschitzians with $\mathcal{D}$-functions ψ and ψ_i, respectively, and
(ii) $A(X)$ and each $B_i(X)$ are bounded.

Then, there is a weakly compact subset K in X such that $K = \sum_{i=1}^{p} A(K) \cdot B_i(K)$ whenever

$$\sum_{i=1}^{p} \|A(X)\|\psi_i(r) + \delta\, \psi(r) + \psi\,(r)\, \psi_i\,(r) < r,$$

where $\delta = \sup\limits_{1 \le i \le p} \|B_i(X)\|$.

Now, we consider the case where the multi-valued operators B_i, $i = 1, \ldots, p$, are weakly completely continuous.

Theorem 3.3. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A : \Omega \longrightarrow \mathcal{P}_{cl,bd}(X)$ and $B_i : \Omega \longrightarrow \mathcal{P}(X)$, $i = 1, \ldots, p$, satisfy:*
(i) A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ψ,
(ii) A is s.w.u.sco and $A(\Omega)$ is bounded,
(iii) each B_i is weakly completely continuous such that $B_i(\Omega)$ is bounded, and
(iv) $\sum_{i=1}^{p} A(\Omega) \cdot B_i(\Omega) \subset \Omega$.
Then, there exists a weakly compact subset K such that

$$K = \sum_{i=1}^{p} A(K) \cdot B_i(K)$$

whenever $p\, \delta\, \psi(r) < r$, for $r > 0$, where $\delta = \sup\limits_{1 \le i \le p} \|B_i(\Omega)\|$.

Proof. Proceeding as in the proof of Theorem 3.1, we can obtain that

$$L = M_0 = \overline{\sum_{i=1}^{p} A(M_0) \cdot B_i(M_0) \cup \{s_0\}}^{w}\ .$$

First observe that L is a weakly compact subset. Otherwise, we have $\beta(L) > 0$. Using the subadditivity property of the measure β together with assumption (iii) one easily sees that

$$\begin{aligned}
\beta(L) &\le \sum_{i=1}^{p} \beta\left(A(L) \cdot B_i(L)\right) \\
&\le \sum_{i=1}^{p} \|A(L)\|\beta\left(\overline{B_i(L)}^{w}\right) + \|B_i(L)\|\beta\left(A(L)\right) \\
&\le \sum_{i=1}^{p} \|B_i(\Omega)\|\psi\left(\beta(L)\right) \\
&\le p\, \delta \psi\left(\beta(L)\right) \\
&< \beta(L),
\end{aligned}$$

which is a contradiction. Now, let $\{x_n,\ n \in \mathbb{N}\}$ be a weakly convergent sequence of Ω such that $x_n \rightharpoonup x \in X$ and let $\{z_n^i,\ n \in \mathbb{N}\}$ be a sequence of X with $z_n^i \in B_i(x_n)$. Since B_i is weakly completely continuous and $\{x_n,\ n \in \mathbb{N}\}$ is a bounded subset of Ω, the sequence $\{z_n^i,\ n \in \mathbb{N}\}$ has a weakly convergent subsequence to some $z \in B_i(x)$

and hence B_i is a s.w.u.sco multi-valued operator. The remained proof follows along the lines of Theorem 3.1.

4. Regular case

In this section, we consider the Definition 2.11 with the goal of weakening the stability hypothesis in the above results (condition (iii) of Theorem 3.1 and condition (iv) of Theorem 3.3). First, we need the following technical lemma.

Lemma 4.1. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A : X \longrightarrow \mathcal{P}_{cl,bd,cv}(X)$ and $B_i : \Omega \longrightarrow \mathcal{P}(X)$, $i = 1, \ldots, p$, satisfy:*
(i) A is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function ψ,
(ii) A is s.w.u.sco and $A(X)$ is bounded, and
(iii) each $B_i(\Omega)$ is bounded.
Then, the multi-valued $\left(\frac{I}{A}\right)^{-1}$ exists on $\sum_{i=1}^{p} B_i(\Omega)$ whenever $p \sup_{1\le i\le p} \|B_i(\Omega)\|\, \psi(r) < r$, for $r > 0$.

