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DECAY ESTIMATES FOR TWO-TERM TIME FRACTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

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Abstract. In this paper, nonlinear differential evolution equations of fractional order in Banach spaces involving unbounded delays are investigated. We aim to prove the existence of mild solutions and demonstrate its polynomial decay by the fixed point principle for condensing maps. An example of the application of abstract results is given for illustration.
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1. Introduction

Let X be a Banach space. We consider the following problem

$$D_C^{\alpha+1}u(t) + \mu D_C^{\beta}u(t) - Au(t) = F(t, u(t), u_t), \quad t \in I = [0, T] \tag{1.1}$$

$$u(s) = g(s), \quad s \leq 0, \tag{1.2}$$

$$u'(0) + h(u) = \psi, \tag{1.3}$$

where $0 < \alpha \leq \beta \leq 1, D_C^{\alpha}$ is the fractional derivative in Caputo's sense of order α, $A : D(A) \subset X \longrightarrow X$ is the closed linear operator, and A generates a strongly continuous family $\{S_{\alpha,\beta}(t)\}$ of bounded and linear operator on X, u is the unknown function defined on I and taking values in X, u_t is the history state defined by $u_t : (-\infty, 0] \to X, u_t(s) = u(t+s)$, for each $t \in I$, $\psi \in X$, $g \in \mathcal{B}$ with $\mathcal{B}$ being an admissible phase space that satisfying some fundamental axioms listed in Subsection 2.1, $F : I \times X \times \mathcal{B} \to X,\ h : BC(I; X) \to X$ are given functions.

Equation (1.1) without delays come from recent investigations [16, 17, 13]. These papers discussed the existence and asymptotic behavior of solutions for linear or semilinear two-term time-fractional differential evolution equations.
In the case $0 < \alpha < 1$, $\beta = 1$, $\mu = 0$, Eq. (1.1) without delays was studied in [27]

and when $0 < \alpha < 1$, $\beta = 1$, $\mu = 0$, $A = \triangle$ is the model of a fractional diffusion-wave equation (see [26]). Eq. (1.1) without delays is the abstract setting of nonlinear time-fractional telegraph equations when $\alpha + 1 = 2\beta$, $A = \triangle$ (see [22, 23]), and nonlinear fractional diffusion-wave with damping when $0 < \alpha < 1$, $\beta = 1$, $A = \triangle$ (see [24]). In the case, $A = \triangle$, in [21] a two-term fractional-order diffusion evolution equation without delays was proposed for the total concentration in solute transport, in order to distinguish explicitly the mobile and immobile status of the solute using fractional dynamics; the kinetic equation with two fractional derivatives of different orders appears also quite naturally when describing subdiffusive motion in velocity fields [19]; and [12] for investigations on the model for wave-type phenomena.

Differential equations with delay are often more realistic mathematical models for practical problems compared to those without delay, for instance, from control theory in which control factor is taken in the form of feedbacks. In such control problems, the presence of delay term is an inherent feature. Numerous papers and monographs have appeared devoted to differential equations with delays (see, e.g., [8, 9, 28, 7, 14, 1, 20, 11] and references therein). These papers contain various types of existence results for initial value problem to differential equations with delays. Recently, in [29], two-term time fractional differential equations with finite delay and the right hand side $f = f(t, u_t)$ has studied, the author established the existence, uniqueness of pseudo asymptotically periodic solutions.

In this paper, based on a fixed point principle for condensing maps for measures of noncompactness on $BC(\mathbb{R}^+; X)$, we prove the existence of mild solution on $[0, T]$ for problem (1.1)-(1.3) and decay estimates of mild solutions u with $\|u(t)\| = O(t^{-\gamma})$ as $t \to \infty$.

The rest of our work is organized as follows. Section 2, we recall some notions, phase space and facts relating to measures of noncompactness and condensing map. Section 3, we prove the existence of mild solutions on $(-\infty, T], T > 0$ under some regular conditions imposed on the nonlinearities h and F. Section 4 is devoted to show that decay mild solutions on $\mathbb{R}$ with the certain decay rate exist if some conditions of the operator A and the phase space $\mathcal{B}$ are added. Section 5, we give an application for partial differential equations to illustrate the obtained results.

2. Preliminaries

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^+, X)$ the Banach space of all locally (Bochner) integrable vector-valued functions. The Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+, X)$ is defined as

$$\mathscr{L}[f](\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}\lambda > \omega.$$

whenever the integral is entirely convergent for $\mathrm{Re}\lambda > \omega$.

The Caputo fractional derivative of order $\alpha > 0$ is ascertained by

$$D^\alpha_C f(t) := \left(\varphi_{m-\alpha} * f^{(m)}\right)(t) := \int_0^t \varphi_{m-\alpha}(t-s) f^{(m)}(s)ds,$$

in which m is the smallest integer greater that or equal to α, and for $\beta > 0$

$$\varphi_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,$$

in which $\Gamma(\cdot)$ denotes the Gamma function. This function satisfies the semigroup property (see [6])

$$\varphi_\alpha * \varphi_\beta = \varphi_{\alpha+\beta}, \quad \forall\, \alpha, \beta > 0.$$

Applying the properties of the Laplace transform, an easy computation shows that for $\alpha > 0$,

$$\mathscr{L}[D_C^\alpha f](\lambda) = \lambda^\alpha \mathscr{L}[f](\lambda) - \sum_{k=0}^{m-1} \lambda^{\alpha-k-1} f^{(k)}(0).$$

Definition 2.1. Let $\mu \geq 0$ and $0 \leq \alpha, \beta \leq 1$ be given. Let A be a closed linear operator with domain $D(A)$ on a Banach space X. Then A is called the generator of an $(\alpha, \beta)_\mu$-regularized family if there exists $\omega \in \mathbb{R}$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}^+ \to \mathscr{L}(X)$ such that $\{\lambda^{\alpha+1} + \mu\lambda^\beta : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^\alpha(\lambda^{\alpha+1} + \mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x dt, \mathrm{Re}(\lambda) > \omega, x \in X.$$

We know that in the case $\mu = 0, \alpha = 0$, this is a C_0-semigroup while if $\mu = 0, \alpha = 1$, we get a cosine family. The existence and characterization of generators of $(\alpha, \beta)_\mu$-regularized families were discussed in [15]. Specifically, let A be a closed and densely defined operator. An operator A is called to be ω-sectorial of angle θ if there exist $\theta \in [0, \frac{\pi}{2})$ and $\omega \in \mathbb{R}$ such that its resolvent family is in the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \theta\} \setminus \{\omega\}, \tag{2.1}$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta. \tag{2.2}$$

Notice that a closed and densely defined operator A is ω-sectorial of angle θ if $A - \omega I$ is sectorial of angle θ. The following results were established in [13].

Lemma 2.2. [13] *Let $0 < \alpha \leq \beta \leq 1, \mu > 0$ and A be an ω-sectorial operator of angle $\frac{\beta\pi}{2}$. Then A generates an exponentially bounded $(\alpha, \beta)_\mu$-regularized family $S_{\alpha\beta}(t)$.*

Lemma 2.3. [13] *Let $0 < \alpha \leq \beta \leq 1, \mu > 0$ and $\omega < 0$. Assume that A is an ω-sectorial operator of angle $\frac{\beta\pi}{2}$. Then A generates an $(\alpha, \beta)_\mu$-regularized family $S_{\alpha\beta}(t)$ satisfying the estimate*

$$\|S_{\alpha\beta}(t)\| \leq \frac{C}{1 + |\omega|(t^{\alpha+1} + \mu t^\beta)}, \quad t \geq 0, \tag{2.3}$$

for some constant $C > 0$ depending only on α, β.

