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# FIXED POINT THEOREMS BASED ON THE DEGREE OF NONDENSIFIABILITY IN $b$-NORM SPACES AND APPLICATIONS TO DIFFERENTIAL PROBLEMS

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**A**bstract. This study aims to introduce a notion of the degree of nondensifiability $b$-Banach spaces and to develop variations of fixed point theorems, such as those by Darbo and its generalizations, Krasnoselskii, and Dhage. These theorems are applicable to operators in Strong $b$-Banach spaces (quasi-Banach spaces) that are not necessarily compact. Furthermore, the study applies these theorems to perturbed conformable differential equations and supports the findings with an illustrative example.
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## 1. Introduction

In the field of topology, the initial theorems concerning fixed points were established by Brouwer and Schauder. This naturally led to the question: 'Is it possible to obtain a fixed point without relying on compactness, as in the famous Schauder fixed point theorem?' A positive answer to this question was provided by Darbo [10] in 1955. He used the measure of noncompactness and extended Schauder's fixed point theorem to the framework of noncompact operators by introducing the concept of $k$-contracting operators. After that, many works have been based on this fixed point theorem and

the measure of noncompactness method in general (refer to [2, 3, 4, 5, 6, 7, 8], as well as the references cited therein).

In [11, 13, 14], the authors highlighted that, in certain cases, fixed point theorems based on measures of noncompactness, such as Darbo's theorem, are not applicable. To address this limitation, they introduced a new approach by defining fixed points using the concept of the degree of nondensifiability based on $\alpha$-dense curves. More recently, several researchers have developed fixed point theorems leveraging this concept, yielding significant results on the existence of solutions to various specific problems (see [12, 13, 14, 18], and the sources given therein).

On the other hand, it is well known that, in certain function spaces, the "natural" norm does not satisfy the triangle inequality. In light of this observation, researchers have proposed generalizations of metric spaces by relaxing the triangle inequality, thus allowing the extension of fixed point theory to these problematic function spaces. There are, however, certain generalized metric spaces, such as quasi-metric spaces (generally referred to as $b$-metric spaces in the context of fixed point theory). The resulting notion of $b$-metric spaces has opened a new direction for the development of fixed point theory, and It seems that the first fixed point theorem in a quasimetric space was established in 1981 by Vulpe et al. [21], who adapted the Picard-Banach contraction mapping principle from metric spaces to the context of quasimetric spaces ($b$-metric space). Since then, many authors have contributed to this development. Recently, the authors of [20] explore fixed-point theory in $b$-metric spaces and they removed the condition $ks < 1$ from Banach's fixed point theorem in $b$-metrics, and this condition has been imposed in works where authors propose versions of Banach's fixed point theorem in $b$-metrics. Today, this field holds a significant position in fixed point theory. However, regarding the origin of these spaces, we recommend reading work [9], as there is some confusion among certain individuals about the date of their emergence. Indeed, many research papers mistakenly attribute the initial work to someone other than its true author.

So a natural question arises: "Can we generalize Schauder's fixed point theorem in $b$-metric spaces?" In other words, "Can we apply Darbo's fixed point theorem in $b$-metric spaces?" Although the notion of MNC (measure of non-compactness) in $b$-metric spaces has not yet found a clear connection. We will try to weaken the compactness condition in Schauder's theorem in $b$-metric spaces by introducing a more general condition. Specifically, the operator should be contracting with respect to the degree of nondensifiability, and we aim to establish versions of other theorems in this context.

The structure of this work is outlined as follows. In Section 2, we focus solely on presenting the essential definitions related to $b$-metric spaces, $b$-Banach spaces, and conformable calculus, which serve as the foundational concepts for the subsequent discussions in this work. Section 3 introduces the notion of the degree of nondensifiability within $b$-metric spaces and explores its properties in $b$-Banach spaces. Additionally, we develop variations of specific fixed-point theorems, incorporating the degree of nondensifiability for noncompact operators in $b$-Banach spaces. These include Darbo's, Krasnoselskii's and Dhage fixed-point theorems, along with their generalizations. In

Section 4, we apply these concepts to establish a theorem on the existence of solutions for a class of perturbed conformable differential equations in $b$-Banach spaces. Finally, we provide an illustrative example that highlights the practical applications of the results obtained in this work.

