Fixed Point Theory, 23(2022), No. 1, 279-292
DOI: 10.24193/fpt-ro.2022.1.18
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPLICATIONS OF FIBRE CONTRACTION PRINCIPLE TO SOME CLASSES OF FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. Let $a < c < b$ real numbers, $(\mathbb{B}, |\cdot|)$ a (real or complex) Banach space, $H \in C([a,b]\times[a,c]\times\mathbb{B},\mathbb{B})$, $K \in C([a,b]^2 \times \mathbb{B},\mathbb{B})$, $g \in C([a,b],\mathbb{B})$, $A : C([a,c],\mathbb{B}) \to C([a,c],\mathbb{B})$ and $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$. In this paper we study the following functional integral equation,

$$x(t) = \int_a^c H(t,s,A(x)(s))ds + \int_a^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b].$$

By a new variant of fibre contraction principle (A. Petruşel, I.A. Rus, M.A. Şerban, Some variants of fibre contraction principle and applications: from existence to the convergence of successive approximations, Fixed Point Theory, 22 (2021), no. 2, 795-808) we give existence, uniqueness and convergence of successive approximations results for this equation. In the case of ordered Banach space $\mathbb{B}$, Gronwall-type and comparison-type results are also given.

Key Words and Phrases: Functional integral equation, Volterra operator, Picard operator, fibre contraction principle, Gronwall lemma, comparison lemma.
2020 Mathematics Subject Classification: 47N05, 47H10, 45D05, 47H09, 54H25.

1. Introduction

In this paper we study the following functional integral equation,

$$x(t) = \int_a^c H(t,s,A(x)(s))ds + \int_a^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b], \tag{1.1}$$

where $a < c < b$ are real numbers, $(\mathbb{B}, |\cdot|)$ is a Banach space, $H \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $K \in C([a,b]^2 \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$ and $A : C([a,c], \mathbb{B}) \to C([a,c], \mathbb{B})$ and $B : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B})$ are given operators.

For some examples of such integral equations see [5], [6], [16], [28], [2], [4], [7].

Let $V : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B})$ be defined by

$$V(x)(t) := \int_a^c H(t,s,A(x)(s))ds + \int_a^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b].$$

In this paper we consider on the spaces of continuous functions max-norms.

Let us suppose that

(C_1) $\exists L_H > 0 : |H(t,s,\eta_1) - H(t,s,\eta_2)| \leq L_H |\eta_1 - \eta_2|$, for all $t \in [a,b]$, $s \in [a,c]$, $\eta_1, \eta_2 \in \mathbb{B}$;

(C_2) $\exists L_K > 0 : |K(t,s,\eta_1) - K(t,s,\eta_2)| \leq L_K |\eta_1 - \eta_2|$, for all $t, s \in [a,b]$, $\eta_1, \eta_2 \in \mathbb{B}$;

(C_3) $\exists L_A > 0 : \max\limits_{[a,c]} |A(y)(t) - A(z)(t)| \leq L_A \max\limits_{[a,c]} |y(t) - z(t)|$, for all $y, z \in C([a,c], \mathbb{B})$;

(C_4) $\exists L_B > 0 : |B(y)(t) - B(z)(t)| \leq L_B \max\limits_{[a,t]} |y(s) - z(s)|$, for all $t \in [a,b]$.

If we apply the contraction principle, in a standard way, for equation (1.1), we have the following result:

Theorem 1.1. *In addition to the above conditions we suppose that:*

(C_5') $L_H L_A (c-a) + L_K L_B (b-a) < 1$.

Then the equation (1.1) has in $C([a,b], \mathbb{B})$ *a unique solutions,* x^* *and* $x^* = \lim\limits_{n\to\infty} x_n$, *where* x_n *is defined by* $x_0 \in C([a,b], \mathbb{B})$, $x_{n+1} = V(x_n)$, $n \in \mathbb{N}$, *i.e.,* V *is a Picard operator.*

The aim of this paper is to improve condition (C_5'), obtaining the same conclusions. In order to do this we shall apply instead of contraction principle, a new variant of fibre contraction principle, variant given in [13].