Proof. Let y be a fixed element in Ω and $z_i \in B_i(y), 1 \le i \le p$.
Define now $T_{z_i} : X \longrightarrow \mathcal{P}(X)$ by

$$T_{z_i}(x) = \sum_{i=1}^{p} Ax \cdot z_i.$$

It is easily verified that $T_{z_i}(x)$ is convex for all $x \in X$. Now we claim that T_{z_i} has a w.s.c.gr. To show this, let $\{x_n,\ n \in \mathbb{N}\}$ be a weakly convergent sequence of X to a point x and $\{y_n,\ n \in \mathbb{N}\}$ be a sequence of X with $y_n \in T_{z_i}(x_n), \forall n \in \mathbb{N}$, such that $y_n \rightharpoonup y$. Therefore, there is a sequence $\{\alpha_n,\ n \in \mathbb{N}\}$ of X with $\alpha_n \in A(x_n), \forall n \in \mathbb{N}$, such that

$$y_n = \sum_{i=1}^{p} \alpha_n \cdot z_i.$$

Using the first part of assumption (ii), there is a renamed subsequence $\{\alpha_n,\ n \in \mathbb{N}\}$ such that $\alpha_n \rightharpoonup \alpha \in Ax$. By considering the condition $(\mathcal{P})$, one easily sees that

$$y_n = \sum_{i=1}^{p} \alpha_n \cdot z_i \rightharpoonup \sum_{i=1}^{p} \alpha \cdot z_i \in T_{z_i}(x).$$

This achieves our claim. Now, let us consider a bounded subset M of X which satisfies $\beta(M) > 0$. Taking into account that $T_{z_i}(M)$ is bounded and using the subadditivity property of β together with Lemma 3.2 in [11], one has

$$\begin{aligned} \beta\left(T_{z_i}(M)\right) &\le \beta\left(\sum_{i=1}^{p} A(M) \cdot z_i\right) \\ &\le \sum_{i=1}^{p} \beta\left(A(M)\right) \|z_i\|. \end{aligned}$$

By using the assumption (i), we deduce that

$$\begin{aligned}\beta\left(T_{z_i}(M)\right) &\leq \psi\left(\beta(M)\right)\sum_{i=1}^{p}\|B_i(\Omega)\| \\ &\leq p\sup_{1\leq i\leq p}\|B_i(\Omega)\|\psi\left(\beta(M)\right).\end{aligned}$$

Accordingly, the operator T_{z_i} has, at least, one fixed point by using Theorem 2.3 in [8]. Or equivalently, there is a point $x\in X$ such that

$$z=\sum_{i=1}^{p}z_i\in\left(\frac{I}{A}\right)(x)\cap\sum_{i=1}^{p}B_i(y).$$

This implies that $\left(\frac{I}{A}\right)^{-1}$ exists on $\sum_{i=1}^{p}B_i(\Omega)$.

Theorem 4.2. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A: X\longrightarrow\mathcal{P}_{cl,bd,cv}(X)$ and $B_i:\Omega\longrightarrow\mathcal{P}(X)$, $i=1,\dots,p$, are s.w.u.sco and satisfy:*
(i) A and each B_i are $\mathcal{D}$-set-Lipschitzians with $\mathcal{D}$-functions ψ and ψ_i respectively,
(ii) $A(X)$ and each $B_i(\Omega)$ are bounded, and
(iii) $x\in\sum_{i=1}^{n}Ax\cdot B_iy,\quad y\in\Omega\Rightarrow x\in\Omega$.
Then, there exists a weakly compact subset K in Ω such that

$$K=\left(\frac{I}{A}\right)^{-1}\sum_{i=1}^{p}B_i(K)$$

whenever

$$\sum_{i=1}^{p}\|A(\Omega)\|\psi_{B_i}(r)+\psi_A(r)\,\psi_{B_i}(r)+p\,\delta\psi_A(r)<r,$$

for $r>0$, where $\delta=\sup_{1\leq i\leq p}\|B_i(\Omega)\|$.