## 2. Preliminaries

**Definition 2.1** ([9])**.** A function $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is called a $b$-metric on $\mathcal{E}$ if it satisfies the following three conditions:

($m_1$) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
($m_2$) $d(x, y) = d(y, x)$, for all $x, y \in \mathcal{E}$;
($m_3$) there exist a constant $s \geq 1$ such that

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in \mathcal{E}.$$

If $d$ is a $b$-metric on $\mathcal{E}$ then $(\mathcal{E}, d, s)$ is called a $b$-metric space (quasi-metric space).

In the specific case when $s = 1$, a $b$-metric $d$ corresponds to the standard definition of a metric (distance). However, when $s \neq 1$, a $b$-metric is not considered a metric.

**Definition 2.2** ([9])**.** Let $\mathcal{E}$ be a real or complex linear space. A map $\|\cdot\| : \mathcal{E} \to [0, \infty)$ is a $b$-norm (quasi-norm) if it satisfies the following conditions:

($m_4$) $\|x\| = 0$ implies $x = 0_{\mathcal{E}}$;
($m_5$) $\|x+y\| \leq s\,(\|x\| + \|y\|)$, for all $x, y \in \mathcal{E}$, where $s \geq 1$ does not depend on $x, y$;
($m_6$) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for all $x \in \mathcal{E}$ and $\lambda \in \mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$).

$(\mathcal{E}, \|\cdot\|, s)$ is called a $b$-normed space (quasi-normed space), and a complete $b$-normed space is called a $b$-Banach space (quasi-Banach space).

**Remark 2.3.** If the mapping $d$, satisfies the conditions ($m_1$) and ($m_2$) and

$$d(x, z) \leq \min\{d(x, y) + sd(y, z); sd(x, y) + d(y, z)\}, \text{ for all } x, y, z \in \mathcal{E}.$$

then, $(\mathcal{E}, d, s)$ is called a strong $b$-metric space, (Similarly for strong $b$-norm spaces).

From [1, 15], We recall two definitions from conformable calculus:

**Definition 2.4.** Let $f : [0, +\infty) \to \mathbb{R}$ be a given function, the conformable derivative of $f$ of order $\beta$ is defined by

$$\mathcal{D}^{\beta}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\beta}\right) - f(t)}{\varepsilon}, \quad \text{for } t > 0 \text{ and } \beta \in (0, 1].$$

**Definition 2.5.** The conformable integral starting from $a$ of the function $f : [a, +\infty) \to \mathbb{R}$ of order $\beta \in (0, 1]$ is defined as

$$I_a^r f(t) = \int_a^t (\tau - a)^{1-\beta} f(\tau) d\tau.$$

## 3. Main results

3.1. **Degree of nondensifiability in $b$-metric space.** In this section, we consider $(\mathcal{E}, d)$ to be a $b$-metric space as previously defined, and $\mathcal{P}_b(\mathcal{E})$ denotes the collection of all non-empty, bounded subsets of $\mathcal{E}$. For any $M \in \mathcal{P}_b(\mathcal{E})$, $\overline{M}$ represents the closure of $M$, and $\mathrm{Diam}(M)$ refers to its diameter. Furthermore, when $\mathcal{E}$ is a $b$-Banach space, $\mathrm{Co}(M)$ denotes the convex hull of $M$, $\mathfrak{F}_T$ is the set of fixed points of the operator $T$, and $\mathcal{J}_1 = [0, 1]$.

From [17], we present the following concepts associated with densifiable sets in a $b$-metric space:

**Definition 3.1.** Let $M \in \mathcal{P}_b(\mathcal{E})$ and $\alpha \geq 0$. A continuous mapping $\gamma : \mathcal{J}_1 \longrightarrow \mathcal{E}$ is said to be an $\alpha$-dense curve in $M$ if the following conditions hold:

(i) $\gamma(\mathcal{J}_1) \subset M$.
(ii) For each $x \in M$ there is $z \in \gamma(\mathcal{J}_1)$ such that $d(x, z) \leq \alpha$.

If for each $\alpha > 0$ there exists an $\alpha$-dense curve in $M$, then $M$ is said to be densifiable.

Now, we introduce the concept of the degree of nondensifiability in $b$-Banach spaces $\mathcal{E}$ (abbreviated as $\mathrm{DND}^b$), building upon a framework similar to that outlined in [13].