In a similar way we study the equation

$$x(t) = \int_b^c H(t,s,A(x)(s))ds + \int_b^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b],$$

with suitable conditions on H, K, A and B.

Throughout this paper we shall use the notations from [28], [22] and [13].

2. Preliminaries

2.1. **Weakly Picard operators.** Let $(X, \to)$ be an L-space, where X is a nonempty set and $\to$ is a convergence structure defined on X. If $T : X \to X$ is an operator, then we denote by $F_T := \{x \in X : x = T(x)\}$ the fixed point set of T.

In the above context, $T : X \to X$ is called a weakly Picard operator (briefly WPO) if, for each $x \in X$, the sequence of Picard iterations $(T^n(x))_{n\in\mathbb{N}}$ converges with respect to $\to$ to a fixed point of T. In particular, if $F_T = \{x^*\}$, then T is called a Picard operator (briefly PO).

If $T : X \to X$ is a WPO, then we define a set retraction $T^\infty : X \to F_T$ by the formula

$$T^\infty(x) := \lim_{n\to\infty} T^n(x).$$

If T is PO with its unique fixed point x^*, then $T^\infty(X) = \{x^*\}$.

For the weakly Picard operator theory see [21], [29], [25], [27], [30].

Theorem 2.1. *(Abstract Gronwall lemma)([21], [29]) Let $(X, \to, \leq)$ be an ordered L-space and $T : X \to X$ be an operator. We suppose that:*

(i) *T is a WPO;*
(ii) *T is increasing.*

Then:

(a) *$x \leq T(x) \Longrightarrow x \leq T^\infty(x)$;*
(b) *$x \geq T(x) \Longrightarrow x \geq T^\infty(x)$.*

Theorem 2.2. *(Abstract Comparison lemma)([21], [29]) Let $(X, \to, \leq)$ be an ordered L-space and $T, U, V : X \to X$ be three operators. We suppose that:*

(i) *$T \leq U \leq V$;*
(ii) *T, U and V are WPOs;*
(iii) *the operator U is increasing.*

Then:

$$x \leq y \leq z \ \Rightarrow\ T^\infty(x) \leq U^\infty(y) \leq V^\infty(z).$$

2.2. **Fibre contraction principle.** The standard fibre contraction principle has the following statement:

Theorem 2.3. *Let $(X_0, \to)$ be an L-space. For $m \in \mathbb{N}^*$, let (X_i, d_i), $i \in \{1, \ldots, m\}$ be complete metric spaces. Let $T_0 : X_0 \to X_0$ be an operator and, for $i \in \{1, \ldots, m\}$, let us consider $T_i : X_0 \times X_1 \times \cdots \times X_i \to X_i$. We suppose that:*

(1) *T_0 is a WPO;*
(2) *for each $i \in \{1, 2, \ldots, m\}$, the operators $T_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i$ are l_i-contractions;*
(3) *for each $i \in \{1, 2, \ldots, m\}$, the operators T_i are continuous.*

Then, the operator $T = (T_0, T_1, \ldots, T_m) : \prod_{i=0}^{m} X_i \to \prod_{i=0}^{m} X_i$, defined by

$$T(x_0, \ldots, x_m) := (T_0(x_0), T_1(x_0, x_1), \ldots, T_m(x_0, \ldots, x_m))$$

is a WPO. Moreover, when T_0 is a PO, then T is a PO too.

For other results regarding fibre contractions, see [11], [23], [30], [25], [34], [31], ..., fibre generalized contractions, see [20], [32], [33], [34], ..., fibre generalized contractions on generalized metric spaces, see [1], [3], [18], [24], [20], ...

In [19] it is obtained a new type of fibre contraction principle in the following settings:

Let (X_i, d_i) ($i \in \{1, ..., m\}$ where $m \geq 2$) be metric spaces and $U_1 \subset X_1 \times X_2$, $U_2 \subset U_1 \times X_3, \ldots, U_{m-1} \subset U_{m-2} \times X_m$, be nonempty subsets.