Proof. Define a multi-valued operator $G:\Omega\longrightarrow\mathcal{P}(\Omega)$ by

$$G(x)=\left(\frac{I}{A}\right)^{-1}\sum_{i=1}^{p}B_i(x). \tag{4.1}$$

From assumption (iii) and Lemma 4.1 it follows that G is well defined on Ω. Let $s_0\in\Omega$ and let $\mathcal{M}$ denote the set of weakly closed subsets M of Ω for which $s_0\in M$ and $G(M)\subset M$. Obviously $\mathcal{M}$ is a non-empty subset since $\Omega\in\mathcal{M}$. Also, it can be shown that $M_0=\bigcap_{M\in\mathcal{M}}M\in\mathcal{M}$. Let $L=\overline{G(M_0)\cup\{s_0\}}^{w}$. It is easy to verify that the subset L is weakly closed and bounded in M_0 and satisfies $s_0\in L$ and $G(L)\subset L$. This demonstrates that $L\in\mathcal{M}$ and so $L=M_0$. Noting that β has the subadditivity property, we have from the inclusion

$$G(L)\subset AG(L)\cdot\sum_{i=1}^{p}B_i(L), \tag{4.2}$$

that

$$\begin{aligned}\beta(L) &= \beta\left(\overline{G(M_0)\cup\{s_0\}}^w\right)\\ &\leq \beta\left(AG(L)\cdot\sum_{i=1}^{p}B_i(L)\right)\\ &\leq \left\|\sum_{i=1}^{p}B_i(L)\right\|\beta\left(AG(L)\right)+\|AG(L)\|\beta\left(\sum_{i=1}^{p}B_i(L)\right)\\ &\quad+\beta\left(AG(L)\right)\beta\left(\sum_{i=1}^{p}B_i(L)\right).\end{aligned}$$

Consequently, we get

$$\begin{aligned}\beta(L) &\leq \left\|\sum_{i=1}^{p}B_i(\Omega)\right\|\beta\left(AG(L)\right)+\|A(L)\|\sum_{i=1}^{p}\beta\left(B_i(L)\right)\\ &\quad+\sum_{i=1}^{p}\beta\left(A(L)\right)\beta\left(B_i(L)\right)\\ &\leq \sum_{i=1}^{p}p\,\delta\psi\left(\beta(G(L))\right)+\|A(X)\psi_i\beta(G(L))+\beta\left(A(L)\right)\beta\left(B_i(L)\right)\\ &\leq \sum_{i=1}^{p}p\,\delta\psi\left(\beta(L)\right)+\|A(X)\|\psi_i\left(\beta(L)\right)+\psi\left(\beta(L)\right)\psi_i\left(\beta(L)\right).\end{aligned}$$

If $\beta(L)>0$, we get a contradiction, and consequently L is weakly compact. Let now

$$\mathcal{M}_L=\{M\subseteq\Omega, M \text{ weakly compact such that } M\subset L \text{ and } G(M)\subset M\}.$$

Observe that $L\in\mathcal{M}_L$, and therefore $\mathcal{M}_L$ is non-empty. By Zorn's lemma, $(\mathcal{M}_L,\supseteq)$ has a minimal element K. By definition of $\mathcal{M}_L$, K is a weakly compact subset of L and satisfies $N=G(K)\subset K$. It should be noted that $\overline{N}^w$ is also weakly compact and verifies $G(\overline{N}^w)\subseteq G(K)\subseteq\overline{N}^w$. Consequently, $\overline{N}^w\in\mathcal{M}_L$ and so $\overline{N}^w=K$ since K is minimal. Next we claim that N is weakly closed. Let $x\in\overline{N}^w$. Since $\overline{N}^w$ is weakly compact, we can apply the well-known Eberlein-Šmulian's theorem, see Corollary 2.7 in [2], in order to ensures the existence of some weakly convergent sequences $\{x_n,\ n\in\mathbb{N}\}\subset G(K)$ to some element $x\in X$. This implies that $x_n\in G(y_n)$, for some $y_n\in K$. Then,