We now are in search of suitable concept of mild solutions to problem (1.1)-(1.3). Denoting by $\mathscr{L}$ the Laplace transform for X-valued functions acting on $\mathbb{R}^+$, putting $v(t) = F(t, u(t), u_t)$ and applying the Laplace transform to (1.1)-(1.3), we have

$$\begin{aligned}(\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)&\mathscr{L}[u](\lambda)\\ &= \lambda^{\alpha}u(0) + \lambda^{\alpha-1}u_t(0) + \mu\lambda^{\beta-1}u(0) + \mathscr{L}[v](\lambda),\ Re(\lambda) > \omega.\end{aligned}$$

Therefore

$$\begin{aligned}\mathscr{L}[u](\lambda) = \lambda^{\alpha}(\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)^{-1}u(0) &+ \lambda^{\alpha-1}(\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)^{-1}u'(0)\\ &+ \mu\lambda^{\beta-1}(\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)^{-1}u(0)\\ &+ (\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)^{-1}\mathscr{L}[v](\lambda),\end{aligned}$$

for all λ such that $Re(\lambda) > \omega, \lambda^{\alpha+1}+\mu\lambda^{\beta} \in \rho(A)$. Let $S_{\alpha,\beta}(t)$ be an $(\alpha, \beta)_{\mu}$-regularized family generated by A, we have

$$\begin{aligned}\mathscr{L}[u](\lambda) = \mathscr{L}[S_{\alpha,\beta}](\lambda)\big(g(0)\big) &+ \mathscr{L}[\varphi_1]\mathscr{L}[S_{\alpha,\beta}](\lambda)[\psi - h(u)]\\ &+ \mu\mathscr{L}[\varphi_{1+\alpha-\beta}]\mathscr{L}[S_{\alpha,\beta}](\lambda)\big(g(0)\big)\\ &+ \mathscr{L}[S_{\alpha,\beta}](\lambda)\mathscr{L}[\varphi_{\alpha}](\lambda)\mathscr{L}[v](\lambda),\ Re(\lambda) > \omega,\end{aligned}$$

where functions $\varphi_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta > 0$. Inversion of the Laplace transform indicates that

$$\begin{aligned}u(t) = S_{\alpha,\beta}(t)\big(g(0)\big) &+ (\varphi_1 * S_{\alpha,\beta})(t)[\psi - h(u)]\\ &+ \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\big(g(0)\big) + (S_{\alpha,\beta} * \varphi_{\alpha} * v)(t). \quad (2.4)\end{aligned}$$

Assume that $f(t, u(t), u_t) = \varphi_{\alpha} * F(t, u(t), u_t)$. Motivated by (2.4), the following definition of mild solutions are given.

Definition 2.4. Let $0 < \alpha \leq \beta \leq 1$ and $\mu \geq 0$. A function $u:\ (-\infty, T] \to X$ is called to be a mild solution of problem (1.1)−(1.3) iff $u(t) = g(t)$, for $t \leq 0$ and

$$\begin{aligned}u(t) = S_{\alpha,\beta}(t)(g(0)) + (\varphi_1 * S_{\alpha,\beta})(t)[\psi - h(u)] &+ \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)(g(0))\\ &+ \int_0^t S_{\alpha,\beta}(t-\tau)f(\tau, u(\tau), u_{\tau})d\tau, \quad (2.5)\end{aligned}$$

for each $t \in \mathbb{R}_+$ and $(g, \psi) \in \mathcal{B} \times X$.

2.1. **Phase space.** Let $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ be a semi-normed linear space, consisting of functions mapping $(-\infty, 0]$ into a Banach space X. The definition of a phase space $\mathcal{B}$, introduced in [8], is based on the following axioms. If $v:\ (-\infty, \sigma + T] \to X$, where $\sigma \in \mathbb{R}$ and T is a positive number, is a function such that $v|_{[\sigma,T+\sigma]} \in C([\sigma, T + \sigma]; X)$ and $v_{\sigma} \in \mathcal{B}$, we have

(B1) $v_t \in \mathcal{B}$ for $t \in [\sigma, T + \sigma]$;
(B2) The function $t \mapsto v_t$ is continuous on $[\sigma, T + \sigma]$;

(B3) $|v_t|_{\mathcal{B}} \le K(t-\sigma)\sup_{s\in[\sigma,t]}\|v(s)\| + M(t-\sigma)|v_\sigma|_{\mathcal{B}}$ for each $t \ge \sigma$, where $K, M: [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded and they are independent of v.

An archetypal example for $\mathcal{B}$ is C_γ defined as follows

$$C_\gamma = \{w \in C((-\infty,0];X) \text{ such that } \lim_{s\to-\infty} e^{\gamma s}w(s) \text{ exists in } X\}.$$

If $\gamma > 0$ then C_γ is a Banach space with the norm

$$|w|_\gamma = \sup_{s\le 0} e^{\gamma s}\|w(s)\|_X.$$

In this case $K(t) = 1, M(t) = e^{-\gamma t}$. For more examples of phase space, see [9].

2.2. **Measure of noncompactness.** Let X be a Banach space. Denote by $\mathcal{P}_b(X)$ the collection of all nonempty bounded subsets in X.

Definition 2.5. A function $\Phi: \mathcal{P}_b(X) \longrightarrow [0,+\infty)$ is called a measure of noncompactness (MNC) in X if

$$\Phi(\overline{co}\Omega) = \Phi(\Omega),\ \forall\Omega \in \mathcal{P}_b(X),$$

where $\overline{co}\Omega$ is the closure of the convex hull of Ω. An MNC Φ in X is called

(i) monotone if for $\forall\,\Omega_1,\Omega_2 \in \mathcal{P}_b(X), \Omega_1 \subset \Omega_2$ implies $\Phi(\Omega_1) \le \Phi(\Omega_2)$,
(ii) nonsingular if $\Phi\big(\{x\}\cup\Omega\big) = \Phi(\Omega)$ for $\forall\, x \in X, \forall\,\Omega \in \mathcal{P}_b(X)$;
(iii) invariant with respect to union with compact set if $\Phi(K\cup\Omega) = \Phi(\Omega)$ for every relatively compact $K \subset X$ and $\Omega \in \mathcal{P}_b(X)$;
(iv) algebraically semi-additive if $\Phi(\Omega_1+\Omega_2) \le \Phi(\Omega_1)+\Phi(\Omega_2)$ for any $\Omega_1,\Omega_2 \in \mathcal{P}_b(X)$;
(v) regular if $\Phi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

A significant example of MNC is the Hausdorff MNC $\chi(\cdot)$, which is defined as follows:

$$\chi(\Omega) = \inf\{\varepsilon > 0:\ \Omega \text{ has a finite } \varepsilon-\text{net}\} \tag{2.6}$$

for $\forall\ \Omega \in \mathcal{P}_b(X)$. It is well known that Hausdorff MNC $\chi(\cdot)$, enjoys the the above properties (i)-(v).