For $x \in X_1$, we define

$$U_{1x} := \{x_2 \in X_2 \mid (x, x_2) \in U_1\},$$

for $x \in U_1$, we define

$$U_{2x} := \{x_3 \in X_3 \mid (x, x_3) \in U_2\}, \ldots,$$

and for $x \in U_{m-2}$, we define

$$U_{m-1x} := \{x_m \in X_m \mid (x, x_m) \in U_{m-1}\}.$$

We suppose that $U_{1x}, U_{2x}, \ldots, U_{m-1x}$ are nonempty.

If $T_1 : X_1 \to X_1$, $T_2 : U_1 \to X_2, \ldots, T_m : U_{m-1} \to X_m$, then we consider the operator

$$T : U_{m-1} \to X_1 \times X_2 \times \ldots \times X_m,$$

defined by

$$T(x_1, \ldots, x_m) := (T_1(x_1), T_2(x_1, x_2), \ldots, T_m(x_1, x_2, \ldots, x_m)).$$

The result is the following.

Theorem 2.4. *([19]) We suppose that:*

(1) (X_i, d_i), $i \in \{2, ..., m\}$ are complete metric spaces and U_i, $i \in \{1, ..., m-1\}$ are closed subsets;

(2) $(T_1, T_2, \ldots, T_{i+1})(U_i) \subset U_i$, $i \in \{1, ..., m-1\}$;

(3) T_1 is a WPO;

(4) there exist $L_i > 0$ and $0 < l_i < 1$, $i \in \{1, ..., m-1\}$ such that

$$d_{i+1}(T_{i+1}(x, y,), T_{i+1}(\widetilde{x}, \widetilde{y})) \leq L_i \widetilde{d}_i(x, \widetilde{x}) + l_i d_{i+1}(y, \widetilde{y}),$$

for all $(x, y), (\widetilde{x}, \widetilde{y}) \in U_i$, $i \in \{1, ..., m-1\}$, where $\widetilde{d}_i$ is a metric induced by $d_1, \ldots, d_i$ on $X_1 \times \cdots \times X_i$, defined by $\tilde{d}_i := \max\{d_1, \ldots, d_i\}$.

Then T is WPO. If T_1 is PO, then T is a PO too.

3. Abstract Volterra operators on spaces of continuous functions of one variable

By definition, an operator $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ is forward Volterra operator if the following implication holds:

$$x, y \in C([a,b],\mathbb{B}),\ x|_{[a,t]} = y|_{[a,t]} \Rightarrow V(x)|_{[a,t]} = V(y)|_{[a,t]}\,,$$

for all $t \in [a,b]$.

An operator $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ is backward Volterra operator iff:

$$x, y \in C([a,b],\mathbb{B}),\ x|_{[t,b]} = y|_{[t,b]} \Rightarrow V(x)|_{[t,b]} = V(y)|_{[t,b]}\,,$$

for all $t \in [a,b]$.

If $a < c < b$ then V is forward Volterra operator w.r.t. the interval $[c,b]$ iff

$$x, y \in C([a,b],\mathbb{B}),\ x|_{[a,t]} = y|_{[a,t]} \Rightarrow V(x)|_{[a,t]} = V(y)|_{[a,t]}\,,\ \text{for all } t \in [c,b].$$

The operator V is backward Volterra operator w.r.t. the interval $[a,c]$ iff:

$$x, y \in C([a,b],\mathbb{B}),\ x|_{[t,b]} = y|_{[t,b]} \Rightarrow V(x)|_{[t,b]} = V(y)|_{[t,b]}\,,\ \text{for all } t \in [a,c].$$

Example 3.1. For $f \in C([a,b] \times \mathbb{R}^p, \mathbb{R}^p)$ let us consider the Cauchy problem

$$x'(t) = f(t, x(t)),\ t \in [a,b],$$
$$x(a) = \alpha.$$

This problem is equivalent with the following integral equation

$$x(t) = \alpha + \int_a^t f(s, x(s))ds,\ t \in [a,b].$$

Let $V : C([a,b],\mathbb{R}^p) \to C([a,b],\mathbb{R}^p)$ be defined by

$$V(x)(t) := \alpha + \int_a^t f(s, x(s))ds,\ t \in [a,b].$$

The operator V is a forward Volterra operator.