**Definition 3.2.** Let $M \in \mathcal{P}_b(\mathcal{E})$, $\alpha \geq 0$ and $\Gamma_{M,\alpha}$ the class of $\alpha$-dense curves in $M$. A map

$$\Phi_s : \mathcal{P}_b(\mathcal{E}) \to [0, +\infty),$$

defined by

$$\Phi_s(M) := \inf \{\alpha \geq 0 : \Gamma_{M,\alpha} \neq \emptyset\},$$

is called a degree of nondensifiability defined on $\mathcal{E}$.

Taking the above considerations into account, we obtain

$$0 \leq \Phi_s(M) \leq \mathrm{Diam}(M),$$

for all $M \in \mathcal{P}_b(\mathcal{E})$. Therefore, $\Phi_s$ is well defined.

**Proposition 3.3.** *The $\mathrm{DND}^b$ $\Phi_s$ satisfies:*

($p_1$) *Regularity on the subfamily $\mathcal{M}_{\mathrm{arc}}(\mathcal{E}) \subset \mathcal{P}_b(\mathcal{E})$ of arc-connected subsets of $\mathcal{P}_b(\mathcal{E})$ : $\Phi_s(M) = 0$ if, and only if, $M$ is totally bounded, for each $M \in \mathcal{M}_{\mathrm{arc}}(\mathcal{E})$.*
($p_2$) *$\Phi_s(M) = \Phi_s(\overline{M})$, for each $M \in \mathcal{P}_b(\mathcal{E})$.*
($p_3$) *$\Phi_s(cM) = |c|\Phi_s(M)$, for each $c \in \mathbb{R}$ and $M \in \mathcal{P}_b(\mathcal{E})$.*
($p_4$) *$\Phi_s(x_0 + M) = \Phi_s(M)$, for each $x_0 \in \mathcal{E}$ and $M \in \mathcal{P}_b(\mathcal{E})$.*
($p_5$) *If $(C_n)_{n\geq 1}$ is a decreasing sequence of non-empty, closed and convex subsets of $\mathcal{E}$ with $\lim_n \Phi_s(C_n) = 0$, then the intersection of all $C_n$ is non-empty, convex and compact.*
($p_6$) *For each $M \in \mathcal{P}_b(\mathcal{E})$, $m \in \mathbb{N}^*$, we have*

$$\Phi_s(\mathrm{Co}(M)) \leq \widehat{s}_m \Phi_s(M),$$

*where*

$$\widehat{s}_m = \sum_{j=1}^{m} s^j \lambda_j + \lambda_m s^{m-1}(1-s),\ \sum_{j=1}^{m} \lambda_j = 1,\ \textit{and } m \geq 2.$$

*Or more generally, for* $s > 1$ *and* $m \in \mathbb{N}_1 = \mathbb{N}^* - \{1\}$

$$\Phi_s(\mathrm{Co}(M)) \leq (s+1)\Phi_s(M) \leq s_m \Phi_s(M),$$

*where* $s_m = \frac{s^m - 1}{s-1}$.

$(p_7)$ *For each* $M_1, M_2 \in \mathcal{P}_b(\mathcal{E})$, *we have*

$$\Phi_s\left(M_1 + M_2\right) \leq s\left(\Phi_s\left(M_1\right) + \Phi_s\left(M_2\right)\right).$$

$(p_8)$ *For each* $M_1, M_2 \in \mathcal{P}_b(\mathcal{E})$, $m \in \mathbb{N}^*$, *and* $\lambda_j \in \mathbb{R}^+; j = \overline{1:m}$, *we have*

$$\Phi_s\left(\mathrm{Co}\left(M_1 \cup M_2\right)\right) \leq \widetilde{s}_m \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\},$$

*where*

$$\widetilde{s}_m = \sum_{j=1}^{m} s^{j+1}\lambda_j + (1-s)(s^r\lambda_r + s^m\lambda_m),\ \textit{and } \sum_{j=1}^{m} \lambda_j = 1,\ 1 \leq r \leq m.$$

*Or more generally, for* $s > 1$ *and* $m \in \mathbb{N}_1$

$$\begin{aligned} \Phi_s\left(\mathrm{Co}\left(M_1 \cup M_2\right)\right) &\leq (s+1)\max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\} \\ &\leq s_m \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\} \end{aligned}$$