Sequencingly, some basic MNC estimates are needed. Based on Hausdorff MNC χ in X, one can define the sequential MNC χ_0 as follows:

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\},$$

in which $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [2]). It is known that

$$\frac{1}{2}\chi(\Omega) \le \chi_0(\Omega) \le \chi(\Omega),$$

for all bounded set $\Omega \subset X$. So the following property is evident.

Proposition 2.6. *Let χ be the Hausdorff MNC on Banach space $X, \Omega \in \mathcal{P}_b(X)$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset \Omega$ such that*

$$\chi(\Omega) \le 2\chi\big(\{x_n\}_{n=1}^\infty\big) + \varepsilon,\ \forall\varepsilon > 0. \tag{2.7}$$

Let $C([0,T];X)$ be the space of all continuous functions defined on the interval $[0,T]$ taking values in X, which is a Banach space with the norm

$$\|u\|_C = \sup_{t\in[0,T]} \|u(t)\|_X.$$

Denote by χ_T the Hausdorff measure of noncompactness of $C([0,T];X)$, we obtain the following proposition.

Proposition 2.7. [5]

(1) *If $D \subset C([0,T];X)$ is bounded, then $\chi(D(t)) \le \chi_T(D)$, for any $t \in [0,T]$, where $D(t) = \{x(t) : x \in D\}$.*
(2) *If D is equicontinuous on $[0,T]$, then $\chi(D(t))$ is continuous for $t \in [0,T]$ and $\chi_T(D) = \sup\limits_{t\in[0,T]} \chi(D(t))$.*
(3) *If D is bounded and equicontinuous on $[0,T]$, then $\chi(D(t))$ is continuous for $t \in [0,T]$ and*

$$\chi\left(\int_0^t D(s)ds\right) \le \int_0^t \chi(D(s))ds, \text{ for all } t \in [0,T],$$

where $\int_0^t D(s)ds = \{\int_0^t x(s)ds : x \in D\}$.

We denote by $\big(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)}\big)$ the space of linear bounded operators from X into itself, χ is the Hausdorff MNC on X. For each $S \in \mathcal{L}(X)$, we define $\chi-$norm of S (see [2]) as follows:

$$\|S\|_\chi = \inf\{k > 0 : \ \chi\big(S(\Omega)\big) \le k\chi(\Omega), \ \Omega \in \mathcal{P}_b(X)\} \tag{2.8}$$

We have following estimate (see [4])

$$\|S\|_\chi \le \|S\|_{\mathcal{L}(X)} \tag{2.9}$$

Consider the space $BC(\mathbb{R}^+;X)$ of bounded continuous functions on $\mathbb{R}^+$ taking values on X. Denote by π_T the restriction operator on this space, i. e, $\pi_T(u)$ is the restriction of u on $[0,T]$. Then

$$\chi_\infty(D) = \sup_{T>0} \chi_T\big(\pi_T(D)\big), \ D \subset BC(\mathbb{R}^+;X). \tag{2.10}$$

is an MNC. Some measures of noncompactness are given as follows

$$d_T(D) = \sup_{u\in D} \sup_{t\ge T} \|u(t)\|_X, \tag{2.11}$$

$$d_\infty(D) = \lim_{T\to\infty} d_T(D), \tag{2.12}$$

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.13}$$

The regularity of MNC χ^* is proved in [3, Lem 2.6].

Before coming to the next section, we recall the fixed point principle for condensing maps which will be used in next sections.

Definition 2.8. [4] Let χ be an MNC on Banach space X, and $\emptyset \neq D \subset X$. A continuous map $F: D \longrightarrow X$ is said to be condensing with respect to χ ($\chi-$condensing) if for $\forall\, \Omega \in \mathcal{P}_b(D)$, the relation

$$\chi(\Omega) \leq \chi\big(F(\Omega)\big)$$

implies the relative compactness of Ω.

Theorem 2.9. [10] *Let D be a bounded convex closed subset of Banach space X and let $F: D \longrightarrow D$ be a $\chi-$condensing map. Then*

$$Fix(F) = \{x \in D : x = F(x)\}$$

is a nonempty compact set.

This theorem is rather typical. Particularly, it covers the contraction mapping principle and Krasnoselkii theorem.

3. Existence result

In formulation of problem (1.1)-(1.3), we suppose that

(B) The phase space $\mathcal{B}$ verifies (B1)-(B3).

(F) The nonlinear function $f: [0,T] \times X \times \mathcal{B} \longrightarrow X$ satisfies:

(i) $t \mapsto f(t,v,w)$ is measurable for each $(v,w) \in X \times \mathcal{B}$ and $(v,w) \mapsto f(t,v,w)$ is continuous for a.e $t \in [0,T]$.

(ii) There exists function $m, m_1 \in L_1(0,T)$ such that

$$\|f(t,v,w)\|_X \leq m(t)\|v\|_X + m_1(t)|w|_{\mathcal{B}}, \tag{3.1}$$

for all $(v,w) \in X \times \mathcal{B}$, and for each $t \in [0,T]$.

(iii) There exists $k \in L_1(0,T)$, which is non-negative such that

$$\chi\big(f(t,\Omega,D)\big) \leq k(t)\big(\chi(\Omega) + \sup_{s\leq 0} \chi(D(s))\big), \tag{3.2}$$

for every $t \in [0,T]$ and for all bounded sets $\Omega \subset X, D \subset \mathcal{B}$.

(H) The function $h: C([0,T];X) \to X$ satisfies following conditions:

(i) There exists a nondecreasing continuous function $\theta: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(u)\|_X \leq \theta(\|u\|_C), \tag{3.3}$$

for all $u \in C([0,T];X)$.

(ii) There exists a non-negative constant η such that

$$\chi\big(h(\Omega)\big) \leq \eta\chi_T(\Omega), \tag{3.4}$$

for all bounded set $\Omega \subset C([0,T];X)$.

For $g \in \mathcal{B}$ and $y \in C([0,T];X)$, we ascertain the function $y[g]: (-\infty,T] \to X$ as follows

$$y[g](t) = \begin{cases} y(t) & \text{for } t \in [0,T], \\ g(t) & \text{for } t < 0. \end{cases}$$

We denote

$$C_g = \{y \in C([0,T];X) : \ y(0) = g(0)\}.$$

Then C_g is a closed subspace of $C([0,T];X)$ with the supremum norm.

We ascertain the solution operator $F : \ C_g \to C_g$ by

$$\Phi(u)(t) = S_{\alpha,\beta}(t)(g(0)) + (\varphi_1 * S_{\alpha,\beta})(t)[\psi - h(u)] \\ + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)(g(0)) + \int_0^t S_{\alpha,\beta}(t-\tau) f(\tau, u(\tau), u_\tau) d\tau, \quad (3.5)$$

for $\forall\, u \in C_g, \ \forall t \in [0,T]$. It is clear that u is a fixed point of F then $u[g]$ is a mild solution of (1.1)$-$(1.3) on $(-\infty, T]$.