If we consider the Cauchy problem

$$x'(t) = f(t, x(t)),\ t \in [a,b],$$
$$x(b) = \beta,$$

then this problem is equivalent with the integral equation,

$$x(t) = \beta + \int_b^t f(s, x(s))ds,\ t \in [a,b].$$

In this case the corresponding operator, $V : C([a,b],\mathbb{R}^p) \to C([a,b],\mathbb{R}^p)$ defined by the second part of this integral equation is backward Volterra operator.

If for $t_0 \in]a,b[$, we consider the Cauchy problem

$$x'(t) = f(t, x(t)),\ t \in [a,b],$$
$$x(t_0) = \gamma,$$

then this problem is equivalent with the integral equation

$$x(t) = \gamma + \int_{t_0}^{t} f(s, x(s))ds,\ t \in [a, b],$$

and the corresponding operator V is a forward Volterra operator with respect to $[t_0, b]$ and is backward Volterra operator w.r.t. the interval $[a, t_0]$.

Example 3.2. Let the operator $V : C[a, b] \to C[a, b]$ defined by $V(x)(t) := x(g(t))$, where $g \in C([a, b], [a, b])$. If $g(t) \leq t$, $\forall t \in [a, b]$, then V is forward Volterra operator and if $g(t) \geq t$, $\forall t \in [a, b]$ then V is a backward Volterra operator.

Example 3.3. $V : C[a, b] \to C[a, b]$, $V(x)(t) := \max\limits_{[a,t]} u(\tau)$, $t \in [a, b]$ is a forward Volterra operator.

Example 3.4. Let $A : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ be a forward Volterra operator. Then the operator $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ defined by

$$V(x)(t) := \int_{a}^{t} A(u)(s)ds$$

is a forward Volterra operator.

Example 3.5. ([2], [16]) Let $V : C([0, b], \mathbb{B}) \to C([0, b], \mathbb{B})$ be such that

$$|V(x)(t) - V(y)(t)| \leq \alpha\, |x(t) - y(t)| + \frac{\gamma}{t^\beta} \int_0^t |x(s) - y(s)|\, ds,$$

$\forall x, y \in C([0, b], \mathbb{B})$, $\forall t \in [0, b]$, where $\alpha, \beta \in [0, 1[$ and $\gamma > 0$ are given real numbers.
The operator V is a forward Volterra operator.

For more considerations on abstract Volterra operator, see [9], [36]. For other examples see [2], [4], [5], [6], [7], [12], [14], [16], [17], [22], [35], ...

4. Basic results

Let us consider the equation (1.1) in the conditions $(C_1) - (C_4)$. For $m \in \mathbb{N}, m \geq 2$ we shall use the following notations:

$$t_0 := c,\ t_k := c + \frac{k(b-c)}{m},\ k = \overline{1, m},\ X_0 = C([a, c], \mathbb{B}),$$

$$X_i = C([t_{i-1}, t_i], \mathbb{B}),\ X = \prod_{i=0}^{m} X_i.$$

We consider the spaces of continuous functions with the max-norms. In order to use the variant of fibre contraction principle given by Theorem 2.4, we need the following subsets:

$$U_i = \{(x_0, x_1, \ldots, x_i) \in \prod_{k=0}^{i} X_k |\ x_k(t_k) = x_{k+1}(t_k),\ k = \overline{1, m-1}\},\ i = \overline{1, m}.$$

For $x \in X_0$,

$$U_{1x} := \{x_1 \in X_1 |\ (x, x_1) \in U_1\},$$

for $x \in X_{i-1}$,

$$U_{ix} := \{x_i \in X_i |\ (x, x_i) \in U_i\},\ i = \overline{2, m}.$$

We remark that, $U_i, U_{ix},\ i = \overline{1, m}$ are nonempty closed subsets.