$(p_9)$ *For each* $M_i \in \mathcal{P}_b(\mathcal{E}); i = 1, 2$ *and* $M_{i,*} = \max_{y \in M_i}\{\|y\|\}$, *we have*

$$\Phi_s\left(M_1 M_2\right) \leq s_\theta = \min(\theta_1, \theta_2, \theta_3),$$

*or more generally,*

$$\Phi_s\left(M_1 M_2\right) \leq s^2\left(M_{2,*}\Phi_s\left(M_1\right) + M_{1,*}\Phi_s\left(M_2\right) + \Phi_s\left(M_1\right)\Phi_s\left(M_2\right)\right),$$

*where*

$$\begin{aligned} \theta_1 &= sM_{2,*}\Phi_s\left(M_1\right) + s^2 M_{1,*}\Phi_s\left(M_2\right) + s^2\Phi_s\left(M_1\right)\Phi_s\left(M_2\right), \\ \theta_2 &= s^2 M_{2,*}\Phi_s\left(M_1\right) + s^2 M_{1,*}\Phi_s\left(M_2\right) + s\Phi_s\left(M_1\right)\Phi_s\left(M_2\right), \\ \theta_3 &= s^2 M_{2,*}\Phi_s\left(M_1\right) + sM_{1,*}\Phi_s\left(M_2\right) + s^2\Phi_s\left(M_1\right)\Phi_s\left(M_2\right), \end{aligned}$$

*and* $(\mathcal{E}, \|\cdot\|)$ *is a b-Banach algebra satisfying the condition* $\|uv\| \leq \|u\|\|v\|$ *for all* $u, v \in \mathcal{E}$.

*Proof.* For the proof of properties $(p_1)$ to $(p_5)$, Since in the proof found in [11, 14, 13], the triangular inequality has no effect, the proof remains valid in $b$-metric space. Now, about:

$(p_6)$ : Let us denote $\alpha_* := \Phi_s\left(M\right)$, then for a given $\epsilon > 0$ there exist a curve $\gamma_*$ in $M$, such that

$$M \subset \gamma_*(\mathcal{J}_1) + (\alpha_* + \epsilon)\, B_{\mathcal{E}}.$$

Now, consider any point $x \in \mathrm{Co}\left(M\right)$, This point can be expressed as

$$x := \sum_{j=1}^{m} \lambda_j x_j, \quad \text{for } \lambda_j \geq 0, \sum_{j=1}^{m} \lambda_j = 1,$$

where $\{x_j\}_{j=1}^m$. By the property of $\gamma_*$, there exist $\{t_j\}_{j=1}^m \in \mathcal{J}_1$ such that

$$\|x_j - \gamma_*(t_j)\| \leq \alpha_* + \epsilon, \text{ for all } j = 1, \ldots, m.$$

Define

$$z := \sum_{j=1}^{m} \lambda_j \gamma_*(t_j).$$

Clearly $z \in \mathrm{Co}(\gamma_*(\mathcal{J}_1)) \subset \mathrm{Co}(M)$ and there exists $t \in \mathcal{J}_1$ such that $\gamma_*(t) = z$. Then, we have

$$\begin{aligned}
\|x - z\| &= \left\| \sum_{j=1}^{m} \lambda_j (x_j - \gamma_*(t_j)) \right\| \\
&\leq \left( \sum_{j=1}^{m-1} s^j \lambda_j \|x_j - \gamma_*(t_j)\| + s^{m-1} \lambda_m \|x_m - \gamma_*(t_m)\| \right) \\
&\leq (\alpha_* + \epsilon) \left( \sum_{j=1}^{m-1} s^j \lambda_j + s^{m-1} \lambda_m \right).
\end{aligned}$$

Thus,

$$\|x - z\| \leq \widehat{s}_m (\alpha_* + \epsilon),$$

where

$$\widehat{s}_m = \sum_{j=1}^{m-1} s^j \lambda_j + s^{m-1} \lambda_m,$$

which proves that $\gamma_*$ is a $\widehat{s}_m(\alpha_* + \epsilon)$-dense curve in $\mathrm{Co}(M)$. Then, since $\epsilon$ is arbitrary, $\gamma_*$ is a $\widehat{s}_m \alpha_*$-dense curve in $\mathrm{Co}(M)$. Therefore, $\Phi_s(\mathrm{Co}(M)) \leq \widehat{s}_m \alpha_*$.
Thus

$$\Phi_s(\mathrm{Co}(M)) \leq \widehat{s}_m \Phi_s(M).$$

Or more generally, for $s > 1$ we have $\widehat{s}_m < s_m$ for all $m \in \mathbb{N}_1$, then

$$\Phi_s(\mathrm{Co}(M)) \leq \left( \inf_{m \in \mathbb{N}_1} s_m \right) \Phi_s(M) = s_2 \Phi_s(M) \leq s_m \Phi_s(M).$$

$(p_7)$ : The proof is simple and similar to that of Proposition 2.1 in [14]. One just needs to use the inequality in condition $(m_3)$.