From the assumptions imposed on f, h, it is seen that Φ is a continuous map on C_g. Set

$$S_T = \sup_{t\in[0,T]} \|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}, \ \Lambda_T = \sup_{t\in[0,T]} \|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)},$$

$$\Theta_T = \sup_{t\in[0,T]} \|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}, \ \Gamma_T = \sup_{t\in[0,T]} \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau) d\tau$$

$$\Upsilon_T = \sup_{t\in[0,T]} \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} \big[m(\tau) + m_1(\tau)K(\tau)\big] d\tau.$$

Lemma 3.1. *Let* $(\mathbf{B}), (\mathbf{F}), (\mathbf{H})$ *hold and*

$$\Upsilon_T + \Lambda_T \liminf_{n\to\infty} \frac{\theta(n)}{n} < 1. \quad (3.6)$$

Then there exists $R > 0$ such that $F(B_R) \subset B_R$, in which B_R is the ball in C_g centered at origin with radius R.

Proof. Assume to the contrary that there exists a sequence $\{u_n\}_{n=1}^\infty \subset B_n$ with $\|u_n\|_C \le n$ but $\|F(u_n)\|_C > n$. From the formulation of Φ, we set

$$\Phi(u_n)(t) = \Phi_1(u_n)(t) + \Phi_2(u_n)(t) + \Phi_3(u_n)(t),$$

where

$$\Phi_1(u)(t) = S_{\alpha,\beta}(t)g(0) + \mu\big(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}\big)(t)\big(g(0)\big);$$
$$\Phi_2(u)(t) = (\varphi_1 * S_{\alpha,\beta})(t)\big(\psi - h(u)\big);$$
$$\Phi_3(u)(t) = \int_0^t S_{\alpha,\beta}(t-\tau) f\big(\tau, u(\tau), u_\tau\big) d\tau.$$

We have

$$\|\Phi(u_n)(t)\|_X \le \|\Phi_1(u_n)(t)\|_X + \|\Phi_2(u_n)(t)\|_X + \|\Phi_3(u_n)(t)\|_X. \tag{3.7}$$

$$\begin{aligned}\|\Phi_1(u_n)(t)\|_X &\le \|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\|g(0)\|_X + \mu\|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\|g(0)\|_X \\ &\le \sup_{t\in[0,T]} \|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\|g(0)\|_X \\ &\qquad + \mu \sup_{t\in[0,T]} \|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\|g(0)\|_X \\ &= S_T\|g(0)\|_X + \mu\Theta_T\|g(0)\|_X. \end{aligned} \tag{3.8}$$

$$\begin{aligned}\|\Phi_2(u_n)(t)\|_X &\le \|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\|\psi - h(u_n)\|_X \\ &\le \sup_{t\in[0,T]} \|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)}\big(\|\psi\|_X + \|h(u_n)\|_X\big) \\ &\le \Lambda_T\big(\|\psi\|_X + \theta(n)\big), \end{aligned} \tag{3.9}$$

owing to the assumption (H).

$$\begin{aligned}\|\Phi_3(u_n)(t)\|_X &\le \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)}\|f\big(\tau, u_n(\tau), u_n[g]_\tau\big)\|_X d\tau \\ &\le \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)}[m(\tau)\|u_n(\tau)\|_X + m_1(\tau)|u_n[g]_\tau|_{\mathcal{B}}]d\tau,\end{aligned}$$

owing to the assumption **(F)**. Noting that

$$\begin{aligned}|u_n[g]_\tau|_{\mathcal{B}} &\le K(\tau)\sup_{r\in[0,\tau]}\|u_n(r)\|_X + M(\tau)|g|_{\mathcal{B}} \\ &\le K(\tau)\|u_n\|_C + M_T|g|_{\mathcal{B}} \le nK(\tau) + M_T|g|_{\mathcal{B}},\end{aligned}$$

we get

$$\begin{aligned}\|\Phi_3(u_n)(t)\|_X &\le \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(x)}[m(\tau)n + m_1(\tau)nK(\tau) + m_1(\tau)M_T|g|_{\mathcal{B}}]d\tau. \\ &\le n\Upsilon_T + \Gamma_T M_T|g|_{\mathcal{B}}. \end{aligned} \tag{3.10}$$

From (3.7)-(3.10), we obtain

$$\begin{aligned}\|\Phi(u_n)(t)\|_X \le \big[S_T\|g(0)\|_X + \Lambda_T(\|\psi\|_X + \theta(n))\big] \\ + \mu\Theta_T\|g(0)\|_X + n\Upsilon_T + \Gamma_T M_T|g|_{\mathcal{B}}.\end{aligned}$$

It is inferred

$$\begin{aligned}\|\Phi(u_n)\|_C \le \big[S_T\|g(0)\|_X + \Lambda_T\|\psi\|_X + \mu\Theta_T\|g(0)\|_X + \Gamma_T M_T|g|_{\mathcal{B}}\big] \\ + \Lambda_T\theta(n) + n\Upsilon_T.\end{aligned}$$

So

$$1 < \frac{1}{n}\|\Phi(u_n)\|_C \leq \frac{1}{n}\Big[S_T\|g(0)\|_X + \Lambda_T\|\psi\|_X + \mu\Theta_T\|g(0)\|_X + \Gamma_T M_T|g|_{\mathcal{B}}\Big] + \Big(\Upsilon_T + \Lambda_T\frac{\theta(n)}{n}\Big). \quad (3.11)$$

Passing to the limit in (3.11), we obtain a contradiction. Thus, Lemma 3.1 is proved. □

Now, we put

$$\Omega_T = \begin{cases} 0, \text{ if } S_{\alpha,\beta}(\cdot) \text{ is compact,} \\ \sup\limits_{t\in[0,T]} \int\limits_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)}k(\tau)d\tau, \text{ otherwise.} \end{cases}$$

Lemma 3.2. *Let* (**B**), (**F**) *and* (**H**) *be satisfied. Then*

$$\chi_T\big(\Phi(D)\big) \leq (\eta\Lambda_T + 4\Omega_T)\chi_T(D), \quad (3.12)$$

for all bounded sets $D \subset C_g$.

Proof. Take the decompositions of Φ as in Lemma 3.1. From the algebraically semi-additive property of χ_T, we have

$$\chi_T\big(\Phi(D)\big) \leq \chi_T\big(\Phi_1(D)\big) + \chi_T\big(\Phi_2(D)\big) + \chi_T\big(\Phi_3(D)\big). \quad (3.13)$$

1. Obviously, we have

$$\chi_T\big(\Phi_1(D)\big) = 0. \quad (3.14)$$

2. For $z_1, z_2 \in \Phi_2(D)$, there exist $u_1, u_2 \in D$ such that

$$z_1(t) = \Phi_2(u_1)(t), z_2(t) = \Phi_2(u_2)(t).$$

Then

$$\begin{aligned}\|z_1(t) - z_2(t)\|_X &= \|\Phi_2(u_1)(t) - \Phi_2(u_2)(t)\|_X \\ &\leq \|(\varphi_1 * S_{\alpha,\beta})(t)\|_{\mathcal{L}(X)}\|h(u_2) - h(u_1)\|_X.\end{aligned}$$

So

$$\chi_T\big(\Phi_2(D)\big) \leq \Lambda_T\chi\big(h(D)\big) \leq \Lambda_T\eta\chi_T(D) \quad (3.15)$$