We also need the following operators:

$$R_i : C([a, t_i], \mathbb{B}) \to \prod_{k=0}^{i} X_k,\ R_i(x) = \left(x|_{[a,t_0]}, x|_{[t_0,t_1]}, \dots, x|_{[t_{i-1},t_i]} \right),\ i = \overline{1, m}.$$

It is clear that, $R_i\,(C([a, t_i], \mathbb{B})) = U_i$ and $R_i : C([a, t_i], \mathbb{B}) \to U_i$ is an increasing homeomorphism.

Since the operator, $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ defined by $V(x)(t) :=$ second part of equation (1.1), is a forward Volterra operator on $[c, b]$, it induces the following operators:

$$\begin{aligned} T_0 \quad &: \quad X_0 \to X_0, \\ T_0(x_0)(t) \quad &= \quad V(x_0)(t),\ t \in [a, c], \end{aligned}$$

$$T_1 : U_1 \to X_1,$$

$$T_1(x_0, x_1)(t) := \int_a^c H(t, s, A(x_0)(s))ds + \int_a^c K(t, s, B(x_0)(s))ds$$
$$+ \int_c^t K(t, s, B(R_1^{-1}(x_0, x_1)(s))ds + g(t),\ t \in [c, t_1],$$

$$T_2 : U_2 \to X_2,$$

$$T_1(x_0, x_1, x_2)(t) := \int_a^c H(t, s, A(x_0)(s))ds + \int_a^c K(t, s, B(x_0)(s))ds$$
$$+ \int_c^{t_1} K(t, s, B(R_1^{-1}(x_0, x_1)(s))ds$$
$$+ \int_{t_1}^t K(t, s, B(R_2^{-1}(x_0, x_1, x_2)(s))ds + g(t),\ t \in [t_1, t_2],$$

$$\dots$$

$$T_m : U_m \to X_m,$$

$$T_m(x_0, x_1, \dots, x_m)(t) := \int_a^c H(t, s, A(x_0)(s))ds + \int_a^c K(t, s, B(x_0)(s))ds + \dots$$
$$+ \int_{t_{m-1}}^t K(t, s, B(R_m^{-1}(x_0, x_1, \dots, x_m)(s))ds + g(t),\ t \in [t_{m-1}, b].$$

Let

$$\begin{aligned} T \quad &: \quad = (T_0, T_1, \dots, T_m), \\ T(x_0, x_1, \dots, x_m) \quad &: \quad = (T_0(x_0),\ T_1(x_0, x_1), \dots, T_m(x_0, x_1, \dots, x_m)). \end{aligned}$$

If on the cartesian product we consider max-norms, the operators $R_i,\ i = \overline{1, m}$ are isometries. From the above definitions, we remark that

$$(T_0, T_1)(U_1) \subset U_1,\ (T_0, T_1, \dots, T_m)(U_m) \subset U_m.$$

In the conditions $(C_1) - (C_4)$ we have that: T_0 is $(L_H L_A + L_K L_B)(c - a)$-Lipschitz.

If we suppose that

(C_5) $(L_H L_A + L_K L_B)(c-a) < 1$

then we are in the conditions of Theorem 2.4 with

$$L_i = \max\left\{(L_H L_A + L_K L_B)(c-a), \frac{L_K L_B(b-c)}{m}\right\}$$

and $l_i = \frac{L_K L_B(b-c)}{m}$, with suitable $m \in \mathbb{N}$.

From this theorem we have that T is PO.

Since $V = R_m^{-1} T R_m$ and $V^n = R_m^{-1} T^n R_m$, it follows that V is PO.

So, we have:

Theorem 4.1. *We consider the equation (1.1) in the condition* $(C_1)-(C_5)$*. Under these conditions we have that:*

(i) *The equation (1.1) has in* $C([a,b],\mathbb{B})$ *a unique solution,* x^*.