$$\left(\frac{I}{A}\right)(x_n)\cap\sum_{i=1}^{p}B_i(y_n)\neq\emptyset$$

and

$$x_n \in A(x_n) \sum_{i=1}^{p} B_i(y_n),$$

which shows that

$$x_n = \sum_{i=1}^{p} \xi_n \cdot \rho_n^i,$$

for some $\xi_n \in A(x_n)$ and $\rho_n^i \in B_i(y_n)$, $i = 1, \dots, p$. The set K being weakly compact, then by the celebrated theorem of Eberlein-Šmulian one easily sees that $\{y_n,\ n \in \mathbb{N}\}$ has a subsequence (y_{n_k}) such that $y_{n_k} \rightharpoonup y$. Since A and B_i are sequentially weakly upper semi-compact, for all $i = 1, \dots, p$, then there exist two renamed subsequences such that

$$\xi_{n_k} \rightharpoonup \xi \in A(x) \quad \text{and} \quad \rho_{n_k}^i \rightharpoonup \rho^i \in B_i(y),$$

then by using condition $(\mathcal{P})$ we get

$$x_{n_k} = \sum_{i=1}^{p} \xi_{n_k} \cdot \rho_{n_k}^i \rightharpoonup \sum_{i=1}^{p} \xi \cdot \rho^i \in \sum_{i=1}^{p} A(x) \cdot B_i(y).$$

Consequently,

$$\left(\frac{I}{A}\right)(x) \cap \sum_{i=1}^{p} B_i(y) \neq \emptyset,$$

and therefore $x \in G(y) \subset G(K)$, which achieves the claim. Hence $K = G(K)$ since K is minimal.

Theorem 4.3. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A : X \longrightarrow \mathcal{P}_{cl,bd,cv}(X)$ and $B_i : \Omega \longrightarrow \mathcal{P}(X)$, $i = 1, \dots, p$, satisfy:*

(i) A is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function ψ,

(ii) A is s.w.u.sco such that $A(X)$ is bounded,

(iii) each B_i is weakly completely continuous and $B_i(\Omega)$ is bounded, and

(iv) $x \in \sum_{i=1}^{p} Ax \cdot B_i y,\quad y \in \Omega \Rightarrow x \in \Omega$.

Then, there exists a weakly compact subset K in S such that

$$K = \left(\frac{I}{A}\right)^{-1} \sum_{i=1}^{p} B_i(K)$$

whenever $p\,\delta\,\psi(r) < r$, for $r > 0$, where $\delta = \sup_{1 \le i \le p} \|B_i(\Omega)\|$.

Proof. Proceeding as in the proof of Theorem 4.2, the multi-valued operator G, given in (4.1), is well defined and there exists a weakly closed subset L such that $L = \overline{G(L) \cup \{s_0\}}^w$ and $G(L) \subset L$. This allows us to have

$$G(L) \subset AG(L) \cdot \sum_{i=1}^{p} B_i(L) \subset A(L) \cdot \sum_{i=1}^{p} B_i(L).$$

Taking into account the subadditivity of β, it follows from our assumptions that

$$\begin{aligned}
\beta(L) &\leq \beta\left(AG(L)\cdot\sum_{i=1}^{p}B_i(L)\right)\\
&\leq \left\|\sum_{i=1}^{p}B_i(L)\right\|\beta\left(AG(L)\right)\\
&\leq \left\|\sum_{i=1}^{p}B_i(\Omega)\right\|\beta\left(AG(L)\right)\\
&\leq \left\|\sum_{i=1}^{p}B_i(\Omega)\right\|\psi\left(\beta(G(L))\right)\\
&\leq p\,\delta\psi(\beta(L))\\
&< \beta(L).
\end{aligned}$$

This contradiction implies the weak compactness of L. The remained proof follows along the lines of Theorem 4.2, by using the fact that each B_i is s.w.u.sco.