3. Using Proposition 2.6, for every $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset D$ such that

$$\chi_T\big(\Phi_3(D)\big) \leq 2\chi_T\big(\{\Phi_3(u_n)\}_{n=1}^{\infty}\big) + \varepsilon. \quad (3.16)$$

$$\begin{aligned}\chi\big(\{\Phi_3(u_n)(t)\}\big) &\leq \int\limits_0^t \chi\big(S_{\alpha,\beta}(t-\tau)\big)f\big(\tau, \{u_n(\tau)\}, \{u_n[g]_\tau\}\big)d\tau \\ &\leq \int\limits_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)}\,\chi\big(f\big(\tau, \{u_n(\tau)\}, \{u_n[g]_\tau\}\big)\big)d\tau.\end{aligned}$$

It is seen that

$$\begin{aligned}\chi\big(f\big(\tau,\{u_n(\tau)\},\{u_n[g]_\tau\}\big)\big) &\le k(\tau)\big[\chi\big(\{u_n(\tau)\}\big)+\sup_{r\le 0}\chi\big(\{u_n[g](\tau+r)\}\big)\big]\\ &\le k(\tau)\big[\chi\big(\{u_n(\tau)\}\big)+\sup_{r\in[0,\tau]}\chi\big(\{u_n(r)\}\big)\big]\\ &\le 2k(\tau)\sup_{r\in[0,\tau]}\chi\big(\{u_n(r)\}\big).\end{aligned}$$

So

$$\chi\big(\{\Phi_3(u_n)(t)\}\big)\le 2\left(\int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)}k(t)d\tau\right)\sup_{r\in[0,\tau]}\chi\big(\{u_n(r)\}\big).$$

We also see that $\{\Phi_3(u_n)\}$ is an equicontinuous set. Therefore

$$\chi_T\big(\{\Phi_3(u_n)\}\big)\le 2\Omega_T\chi_T(D).$$

So

$$\chi_T\big(\Phi_3(D)\big)\le 4\Omega_T\chi_T(D). \tag{3.17}$$

The combination of (3.13)-(3.17) brings in

$$\chi_T\big(\Phi(D)\big)\le(\eta\Lambda_T+4\Omega_T)\chi_T(D). \tag{3.18}$$

Lemma 3.2 is proved. □

Theorem 3.3. *Let* $(\mathbf{B}),(\mathbf{F})$ *and* $(\mathbf{H})$ *be satisfied. Then problem (1.1)-(1.3) has at least one mild solution on* $(-\infty,T]$ *on condition that*

$$l:=\eta\Lambda_T+4\Omega_T<1, \tag{3.19}$$

$$\Upsilon_T+\Lambda_T\liminf_{n\to\infty}\frac{\theta(n)}{n}<1. \tag{3.20}$$

Proof. By inequality (3.19), the solution operator F is a χ_T-condensing. Indeed, let $D\subset M$ be a bounded set such that $\chi_T(D)\le\chi_T\big(F(D)\big)$. Applying Lemma 3.2, we have

$$\chi_T(D)\le\chi_T\big(F(D)\big)\le l\chi_T(D).$$

So $\chi_T(D)=0$, and thus, D is relatively compact.

Applying Lemma 3.1, bearing in mind (3.20), we get $F(B_R)\subset B_R$. Next, applying Theorem 2.9, the χ_T-condensing map F defined by (3.5) has fixed point set $Fix(F)\subset B_R$ which is nonempty and compact. This indicates that the problem (1.1)−(1.3) has at least one mild solution $u[g]$ with $u\in Fix(F)$. □

4. Polynomial Decay of mild solutions

In this section, we look at solution operator Φ on the following space:

$$B_R^\gamma(\rho)=B_R\cap\{u\in BC_g \text{ and } \sup_{t\in\mathbb{R}^+}t^\gamma\|u(t)\|_X\le\rho\},$$

where $BC_g=\{y\in BC(\mathbb{R}^+;X):\ y(0)=g(0)\}$, B_R is the ball in BC_g centered at the origin with radius $R>0$, and γ,ρ are positive numbers with $\gamma<\min\{\alpha,\beta-\alpha\}$. We can see that $B_R^\gamma(\rho)$ is nonempty, bounded convex closed subset of $BC(\mathbb{R}^+;X)$.

Our following task is to prove that F keeps $B_R^\gamma(\rho)$ invariant, i.e. $\Phi\big(B_R^\gamma(\rho)\big) \subset B_R^\gamma(\rho)$, and F is χ^*-condensing on $B_R^\gamma(\rho)$. To this end, we suppose that

($\mathbf{B}^*$) The phase $\mathcal{B}$ satisfies ($\mathbf{B}$) with $K \in BC(\mathbb{R}^+;\mathbb{R}^+)$ and M being such that $t^\gamma M(t) = O(1)$ as $t \to \infty$.

($\mathbf{A}^*$) A is an $\omega-$sectorial operator of angle $\beta\pi/2$ with $\omega < 0$.

($\mathbf{F}^*$) f satisfies ($\mathbf{F}$) for all $T > 0$. Moreover, we assume that $k, m, m_1 \in L^1_{loc}(\mathbb{R}^+)$, such that

$$N = \sup_{t\geq 0}\int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m(\tau)d\tau < +\infty,$$

$$N_1 = \sup_{t\geq 0}\int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)d\tau < +\infty,$$

$$M_1^\infty = \sup_{t\geq 0} m_1(t) < +\infty, \quad M^\infty = \sup_{t\geq 0} m(t) < +\infty.$$

($\mathbf{H}^*$) In ($\mathbf{H}$), space $C([0,T];X)$ is replaced by space $BC(\mathbb{R}^+;X)$.

Lemma 4.1. [17] *Let $0 < \alpha \leq \beta \leq 1, \mu > 0$ and A be an ω-sectorial operator of angle $\frac{\beta\pi}{2}$. Then*

$$t^\gamma\|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)} = O(1),\ t^\gamma\|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)} = O(1),\ \text{and}$$
$$t^\gamma\|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)} = O(1)\ \text{as}\ t\to+\infty. \tag{4.1}$$

Set

$$K^\infty = \sup_{t\geq 0} K(t),\ M^\infty = \sup_{t\geq 0} M(t), S^\infty = \sup_{t\geq 0}\|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)},$$
$$\Lambda^\infty = \sup_{t\geq 0}\|\varphi_1 * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)},\ \Theta^\infty = \sup_{t\geq 0}\|\varphi_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)},$$
$$\Gamma^\infty = \sup_{t\geq 0}\int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)M(\tau)d\tau,$$
$$\Upsilon^\infty = \sup_{t\geq 0}\int_0^t \|S_{\alpha,\beta}(t-s)\|_{\mathcal{L}(X)}\big[m(\tau) + m_1(\tau)K(\tau)\big]d\tau.$$

Lemma 4.2. *Let* $(\mathbf{B}^*), (\mathbf{A}^*), (\mathbf{F}^*), (\mathbf{H}^*)$ *hold and*

$$\Lambda^\infty \liminf_{n\to\infty}\frac{\theta(n)}{n} + \Upsilon^\infty < 1, \tag{4.2}$$
$$N + 2^\gamma N_1 K^\infty < 1. \tag{4.3}$$

Then $\Phi(B_R^\gamma(\rho)) \subset B_R^\gamma(\rho)$ for all $0 < \gamma < \min\{\alpha, \beta - \alpha\}$.