(ii) *The sequence,* $(x_n)_{n\in\mathbb{N}}$*, defined by*

$$\begin{aligned} x^0 &\in C([a,b],\mathbb{B}), \\ x^{n+1}(t) &= \int_a^c H(t,s,A(x^n)(s))ds + \int_a^t K(t,s,B(x^n)(s))ds + g(t),\ t \in [a,b], \end{aligned}$$

converges to x^**, i.e., the operator* V *is PO.*

Remark 4.2. If we take, $\mathbb{B} = \mathbb{R}^p$ or $\mathbb{C}^p$ or another finite dimensional Banach space, then Theorem 4.1 is a result for a system of functional integral equations.

Remark 4.3. If we take, $\mathbb{B} := l^p(\mathbb{C})$ or $\mathbb{B} := l^p(\mathbb{R})$, $1 \le p \le \infty$, or another Banach space of sequences, Theorem 4.1 is a result for an infinite system of functional integral equations.

Remark 4.4. For some particular cases of A and B our result is in connection with some result given in [2], [4], [5], [6], [7], [10], [14], [26], [28], [35], [8], [12], [13], [15].

5. Equations with backward Volterra operators

In this section we consider the following integral equation

$$x(t) = \int_b^c H(t,s,A(x)(s))ds + \int_b^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b], \tag{5.1}$$

where $a < c < b$ are real numbers, $(\mathbb{B}, |\cdot|)$ is a Banach space, $H \in C([a,b]\times[c,b]\times\mathbb{B},\mathbb{B})$, $K \in C([a,b]^2\times\mathbb{B},\mathbb{B})$, $g \in C([a,b],\mathbb{B})$ and $A : C([c,b],\mathbb{B}) \to C([c,b],\mathbb{B})$ and $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ are operators. We suppose that:

$(\widetilde{C}_1)$ $\exists L_H > 0 : |H(t,s,\eta_1) - H(t,s,\eta_2)| \le L_H |\eta_1 - \eta_2|$, for all $t \in [a,b]$, $s \in [c,b]$, $\eta_1,\eta_2 \in \mathbb{B}$;

$(\widetilde{C}_2)$ $\exists L_K > 0 : |K(t,s,\eta_1) - K(t,s,\eta_2)| \le L_K |\eta_1 - \eta_2|$, for all $t,s \in [a,b]$, $\eta_1,\eta_2 \in \mathbb{B}$;

$(\widetilde{C}_3)$ $\exists L_A > 0 : \max_{[c,b]} |A(y)(t) - A(z)(t)| \le L_A \max_{[c,b]} |y(t) - z(t)|$, for all $y,z \in C([c,b],\mathbb{B})$;

$(\widetilde{C}_4)$ $\exists L_B > 0 : |B(y)(t) - B(z)(t)| \le L_B \max_{[t,b]} |y(s) - z(s)|$, for all $t \in [a,b]$;

$(\widetilde{C}_5)$ $(L_H L_A + L_K L_B)(b-c) < 1.$

For $m \in \mathbb{N}, m \geq 2$ we shall use the following notations:

$$t_0 := c,\ t_k := c - \frac{k(c-a)}{m},\ k = \overline{1,m},\ X_0 = C([c,b],\mathbb{B}),$$

$$X_i = C([t_{i+1}, t_i], \mathbb{B}), X = \prod_{i=0}^{m} X_i.$$

We will apply again Theorem 2.4 in the following settings. The continuous functions spaces are endowed with the max-norms. We consider the following subsets:

$$U_i = \{(x_0, x_1, \ldots, x_i) \in \prod_{k=0}^{i} X_k |\ x_k(t_k) = x_{k+1}(t_k),\ k = \overline{0, m-1}\},\ i = \overline{1,m},$$

$$U_{1x} := \{x_1 \in X_1 |\ (x, x_1) \in U_1\},\quad \text{for } x \in X_0.$$

For $x \in X_0$, for $x \in X_{i-1}$, $U_{ix} := \{x_i \in X_i |\ (x, x_i) \in U_i\}$, $i = \overline{2,m}$.
We remark that, U_i, U_{ix}, $i = \overline{1,m}$ are nonempty closed subsets.
We also need the following operators:

$$R_i : C([t_i, b], \mathbb{B}) \to \prod_{k=0}^{i} X_k,\ R_i(x) = \left(x|_{[t_0,b]}, x|_{[t_1,t_0]}, \ldots, x|_{[t_i,t_{i-1}]} \right),\ i = \overline{1,m}.$$

It is clear that, $R_i\left(C([t_i,b],\mathbb{B})\right) = U_i$ and $R_i : C([t_i,b],\mathbb{B}) \to U_i$ is an increasing homeomorphism.