In the particular case when A is a weakly sequentially continuous single-valued operator, we obtain the following result:

Theorem 4.4. *Let Ω be a nonempty and weakly closed subset of X. Assume that the multi-valued operators $A : X \longrightarrow X$ and $B_i : \Omega \longrightarrow \mathcal{P}(X)$ satisfy:*
(i) A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ψ,
(ii) A is weakly sequentially continuous such that $A(X)$ is bounded,
(iii) each B_i is s.w.u.sco such that $B_i(\Omega)$ is bounded, and
(iv) $x \in \sum_{i=1}^{p} Ax\cdot B_iy,\quad y\in\Omega \Rightarrow x\in\Omega$.
Then, there exists a weakly compact subset K in Ω such that

$$\left(\frac{I}{A}\right)(K)=\sum_{i=1}^{p}B_i(K)$$

whenever $p\,\delta\psi(r) < r$, for $r > 0$, where $\delta = \sup\limits_{1\leq i\leq p}\|B_i(\Omega)\|$.

Proof. Let y_i be a fixed element in $B_i(\Omega)$, $i = 1,\ldots,p$, and define φ_y by

$$\begin{cases}\varphi_y : X \longrightarrow X\\ x\mapsto \displaystyle\sum_{i=1}^{p}Ax\cdot y_i\end{cases}$$

Using assumption (i), it is easy to verify that φ_y is a nonlinear contraction, and so it has a unique fixed point in view of the Boyd and Wong's fixed point theorem [12]. This proves that the multi-valued operator G, given in (4.1), is well defined on Ω and there exists a minimal K weakly compact such that $K=\overline{G(K)}^{w}$. Let $x\in\overline{G(K)}^{w}$ be fixed.

Then, there is a sequence $\{x_n,\ \ n \in \mathbb{N}\}$ of elements in $G(K)$ which converges weakly to some x and a sequence $\{y_n,\ \ n \in \mathbb{N}\}$ of element in K with $x_n \in G(y_n), \forall n \in \mathbb{N}$. From the above discussion, one has

$$\left(\frac{I}{A}\right)(x_n) \cap \sum_{i=1}^{p} B_i(y_n) \neq \emptyset,\ n \in \mathbb{N},$$

or equivalently, there is a sequence $z_n^i \in B_i(y_n)$ such that $x_n = \sum_{i=1}^{p} Ax_n \cdot z_n^i$. Since K is a weakly compact subset of Ω and B_i is s.w.u.sco, one easily sees that $\{y_n,\ \ n \in \mathbb{N}\}$ possesses a weakly convergent subsequence $(y_{n_k})_k$ to a point $y \in K$, so that $\left\{z_{n_k}^i,\ \ k \geq n\right\}$ has a subsequence, say $\left(z_{n_{k_j}}^i\right)_j$, that weakly converges to z^i, for some $z^i \in B_i(y)$. If we now use the sequential weak continuity of the operator A and deploy the condition $(\mathcal{P})$, we get

$$x_{n_{k_j}} = \sum_{i=1}^{p} Ax_{n_{k_j}} \cdot z_{n_{k_j}}^i \rightharpoonup \sum_{i=1}^{p} Ax \cdot z^i \in \sum_{i=1}^{p} A(K) \cdot B_i(K).$$

Then $K = \left(\frac{I}{A}\right)^{-1} \sum_{i=1}^{p} B_i(K)$, which achieves the result since $\left(\frac{I}{A}\right)^{-1}$ is a single-valued operator.