Proof. 1. By using similar argument to the proof in Lemma 3.1, we have $\Phi(B_R) \subset B_R$ only if (4.2), where B_R is the ball in BC_g.

2. Next, we prove that there is a positive number ρ such that $\Phi(B_R^\gamma(\rho)) \subset B_R^\gamma(\rho)$. Indeed, assume to the contrary that for each $n \in \mathbb{N}$ there exists $u_n \in B_R^\gamma(\rho)$ such that $\sup\limits_{t\in\mathbb{R}^+} t^\gamma \|u_n(t)\|_X \leq n$ but $\sup\limits_{t\in\mathbb{R}^+} t^\gamma \|\Phi(u_n)(t)\|_X > n$. We have

$$\begin{aligned}
\|\Phi(u_n)(t)\|_X \leq \Big\| & S_{\alpha,\beta}(t)\big(g(0)\big) + (\varphi_1 * S_{\alpha,\beta})(t)\big(\psi - h(u_n)\big) \\
& + \mu(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\big(g(0)\big)\Big\| \\
& + \Big\| \int_0^t S_{\alpha,\beta}(t-\tau) f\big(\tau, u_n(\tau), u_{n\tau}\big) d\tau \Big\| \\
:= {} & P(u_n)(t) + Q(u_n)(t). \qquad (4.4)
\end{aligned}$$

We also have

$$\begin{aligned}
t^\gamma P(u_n)(t) \leq {} & t^\gamma \|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)} \|g(0)\|_X + t^\gamma \|(\varphi_1 * S_{\alpha,\beta})(t)\|_{\mathcal{L}(X)} \big(\|\psi\|_X \\
& + \|h(u_n)\|_X\big) + t^\gamma \mu \|(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\|_{\mathcal{L}(X)} \|g(0)\|_X \\
\leq {} & t^\gamma \|S_{\alpha,\beta}(t)\|_{\mathcal{L}(X)} \|g(0)\|_X + t^\gamma \|(\varphi_1 * S_{\alpha,\beta})(t)\|_{\mathcal{L}(X)} \big(\|\psi\|_X + \theta(R)\big) \\
& t^\gamma \mu \|(\varphi_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\|_{\mathcal{L}(X)} \|g(0)\|_X
\end{aligned}$$

Applying Lemma 4.1, we achieve

$$\sup_{t\geq 0} t^\gamma P(u_n)(t) = C_1 < +\infty. \qquad (4.5)$$

By using the assumption $(\mathbf{F}^*)$, we have

$$\begin{aligned}
t^\gamma Q(u_n)(t) \leq {} & t^\gamma \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} \left\| f\big(\tau, u_n(\tau), u_{n\tau}\big)\right\|_X d\tau \\
= {} & t^\gamma \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m(\tau) \|u_n(\tau)\|_X d\tau \\
& + t^\gamma \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau) |u_n[g]_\tau|_{\mathcal{B}} d\tau \\
:= {} & R_1(u_n)(t) + R_2(u_n)(t) \qquad (4.6)
\end{aligned}$$

We also have

$$\begin{aligned}
R_1(u_n)(t) = {} & t^\gamma \int_0^{\frac{t}{2}} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m(\tau) \|u_n(\tau)\|_X d\tau \\
& + t^\gamma \int_{\frac{t}{2}}^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m(\tau) \|u_n(\tau)\|_X d\tau \\
:= {} & L_1(u_n)(t) + L_2(u_n)(t). \qquad (4.7)
\end{aligned}$$

Looking at L_1, by applying Lemma 2.3, we have

$$\begin{aligned}L_1(u_n)(t) &\leq t^\gamma \int_0^{\frac{t}{2}} \frac{C}{1+|\omega|\big((t-\tau)^{\alpha+1}+\mu(t-\tau)^\beta\big)} m(\tau)\|u_n(\tau)\|_X d\tau \\ &\leq \frac{CRM^\infty t^\gamma}{|\omega|}\int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\alpha+1}} d\tau \\ &\leq \frac{2^\alpha CRM^\infty}{\alpha|\omega|} t^{\gamma-\alpha}.\end{aligned}$$

As $\gamma < \alpha$, the last estimate deduces that

$$L_1(u_n)(t) = O(1) \text{ as } t \to +\infty.$$

Thus

$$\sup_{t\geq 0} L_1(u_n)(t) = C_1^* \tag{4.8}$$

Looking at L_2, we have

$$\begin{aligned}L_2(u_n)(t) &= t^\gamma \int_{\frac{t}{2}}^{t} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m(\tau)\tau^{-\gamma}\tau^\gamma \|u_n(\tau)\|_X d\tau \\ &\leq n2^\gamma N, \end{aligned}\tag{4.9}$$

owing to $\tau^{-\gamma} \leq \left(\frac{t}{2}\right)^{-\gamma}$. From (4.7)-(4.9), we have

$$\sup_{t\geq 0} R_1(u_n)(t) \leq C_1^* + n2^\gamma N. \tag{4.10}$$

Next, we have

$$\begin{aligned}R_2(u_n)(t) &= t^\gamma \int_0^{\frac{t}{2}} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)|u_n[g]_\tau|_{\mathcal{B}} d\tau \\ &\qquad + t^\gamma \int_{\frac{t}{2}}^{t} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)|u_n[g]_\tau|_{\mathcal{B}} d\tau \\ &:= L_3(u_n)(t) + L_4(u_n)(t).\end{aligned}\tag{4.11}$$

We see that

$$\begin{aligned}L_3(u_n)(t) &\leq t^\gamma \int_0^{\frac{t}{2}} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)\Big[K(\tau)\sup_{r\in[0,\tau]}\|u_n(r)\|_X + M(\tau)|g|_{\mathcal{B}}\Big] d\tau \\ &\leq M_1^\infty\big(K^\infty R + M^\infty |g|_{\mathcal{B}}\big) t^\gamma \int_0^{t/2} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} d\tau.\end{aligned}$$

By applying Lemma 2.3, we obtain

$$L_3(u_n)(t) \le Ct^\gamma \int_0^{t/2} \frac{1}{\omega(t-\tau)^{\alpha+1}} d\tau = C\frac{2^\alpha - 1}{\alpha|\omega|} t^{\gamma-\alpha}.$$

Since $\gamma < \alpha$, the last estimate deduces that

$$L_3(u_n)(t) = O(1) \text{ as } t \to +\infty.$$

Thus

$$\sup_{t\ge 0} L_3(u_n)(t) = \widetilde{C}_1. \tag{4.12}$$

Looking at L_4, we have

$$|u_n[g]_\tau|_{\mathcal{B}} \le K(\tau/2) \sup_{r\in[\tau/2,\tau]} \|u_n(r)\|_X + M(\tau/2)|u_n[g]_{\tau/2}|_{\mathcal{B}}.$$

$$\begin{aligned} |u_n[g]_{\tau/2}|_{\mathcal{B}} &\le K(\tau/2) \sup_{r\in[0,\tau/2]} \|u_n(r)\|_X + M(\tau/2)|g|_{\mathcal{B}} \\ &\le K^\infty R + M^\infty |g|_{\mathcal{B}} = C^*. \end{aligned}$$