$(p_8)$ : Based on [13], let us define $\alpha_i := \Phi_s(\mathrm{Co}(M_i)); i = 1, 2$, according to the definition for a given $\epsilon > 0$ there exist a curves $\gamma_i$ in $\mathrm{Co}(M_i)$, such that

$$\mathrm{Co}(M_i) \subset \gamma_1(\mathcal{J}_1) + (\alpha_i + \epsilon) B_{\mathcal{E}}.$$

Then,

$$\mathrm{Co}(\gamma_1(\mathcal{J}_1) \cup \gamma_2(\mathcal{J}_1)) \subset \mathrm{Co}(M_1 \cup M_2),$$

and there exists a continuous map $\gamma : \mathcal{J}_1 \longrightarrow \mathcal{E}$ such that

$$\gamma(\mathcal{J}_1) = \mathrm{Co}(\gamma_1(\mathcal{J}_1) \cup \gamma_2(\mathcal{J}_1)).$$

Now, let $x$ be a point of $\mathrm{Co}\,(M_1 \cup M_2)$, then

$$x := \sum_{j=1}^{m} \lambda_j x_j, \quad \lambda_j \geq 0, \sum_{j=1}^{m} \lambda_j = 1,$$

with $\{x_j\}_{j=1}^{r} \in M_1$ and $\{x_j\}_{j=r+1}^{m} \in M_2$ for some $1 < r < m$. then, there exist $\{t_j\}_{j=1}^{m} \in \mathcal{J}_1$, such that

$$\|x_j - \gamma_1(t_j)\| \leq \alpha_1 + \epsilon \text{ for all } j = \overline{1:r}.$$

and

$$\|x_j - \gamma_2(t_j)\| \leq \alpha_2 + \epsilon \text{ for all } j = \overline{r+1:m}.$$

Now by defining

$$z := \sum_{j=1}^{r} \lambda_j \gamma_1(t_j) + \sum_{j=r+1}^{m} \lambda_j \gamma_2(t_j).$$

Consequently $z$ belongs to the convex hull of $(\gamma_1(\mathcal{J}_1) \cup \gamma_2(\mathcal{J}_1))$ and there exists $t \in \mathcal{J}_1$ such that $\gamma(t) = z$. Therefore, we have

$$\begin{aligned}
\|x - z\| &\leq s \left\| \sum_{j=1}^{r} \lambda_j (x_j - \gamma_1(t_j)) \right\| + s \left\| \sum_{j=r+1}^{m} \lambda_j (x_j - \gamma_2(t_j)) \right\| \\
&\leq s \left( \sum_{j=1}^{r-1} s^j \lambda_j \|x_j - \gamma_1(t_j)\| + s^{r-1} \lambda_r \|x_r - \gamma_1(t_r)\| \right) \\
&\quad + s \left( \sum_{j=r+1}^{m-1} s^j \lambda_j \|x_j - \gamma_2(t_j)\| + s^{m-1} \lambda_n \|x_m - \gamma_2(t_m)\| \right) \\
&\leq (\alpha_1 + \epsilon) \left( \sum_{j=1}^{r-1} s^{j+1} \lambda_j + s^r \lambda_r \right) + (\alpha_2 + \epsilon) \left( \sum_{j=r+1}^{m-1} s^{j+1} \lambda_j + s^m \lambda_m \right) \\
&\leq (\max\{\alpha_1, \alpha_2\} + \epsilon) \left( \sum_{j=1}^{m} s^{j+1} \lambda_j + (s^r - s^{r+1}) \lambda_r + (s^m - s^{m+1}) \lambda_m \right).
\end{aligned}$$