5. Self-similar sets

This section is dedicated to prove the applicability of some results developed in the above sections to the theory of self similar sets. Let $\mathfrak{X}$ be a Banach space and $\mathfrak{D}$ be a nonempty weakly closed subset of $\mathfrak{X}$. Let Ψ be an arbitrary family of self maps of $\mathfrak{D}$. For any $x \in \mathfrak{D}$, we denote

$$\Psi(x) = \{\psi(x), \psi \in \Psi\}$$

and

$$\Psi(\mathfrak{D}) = \bigcup_{\psi \in \Psi} \psi(\mathfrak{D}).$$

Let ω be a non empty subset of $\mathfrak{D}$. Then, we say that ω is a self similar with respect to Ψ if $\Psi(\omega) = \omega$. In the case when $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a finitely family of self maps, then we say that $(\mathfrak{D}, \{\psi_1, \ldots, \psi_n\})$ is an iterated function system $(I.F.S)$. Moreover, we say that an $(I.F.S)$ is continuous (resp., contraction, condensing with respect to a measure of noncompactness Φ or Φ-condensing, $\mathcal{D}$-set-lipschitzian etc.), if each ψ_i is so.

Notice that, the theory of self-similar sets play a crucial role in various fields of applied mathematics, and have been used by many authors to model various physical phenomena, see [32], [34].

It is well known that for any contractive $(I.F.S)$, there exists a unique compact self-similar set K [20], i.e.

$$K = \psi_1(K) \cup \ldots \cup \psi_n(K).$$

In [34], E. A. Ok has extended this result to a continuous and μ-condensing $(I.F.S)$. In weak topology setting, Ben Amar et al. [2] have discussed the existence of a weakly compact self-similar set of an $(I.F.S)$ of the form $(\mathfrak{D}, \{A\psi_1, \dots, A\psi_n\})$. In this section, our main interest is dealing with some fixed set results for $\mathcal{D}$-set-Lipschitzian operators.

Theorem 5.1. *Let $\mathfrak{D}$ be a nonempty and weakly closed subset of a Banach algebra $\mathfrak{X}$ satisfying the condition $(\mathcal{P})$. Let $(\mathfrak{D}, \{A\psi_{1,1}, \dots, A\psi_{n,p}\})$ be a $(I.F.S)$ such that:*
(i) A is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function ϕ_A,
(ii) A is weakly sequentially continuous and $A(\mathfrak{D})$ is bounded,
(iii) each $\psi_{i,j}$ is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function $\phi_{i,j}$,
(iv) each $\psi_{i,j}$ is sequentially weakly continuous and $\psi_{i,j}(\mathfrak{D})$ is bounded, and
(v) $A(\mathfrak{D})\sum_{j=1}^{p}\bigcup_{i=1}^{n}\psi_{i,j}(\mathfrak{D}) \subset \mathfrak{D}$.
Then, there exists a weakly compact subset K in $\mathfrak{D}$ such that

$$K = A(K) \cdot \sum_{j=1}^{p}\bigcup_{i=1}^{n}\psi_{i,j}(K)$$

whenever

$$\sum_{j=1}^{p}\left[\|A(\mathfrak{D})\|\phi_{i,j}(r) + \phi_A(r)\phi_{i,j}(r)\right] + p\,\delta\phi_A(r) < r,$$

for $r > 0$ and $i \in \{1, \dots, n\}$, where $\delta = \sup_{1\le j\le p}\max_{1\le i\le n}\|\psi_{i,j}(\mathfrak{D})\|$.

Proof. For $j \in \{1, 2, \dots, p\}$, define B_j by

$$\begin{cases} B_j : \mathfrak{D} \longrightarrow \mathcal{P}(\mathfrak{X}) \\ \xi \mapsto \bigcup_{i=1}^{n}\{\psi_{i,j}(\xi)\} \end{cases}$$