It is inferred

$$|u_n[g]_\tau|_{\mathcal{B}} \le K(\tau/2) \sup_{r\in[\tau/2,\tau]} \|u_n(r)\|_X + C^* M(\tau/2).$$

So

$$\begin{aligned} (\tau/2)^\gamma |u_n[g]_\tau|_{\mathcal{B}} &\le K(\tau/2)\big(\tau/2\big)^\gamma \sup_{r\in[\tau/2,\tau]} \|u_n(r)\|_X + C^*\big(\tau/2\big)^\gamma M(\tau/2) \\ &\le K(\tau/2) \sup_{r\in[\tau/2,\tau]} r^\gamma \|u_n(r)\|_X + C^*\widetilde{C} \le K^\infty n + C^*\widetilde{C}. \end{aligned}$$

Hence

$$\begin{aligned} L_4(u_n)(t) &= t^\gamma \int_{\frac{t}{2}}^{t} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau)|u_n[g]_\tau|_{\mathcal{B}} d\tau \\ &\le 2^\gamma (K^\infty n + C^*\widetilde{C}) \int_{\frac{t}{2}}^{t} \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} m_1(\tau) d\tau \\ &\le 2^\gamma (K^\infty n + C^*\widetilde{C}) N_1. \end{aligned} \tag{4.13}$$

From (4.11)-(4.13), we have

$$\sup_{t\ge 0} R_2(u_n)(t) \le C_2 + 2^\gamma (K^\infty n + C^*\widetilde{C}) N_1 \le C_3 + 2^\gamma N_1 K^\infty n. \tag{4.14}$$

It follows from (4.6), (4.10), and (4.14) that

$$\sup_{t\ge 0} t^\gamma Q(u_n)(t) \le C_4 + (N + 2^\gamma N_1 K^\infty) n. \tag{4.15}$$

Gathering the results of (4.4), (4.5), and (4.15), we have

$$\sup_{t\geq 0} t^\gamma \|\Phi(u_n)(t)\|_X \leq C_5 + (N + 2^\gamma N_1 K^\infty)n.$$

It is inferred that

$$1 < \frac{1}{n}\sup_{t\geq 0} t^\gamma \|\Phi(u_n)(t)\|_X \leq \frac{C_5}{n} + (N + 2^\gamma N_1 K^\infty).$$

Passing the last relation into limits as $n \to +\infty$, we get a contradiction to (4.3). □

Now, we set

$$\Omega^\infty = \begin{cases} 0, \text{ if } S_{\alpha,\beta}(\cdot) \text{ is compact}, \\ \sup\limits_{t\geq 0}\int\limits_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} k(\tau)d\tau, \text{ otherwise}. \end{cases}$$

Lemma 4.3. *Let* $(\mathbf{B}^*), (\mathbf{A}^*), (\mathbf{F}^*), (\mathbf{H}^*)$ *hold and*

$$l_\infty := \Lambda^\infty \eta + 4\Omega^\infty < 1. \tag{4.16}$$

Then the solution operator Φ *is* χ^*-*condensing.*

Proof. Let $D \subset B_R^\gamma(\rho)$ be a bounded set. We have

$$\chi^*\big(\Phi(D)\big) = \chi_\infty\big(\Phi(D)\big) + d_\infty\big(\Phi(D)\big). \tag{4.17}$$

1. Applying the Hausdorf MNC χ and using the results in Lemma 3.2, we have

$$\begin{aligned} &\chi_\infty(\Phi_1(D)) = 0 \\ &\chi_\infty(\Phi_2(D)) \leq \Lambda^\infty \eta \chi_\infty(D) \\ &\chi_\infty(\Phi_3(D)) \leq 4\Omega^\infty \chi_\infty(D). \end{aligned}$$

Take the decompositions of Φ as in Lemma 3.2, we obtain

$$\begin{aligned} \chi_\infty(\Phi(D)) &\leq \chi(\Phi_1(D)) + \chi_\infty(\Phi_2(D)) + \chi_\infty(\Phi_3(D)) \\ &\leq (\Lambda^\infty \eta + 4\Omega^\infty)\chi_\infty(D). \end{aligned} \tag{4.18}$$

2. Let $D \subset B_R^\gamma(\rho)$ be a bounded set. Then, for all $u \in D$, we have

$$t^\gamma \|\Phi(u)(t)\|_X \leq \rho \quad \text{as } t \to \infty.$$

This means that $\|\Phi(u)(t)\|_X \leq \rho t^{-\gamma}$, $\forall u \in D$, for all large t. Similarly, for a large T, one has $d_T(\Phi(D)) \leq \rho T^{-\gamma}$. Thus

$$d_\infty\big(\Phi(D)\big) = \lim_{T\to\infty} d_T(\Phi(D)) = 0. \tag{4.19}$$

From (4.17), (4.18) and (4.19), the proof is completed. □

From the combination of Lemma (4.2) and Lemma (4.3), we obtain the following theorem.

Theorem 4.4. *Under the assumptions of Lemma 4.2 and Lemma 4.3, problem (1.1)-(1.3) has at least a mild solution on* $\mathbb{R}$*, such that* $t^\gamma\|u(t)\| = O(1)$ *as* $t \to +\infty$*.*

Proof. By the inequality (4.16), the solution operator Φ is a χ^*-condensing, owing to Lemma 4.3. Indeed, if $D \subset B_R^\gamma(\rho)$ is bounded such that $\chi^*(D) \leq \chi^*\big(\Phi(D)\big)$. Applying Lemma 4.3, we obtain

$$\chi^*(D) \leq \chi^*\big(\Phi(D)\big) \leq l_\infty \chi^*(D).$$

Therefore $\chi^*(D) = 0$, and so D is relatively compact.
From assumptions (4.2), (4.3) and Lemma 4.2, we have $\Phi(B_R^\gamma(\rho)) \subset B_R^\gamma(\rho)$. So applying Theorem 2.9, the solution operator Φ defined by (3.5) has a compact and nonempty fixed point set $Fix(\Phi) \subset B_R^\gamma(\rho)$. Hence, the problem (1.1)$-$(1.3) has a mild solution $u[g](t), t \in \mathbb{R}$ with $u \in BC_R^\gamma(\beta)$. □

5. An example

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ and

$$L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial_i \partial_j}$$

be a uniformly elliptic second order differential operator, i.e, there exists a positive constant c such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n, x \in \Omega.$$

With $0 < \alpha \leq \beta \leq 1, \mu > 0, a_0 > 0$, and each $t \in [0,T], x \in \Omega$, we consider the following problem:

$$D_C^{\alpha+1}u(t,x) + \mu D_C^\beta u(t,x) - Lu(t,x) + a_0 u(t,x) = F\big(t, u(t,x), u_t\big), \tag{5.1}$$
$$u(s) = g(s),\ s \leq 0, \tag{5.2}$$
$$u'(0) + h(u) = \psi. \tag{5.3}$$

Let $X = L^2(\Omega)$, $A = L - a_0$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in \mathcal{B} = C_\gamma$, $\gamma \in (0,\alpha)$, $\psi \in X$; $0 < t_1 < t_2 < \cdots < t_N < \infty$; C_i $(i = 1, \cdots, N)$ are constants. Then problems (5.1)-(5.3) is in the form of the abstract model (1.1)-(1.3) with

$$\begin{aligned} f\big(t,x,u(t,x),u_t\big) &:= \varphi_\alpha * F\big(t,x,u(t,x),u_t\big) \\ &= \widehat{f}_1\big(t,x,u(t,x)\big) + \mu(t,x) \int_{-\infty}^{0} \int_\Omega \nu(s,y) \widehat{f}_2\big(y, u(t+s,y)\big) dy ds \end{aligned}$$

$$\text{and} \quad h(u) = \sum_{i=1}^N C_i u(t,\cdot).$$

(**A**) It is known that (see [25, Theorem 3.6]), L is a sectorial operator of angle $\frac{\pi}{2}$ (and hence of angle $\frac{\beta\pi}{2}$). Therefore, we have A is an $\omega-$sectorial operator of angle $\frac{\beta\pi}{2}$ with $\omega = -a_0 < 0$.