Since the operator, $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ defined by $V(x)(t) :=$ second part of equation (5.1), is a backward Volterra operator on $[a,c]$, it induces the following operators:

$$\begin{aligned} T_0 &: X_0 \to X_0, \\ T_0(x_0)(t) &: = V(x_0)(t),\ t \in [c,b], \end{aligned}$$

$$T_1 : U_1 \to X_1,$$
$$T_1(x_0,x_1)(t) := \int_b^c H(t,s,A(x_0)(s))ds + \int_b^c K(t,s,B(x_0)(s))ds$$
$$+ \int_c^t K(t,s,B(R_1^{-1}(x_0,x_1)(s))ds + g(t),\ t \in [t_1,c],$$

$$T_2 : U_2 \to X_2,$$
$$T_1(x_0,x_1,x_2)(t) := \int_b^c H(t,s,A(x_0)(s))ds + \int_b^c K(t,s,B(x_0)(s))ds$$
$$+ \int_c^{t_1} K(t,s,B(R_1^{-1}(x_0,x_1)(s))ds$$
$$+ \int_{t_1}^t K(t,s,B(R_2^{-1}(x_0,x_1,x_2)(s))ds + g(t),\ t \in [t_2,t_1],$$

$$\ldots$$

$$T_m : U_m \to X_m,$$
$$T_m(x_0, x_1, \ldots, x_m)(t) := \int_b^c H(t, s, A(x_0)(s))ds + \int_b^c K(t, s, B(x_0)(s))ds + \ldots$$
$$+ \int_{t_{m-1}}^t K(t, s, B(R_m^{-1}(x_0, x_1, \ldots, x_m)(s))ds + g(t),\ t \in [a, t_{m-1}].$$

Let

$$T \ : \ = (T_0, T_1, \ldots, T_m),$$
$$T(x_0, x_1, \ldots, x_m) \ : \ = (T_0(x_0),\ T_1(x_0, x_1), \ldots, T_m(x_0, x_1, \ldots, x_m))).$$

If on the cartesian product we consider max-norms, the operators $R_i,\ i = \overline{1, m}$ are isometries. From the above definitions, we remark that

$$(T_0, T_1)(U_1) \subset U_1,\ (T_0, T_1, \ldots, T_m)(U_m) \subset U_{m.}$$

In the conditions $(C_1) - (C_4)$ we have that: T_0 is $(L_H L_A + L_K L_B)(b - c)$-contraction and T_i, $i = 1, ..., m$, satisfy the condition (5) from the Theorem 2.4 with $L_i = \max\left\{(L_H L_A + L_K L_B)(b - c), \frac{L_K L_B(c - a)}{m}\right\}$ and $l_i = \frac{L_K L_B(c - a)}{m}$, with suitable $m \in \mathbb{N}$.

From this theorem we have that T is PO.

Since $V = R_m^{-1} T R_m$ and $V^n = R_m^{-1} T^n R_m$, it follows that V is PO.

So, we have:

Theorem 5.1. *We consider the equation (5.1) in the condition* $(\widetilde{C}_1) - (\widetilde{C}_5)$. *Under these conditions we have that:*

(i) *The equation (5.1) has in* $C([a, b], \mathbb{B})$ *a unique solution,* x^*.