Let j be arbitrary in $\{1, 2, \dots, p\}$. First we begin by proving that B_j is a sequentially weakly upper semi-compact multi-valued operator. To see this, take an arbitrary sequence $\{\xi_n,\ n \in \mathbb{N}\}$ of element in $\mathfrak{D}$ converging to some element ξ and $\zeta_n \in B_j(\xi_n)$. By the weak sequential continuity of $\psi_{i,j}$, $i = 1, \dots, n$, we results that $B_j(\xi_n) = \bigcup_{i=1}^{n}\psi_{i,j}(\xi_n)$ is relatively weakly compact. We now apply the Eberlein-Šmulian's theorem [14] in order to have $\zeta_{n_k} \rightharpoonup \zeta$, for some subsequence $(\zeta_{n_k})_k$ of $\{\zeta_n,\ n \in \mathbb{N}\}$. Let $i \in \{1, \dots, n\}$ such that the set $\{k, \zeta_{n_k} = \psi_{i,j}(\xi_{n_k})\}$ is infinitely. Using again the fact that $\psi_{i,j}$ is sequentially weakly continuous, one has $\zeta_{n_k} \rightharpoonup \psi_{i,j}(\xi) \in B_j(\xi)$, and hence B_j is a sequentially weakly upper semi-compact multi-valued operator. Next, we prove that the multi-valued operator B_j is $\mathcal{D}$-set-lipschitzian on $\mathfrak{D}$. So, let M be a bounded subset of $\mathfrak{D}$ such that $B_j(M) \in \mathcal{P}_{bd}(\mathfrak{X})$. Based on the maximum property of β, we have

$$\begin{aligned} \beta(B_j(M)) &\le \max\{\beta(\psi_{1,j}(M)), \dots, \beta(\psi_{n,j}(M))\} \\ &\le \max\{\phi_{1,j}(\beta(M)), \dots, \phi_{n,j}(\beta(M))\}. \end{aligned}$$

This implies that B_j is a $\mathcal{D}$-set-lipschitzian multi-valued operator with $\mathcal{D}$-function $\phi_{max,j}$, which assigns for $r > 0$ the values $\phi_{max,j}(r) = \max\{\phi_{1,j}(r), \ldots, \phi_{n,j}(r)\}$. The remained proof follows along the lines of Theorem 3.1, using Remark 2.6.

When $p = 1$ and each $\psi_{i,1}(\mathfrak{D}), i = 1, \ldots, n$, is relatively weakly compact, the following result represents a generalization of Theorem 4.1 in [2].

Corollary 5.2. *Let $\mathfrak{D}$ be a nonempty and weakly closed subset of a Banach algebra $\mathfrak{X}$ satisfying the condition $(\mathcal{P})$. Let $(\mathfrak{D}, \{A\psi_1, \ldots, A\psi_n\})$ be an $(I.F.S)$ such that:*
(i) A is weakly sequentially continuous $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_A,
(ii) each ψ_j is weakly sequentially continuous such that $\psi_j(\mathfrak{D})$ is relatively weakly compact,
(iii) $A(\mathfrak{D})\psi_j(\mathfrak{D}) \subset \mathfrak{D}$, for $j = 1, \ldots, n$.
Then, $(\mathfrak{D}, \{A\psi_1, \ldots, A\psi_n\})$ has a weakly compact self-similar set whenever

$$\delta\phi_A(r) < r,$$

for $r > 0$, where $\delta = \max_{1\leq j\leq n} \|\psi_j(\mathfrak{D})\|$.

Next, we can modify the assumption (v) of Theorem 5.1 in order to obtain the following result.

Theorem 5.3. *Let $\mathfrak{X}$ be a Banach algebra satisfying the condition $(\mathcal{P})$ and $\mathfrak{D}$ be a nonempty and weakly closed subset of $\mathfrak{X}$. Let $p \geq 2$, $n \geq p$ and $(\mathfrak{D}, \{A\psi_1, \ldots, A\psi_n\})$ be an $(I.F.S)$ such that:*
(i) A is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-functions ϕ_A,
(ii) A is weakly sequentially continuous such that $A(\mathfrak{X})$ is bounded,
(iii) each ψ_i is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function ϕ_i,
(iv) each ψ_i is sequentially weakly continuous such that $f_i(\mathfrak{D})$ is bounded, and
(v) $A(\mathfrak{D})\sum_{i=1}^{p-1}\psi_i(\mathfrak{D}) + A(\mathfrak{D})\psi_j(\mathfrak{D}) \subset \mathfrak{D}$ for all $p \leq j \leq n$.