(**F**) The nonlinear function f, in which

$$\widehat{f}_1 : \mathbb{R}^+ \times \Omega \times \mathbb{R} \to \mathbb{R}, \quad \mu : \ (-\infty, 0] \times \Omega \to \mathbb{R},$$

$$\nu : (-\infty; 0] \times \Omega \to \mathbb{R}, \quad \widehat{f}_2 : \ \Omega \times \mathbb{R} \to \mathbb{R}$$

such that

(a) $\widehat{f}_1$ is a continuous function such that $\widehat{f}_1(t, x, 0) = 0$ and

$$|\widehat{f}_1(x, t, z_1) - \widehat{f}_1(t, x, z_2)| \le p(t)|z_1 - z_2|$$

for all $x \in \Omega$ and $\forall\ z_1, z_2 \in \mathbb{R}$, where $p \in L^1_{\text{loc}}(\mathbb{R}^+)$.

(b) $\mu \in BC(\mathbb{R}^+; L^2(\Omega))$.

(c) ν is continuous and satisfies $|\nu(s, y)| \le C_\nu e^{\nu_0 s}$, $\forall\, s \in (-\infty, 0], y \in \Omega$, where $\nu_0 > \gamma$.

(d) $\widehat{f}_2$ is continuous and $|\widehat{f}_2(y, z)| \le q(y)|z|$ for $q \in L^2(\Omega)$.

Let $f : \mathbb{R}^+ \times X \times \mathcal{B} \to X$ such that

$$f(t, v, w)(x) = f_1(t, v)(x) + f_2(t, w)(x)$$

where

$$f_1(t, v)(x) = \widehat{f}_1(t, x, v(x))$$
$$f_2(t, w)(x) = \mu(t, x) \int_{-\infty}^{0} \int_{\Omega} \nu(s, y) \widehat{f}_2(y, w(s, y)) dy ds.$$

Concerning f_1, we have

$$\|f_1(t, v_1) - f_1(t, v_2)\|_X \le p(t)\|v_1 - v_2\|_X, \ \forall v_1, v_2 \in X.$$

This implies

$$\chi(f_1(t, V)) \le p(t)\chi(V), \ \text{ for all bounded set } V \subset X.$$

Regarding f_2, using the Hölder inequality, we have

$$\begin{aligned}
\|f_2(t, w)\|_X^2 &= \|\mu(t, \cdot)\|_X^2 \left(\int_{-\infty}^{0} \int_{\Omega} \nu(s, y) \widehat{f}_2(y, w(s, y)) dy ds \right)^2 \\
&\le \|\mu(t, \cdot)\|_X^2 C_\nu^2 \left(\int_{-\infty}^{0} e^{\nu_0 s} \int_{\Omega} q(y)|w(s, y)| dy ds \right)^2 \\
&\le \|\mu(t, \cdot)\|_X^2 C_\nu^2 \|q\|_X^2 \left(\int_{-\infty}^{0} e^{\nu_0 s} \|w(s)\|_X ds \right)^2 \\
&\le \|\mu(t, \cdot)\|_X^2 C_\nu^2 \|q\|_X^2 |w|_{\mathcal{B}}^2 \left(\int_{-\infty}^{0} e^{(\nu_0 - \gamma)s} ds \right)^2 \\
&\le \frac{1}{(\nu_0 - \gamma)^2} \|\mu(t, \cdot)\|_X^2 C_\nu^2 \|q\|_X^2 |w|_{\mathcal{B}}^2,
\end{aligned}$$

where we have taken the hypothesis $\nu_0 > \gamma$ into consideration. Then we obtain

$$\|f_2(t, w)\|_X \le \frac{1}{\nu_0 - \gamma} \|\mu(t, \cdot)\|_X C_\nu \|q\|_X |w|_{\mathcal{B}}.$$

On the other hand, for any bounded set $W \subset \mathcal{B}$, we see that

$$f_2(t, W) \subset \{\lambda\mu(t,\cdot) : \lambda \in \mathbb{R}\},$$

that is, $f_2(t, W)$ lies in an one dimensional subspace of X. Hence

$$\chi(f_2(t, W)) = 0.$$

Therefore, f fulfills (F^*) with $k(t) = p(t)$ and

$$m(t) = p(t), m_1(t) = \frac{1}{\nu_0 - \gamma}\|\mu(t,\cdot)\|_X C_\nu \|q\|_X.$$

Therefore

$$N = \sup_{t\geq 0} \int_0^t \|S_{\alpha,\beta}(t-\tau)\|_{\mathcal{L}(X)} p(\tau) d\tau,$$

$$N_1 = \frac{C_\nu \|q\|_X}{\nu_0 - \gamma} \sup_{t\geq 0} \int_0^t \|S_{\alpha,\beta}\|_{\mathcal{L}(X)} \|\mu(\tau,\cdot)\|_X d\tau$$

(**H**) Regarding the function h, we have

$$\|h(u_1) - h(u_2)\|_X \leq \sum_{i=1}^N |C_i| \|u_1(t_i,\cdot) - u_2(t_i,\cdot)\|_X \leq \left(\sum_{i=1}^N |C_i|\right) \|u_1 - u_2\|_C.$$

Then

$$\chi(h(\Omega)) \leq \left(\sum_{i=1}^N |C_i|\right) \chi_\infty(\Omega).$$

The assumption (H*)(ii) is satisfied with $\eta = \sum_{i=1}^N |C_i|$. On the other hand, it is easily seen that

$$\|h(u)\| \leq \left(\sum_{i=1}^N |C_i|\right) \|u\|_\infty,$$

which implies (H*)(i), we have

$$\theta(\|u\|_\infty) = \left(\sum_{i=1}^N |C_i|\right) \|u\|_\infty.$$

By the above settings and simple computations, we get

$$K^\infty = 1$$

Applying Theorem 4.4, we can conclude that, the problem (5.1)-(5.3) has at least one mild solution on $\mathbb{R}$ satisfying $t^\gamma \|u(t)\|_{L^2(\Omega)} = O(1)$ as $t \to +\infty$ provided that

$$\Lambda^\infty \sum_{i=1}^N C_i + \Upsilon^\infty < 1, \quad N + 2^\gamma N_1 < 1, \quad \Lambda^\infty \sum_{i=1}^N C_i + \Omega^\infty < 1.$$

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