(ii) *The sequence,* $(x_n)_{n \in \mathbb{N}}$, *defined by*

$$x^0 \in C([a, b], \mathbb{B}),$$
$$x^{n+1}(t) = \int_a^c H(t, s, A(x^n)(s))ds + \int_a^t K(t, s, B(x^n)(s))ds + g(t),\ t \in [a, b],$$

converges to x^*, *i.e., the operator* V *is PO.*

6. Gronwall-type results

In this section we consider $(\mathbb{B}, |\cdot|, \leq)$ an ordered Banach space. Related to the equation (1.1)

$$x(t) = \int_a^c H(t, s, A(x)(s))ds + \int_a^t K(t, s, B(x)(s))ds + g(t),\ t \in [a, b],$$

we consider the inequalities:

$$x(t) \leq \int_a^c H(t, s, A(x)(s))ds + \int_a^t K(t, s, B(x)(s))ds + g(t),\ t \in [a, b] \qquad (6.1)$$

and

$$x(t) \geq \int_a^c H(t,s,A(x)(s))ds + \int_a^t K(t,s,B(x)(s))ds + g(t),\ t \in [a,b]. \qquad (6.2)$$

As an application of the Theorem 2.1 we have

Theorem 6.1. *We consider the equation (1.1) under the hypotheses* $(C_1)-(C_5)$ *of the Theorem 4.1. In addition, we suppose that*

(C_6) *$H(t,s,\cdot)$, $K(t,s,\cdot)$, A and B are increasing.*

Then

(a) *$x \leq x^*$ for any x solution of (6.1);*
(b) *$x \geq x^*$ for any x solution of (6.2);*

where x^* is the unique solution of (1.1).

Proof. It follows from Theorem 4.1 that the operator $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ defined by $V(x)(t) :=$ second part of equation (1.1) is a PO and from (C_6) we have that V is an increasing operator, so the conclusion is obtained from Theorem 2.1. □

7. Comparison-type results

We consider the functional integral equations:

$$x_i(t) = \int_a^c H_i(t,s,A(x)(s))ds + \int_a^t K_i(t,s,B(x)(s))ds + g_i(t), \qquad (7.1)$$
$$t \in [a,b],\ i = \overline{1,3},$$

where $a < c < b$ are real numbers, $(\mathbb{B}, |\cdot|, \leq)$ an ordered Banach space, $H_i \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $K_i \in C([a,b]^2 \times \mathbb{B}, \mathbb{B})$, $g_i \in C([a,b],\mathbb{B})$, $i = \overline{1,3}$, and $A : C([a,c],\mathbb{B}) \to C([a,c],\mathbb{B})$ and $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ are given operators. We have the following comparison result:

Theorem 7.1. *We suppose that:*

(i) *H_i, K_i, g_i, $i = \overline{1,3}$, A, B satisfy the conditions $(C_1)-(C_5)$;*
(ii) *$H_1 \leq H_2 \leq H_3$ and $K_1 \leq K_2 \leq K_3$;*
(iii) *$H_2(t,s,\cdot)$, $K_2(t,s,\cdot)$, A and B are increasing.*

If $x_1(a) \leq x_2(a) \leq x_3(a)$ then $x_1^* \leq x_2^* \leq x_3^*$, where x_i^* is the unique solution of (7.1), $i = \overline{1,3}$.

Proof. From Theorem 4.1 we have that operator $V_i : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ defined by

$$V_i(x)(t) := \int_a^c H_i(t,s,A(x)(s))ds + \int_a^t K_i(t,s,B(x)(s))ds + g_i(t),\ t \in [a,b]$$

is PO, $i = \overline{1,3}$. Let $F_{V_i} = \{x_i^*\}$, $i = \overline{1,3}$.

If $u \in \mathbb{B}$ then we denote by $\tilde{u}$ the constant function

$$\tilde{u} : [a,b] \to \mathbb{B}, \tilde{u}(t) = u.$$

It is clear that
$$V_i^{\infty}(\widetilde{x_i\,(a)}) = x_i^*,\ i = \overline{1,3},$$
and from (ii) we get that
$$V_1(x) \leq V_2(x) \leq V_3(x),\ \forall x \in C([a,b],\mathbb{B}).$$
From condition (iii) we have that operator V_2 is an increasing operator, so, the conclusion is obtained from Theorem 2.2. □

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Received: January 11, 2021; Accepted: March 4, 2021.