Then, there exists a weakly compact subset K of $\mathfrak{D}$ such that

$$K = \left(\frac{I}{A}\right)^{-1}\left(\sum_{i=1}^{p-1}\psi_i(K) + \bigcup_{i=p}^{n}\psi_i(K)\right)$$

whenever

$$\sum_{i=1}^{p-1}\|A(\mathfrak{D})\|\phi_i(r) + \phi_A(r)\,\phi_i(r) + p\,\delta\phi_A(r) + \left[\|A(\mathfrak{D})\|\phi_j(r) + \phi_A(r)\phi_j(r)\right] < r,$$

for $r > 0$ and $p \leq j \leq n$, where $\delta = \max_{1\leq i\leq n}\|\psi_i(\mathfrak{D})\|$.

Proof. For $j \in \{1, 2, \ldots, p-1\}$, define B_i by

$$\begin{cases} B_j : \mathfrak{D} \longrightarrow \mathfrak{X} \\ \xi \mapsto \psi_i(\xi), \end{cases}$$

and B_p by

$$\begin{cases} B_p : \mathfrak{D} \longrightarrow \mathcal{P}(\mathfrak{X}) \\ \xi \mapsto \displaystyle\bigcup_{i=p}^{n}\{\psi_i(\xi)\}. \end{cases}$$

Considering the assumption (iv), it follows from Remark 2.6 that B_j is a sequentially weakly upper semi-compact multi-valued operator for all $j \in \{1, 2, \dots, p-1\}$. Proceeding as in the proof of Theorem 5.1, B_p is, also, sequentially weakly upper semi-compact. Now, we claim that B_j are $\mathcal{D}$-set-Lipschitzian multi-valued operators. The claim regarding B_j for $j = 1, 2, \dots, p-1$ is clear in view of assumption (ii). We corroborate now the claim for B_p. Let M be a bounded subset of $\mathfrak{D}$. We have

$$\begin{aligned} \beta(B_p(M)) &\leq \max\{\beta(\psi_p(M)), \dots, \beta(\psi_n(M))\} \\ &\leq \max\{\phi_p(\beta(M)), \dots, \phi_n(\beta(M))\}. \end{aligned}$$

This implies that B_p is a $\mathcal{D}$-set-Lipschitzian multi-valued operator with $\mathcal{D}$-function ϕ_{max}, which assigns for all $r > 0$ the values $\phi_{max}(r) = \max\{\phi_p(r), \dots, \phi_n(r)\}$. The result follows from Theorem 4.2.

In the particular case when $n = p$, we obtain the following result:

Corollary 5.4. *Let $\mathfrak{X}$ be a Banach algebra satisfying the condition $(\mathcal{P})$ and $\mathfrak{D}$ be a nonempty and weakly closed subset of $\mathfrak{X}$. Let $(\mathfrak{D}, \{A\psi_1, \dots, A\psi_n\})$ be an $(I.F.S)$ such that:*

(i) A is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-functions ϕ_A,
(ii) A is weakly sequentially continuous such that $A(\mathfrak{X})$ is bounded,
(iii) each ψ_i is $\mathcal{D}$-set-Lipschitzian with $\mathcal{D}$-function ϕ_i,
(iv) each ψ_i is sequentially weakly continuous such that $\psi_i(\mathfrak{D})$ is bounded, and
(v) $A(\mathfrak{D}) \sum_{i=1}^{n} \psi_i(\mathfrak{D}) \subset \mathfrak{D}$.

Then, there exists a weakly compact subset K of $\mathfrak{D}$ such that

$$K = \left(\frac{I}{A}\right)^{-1} \sum_{i=1}^{n} \psi_i(K),$$

or equivalently

$$\left\{\frac{x}{Ax}, x \in K\right\} = \sum_{i=1}^{n} \psi_i(K)$$

whenever $\sum_{i=1}^{n} \|A(\mathfrak{D})\|\phi_i(r) + \phi_A(r)\,\phi_i(r) + n\,\delta\phi_A(r) < r$ for $r > 0$, where

$$\delta = \max_{1 \leq i \leq n} \|\psi_i(\mathfrak{D})\|.$$

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