Thus

$$\|x - z\| \leq \widetilde{s}_m (\max\{\alpha_1, \alpha_2\} + \epsilon),$$

where

$$\widetilde{s}_m = \sum_{j=1}^{m} s^{j+1} \lambda_j + (1 - s)(s^r \lambda_r + s^m \lambda_m),$$

This establishes that $\gamma$ is a $\widetilde{s}_m (\max\{\alpha_1, \alpha_2\} + \epsilon)$-dense curve in $\mathrm{Co}(B_1 \cup M_2)$. Since $\epsilon$ can be made arbitrarily small, it follows that $\gamma$ is a $\widetilde{s}_m \max\{\alpha_1, \alpha_2\}$-dense curve in $\mathrm{Co}(M_1 \cup M_2)$. Consequently, we have $\Phi_s(\mathrm{Co}(M_1 \cup M_2)) \leq \widetilde{s}_m \max\{\alpha_1, \alpha_2\}$.
Thus

$$\Phi_s(\mathrm{Co}\,(M_1 \cup M_2)) \leq \widetilde{s}_m \max\{\Phi_s(\mathrm{Co}\,(M_1)), \Phi_s(\mathrm{Co}\,(M_2))\}.$$

Or more generally, for $s > 1$ we have $\widetilde{s}_m < s_m$ for all $m \in \mathbb{N}_1$, then

$$\begin{aligned}\Phi_s\left(\mathrm{Co}\left(M_1 \cup M_2\right)\right) &\leq \left(\inf_{m\in\mathbb{N}_1} s_m\right) \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\} \\ &= s_2 \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\} \\ &\leq s_m \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\}.\end{aligned}$$

$(p_9)$ : Let $x := x_1x_2 \in M_1M_2$ with $x_i \in M_i, i = 1, 2$, from Definition 3.1, we have $M_i \subset \gamma_i(\mathcal{J}_1) + \alpha_i B_{\mathcal{E}}, \quad$ for $i = 1, 2$, then it follow that $x = (z_1 + \alpha_1 u_1)(z_2 + \alpha_2 u_2)$ for some $z_i \in \gamma_i(\mathcal{J}_1)$ and $u_i \in B_{\mathcal{E}}, i = 1, 2$. then we get

$$\|x - z_1z_2\| \leq \widehat{\theta}_s = \min(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3), \tag{3.1}$$

where

$$\begin{aligned}\widehat{\theta}_1 &= s\|z_2\|\alpha_1 + s^2\|z_1\|\alpha_2 + s^2\alpha_1\alpha_2, \\ \widehat{\theta}_2 &= s^2\|z_2\|\alpha_1 + s^2\|z_1\|\alpha_2 + s\alpha_1\alpha_2, \\ \widehat{\theta}_3 &= s^2\|z_2\|\alpha_1 + s\|z_1\|\alpha_2 + s^2\alpha_1\alpha_2.\end{aligned}$$

Define $\omega : \mathcal{J}_1 \to \mathcal{E}$ by $\omega(t) := \gamma_1(r_1(t))\gamma_2(r_2(t))$ for $t \in \mathcal{J}_1$. $\omega$ is continuous and $\omega(\mathcal{J}_1) \subset M_1M_2$. By choosing $t_i \in \mathcal{J}_1$ such that $\gamma_i(r_i(t_i)) = z_i$ for $i = 1, 2$, we can conclude from (3.1) that $\omega$ forms a $\widetilde{\theta}_s$-dense curve in $M_1M_2$, where $\widetilde{\theta}_s = \min(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3)$, where

$$\begin{aligned}\widetilde{\theta}_1 &= sM_{2,*}\alpha_1 + s^2M_{1,*}\alpha_2 + s^2\alpha_1\alpha_2, \\ \widetilde{\theta}_2 &= s^2M_{2,*}\alpha_1 + s^2M_{1,*}\alpha_2 + s\alpha_1\alpha_2, \\ \widetilde{\theta}_3 &= s^2M_{2,*}\alpha_1 + sM_{1,*}\alpha_2 + s^2\alpha_1\alpha_2.\end{aligned}$$

Therefore, $\Phi_s(M_1M_2) \leq \widetilde{\theta}_s$. Finally, observing the arbitrariness of $\alpha_i > \Phi_s(M_i)$, for $i = 1, 2$, we obtain

$$\Phi_s\left(M_1M_2\right) \leq s_\theta. \qquad \square$$

**Remark 3.4.** In a strong $b$-Banach space, $(p_1)$ to $(p_5)$ remain valid, and moreover, we have

$(p_6^*)$ $\Phi_s(\mathrm{Co}(M)) \leq s\Phi_s(M)$, for each $M \in \mathcal{P}_b(\mathcal{E})$.

$(p_7^*)$ For each $M_1, M_2 \in \mathcal{P}_b(\mathcal{E})$, we have

$$\Phi_s\left(M_1 + M_2\right) \leq \min\left\{s\Phi_s\left(M_1\right) + \Phi_s\left(M_2\right), \Phi_s\left(M_1\right) + s\Phi_s\left(M_2\right)\right\}.$$

$(p_8^*)$ For each $M_1, M_2 \in \mathcal{P}_b(\mathcal{E})$, we have

$$\Phi_s\left(\mathrm{Co}\left(M_1 \cup M_2\right)\right) \leq s^2 \max\left\{\Phi_s\left(\mathrm{Co}\left(M_1\right)\right), \Phi_s\left(\mathrm{Co}\left(M_2\right)\right)\right\}.$$

$(p_9^*)$ For each $M_1, M_2 \in \mathcal{P}_b(\mathcal{E})$, we have

$$\Phi_s\left(M_1M_2\right) \leq \widetilde{s_\theta} = \min_{i=\overline{1:6}}(\theta_i),$$

where

$$\begin{aligned}\theta_1 &= M_{2,*}\Phi_s\left(M_1\right) + sM_{1,*}\Phi_s\left(M_2\right) + s^2\Phi_s\left(M_1\right)\Phi_s\left(M_2\right), \\ \theta_2 &= M_{2,*}\Phi_s\left(M_1\right) + s^2M_{1,*}\Phi_s\left(M_2\right) + s\Phi_s\left(M_1\right)\Phi_s\left(M_2\right), \\ \theta_3 &= sM_{2,*}\Phi_s\left(M_1\right) + M_{1,*}\Phi_s\left(M_2\right) + s^2\Phi_s\left(M_1\right)\Phi_s\left(M_2\right). \\ \theta_4 &= sM_{2,*}\Phi_s\left(M_1\right) + s^2M_{1,*}\Phi_s\left(M_2\right) + \Phi_s\left(M_1\right)\Phi_s\left(M_2\right),\end{aligned}$$

$$\theta_5 = s^2 M_{2,*}\Phi_s(M_1) + M_{1,*}\Phi_s(M_2) + s\Phi_s(M_1)\Phi_s(M_2),$$
$$\theta_6 = s^2 M_{2,*}\Phi_s(M_1) + sM_{1,*}\Phi_s(M_2) + \Phi_s(M_1)\Phi_s(M_2).$$

Or more generally,

$$\Phi_s(M_1M_2) \leq s^2\left(M_{2,*}\Phi_s(M_1) + M_{1,*}\Phi_s(M_2) + \Phi_s(M_1)\Phi_s(M_2)\right).$$

Now, we consider

$$\mathcal{Y}_1 = \left\{\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+, \text{ continuous nondecreasing functions with } \mathcal{K}(0) = 0\right\},$$
$$\mathcal{Y}_2 = \left\{\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+, \text{ such that } \lim_{n\to\infty} \mathcal{K}^n(t) = 0, \forall t \in [0, +\infty)\right\},$$
$$\mathcal{Y}_3 = \{\mathcal{K} \in \mathcal{Y}_1, \text{ with } \mathcal{K}(t) = kt, \quad k \in [0, 1)\}.$$

**Definition 3.5.** Let $\mathcal{E}$ be a strong $b$-Banach space and a mapping $T : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ that satisfies

$$s\Phi_s(T(M)) \leq \mathcal{K}\left(\Phi_s(M)\right), \text{ for all } M \in \mathcal{P}_b(\mathcal{E}).$$

Then $T$ is called

(1) $\mathcal{K}$-Lipschitz$^{\Phi_s}$ if $\mathcal{K} \in \mathcal{Y}_1$.
(2) $k$-contraction$^{\Phi_s}$ on $\mathcal{E}$, if $\mathcal{K} \in \mathcal{Y}_3$.
(3) A nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$ on $\mathcal{E}$, if $\mathcal{K} \in \mathcal{Y}_1$ and $\mathcal{K}(v) < v$ for $v > 0$ ( which implies that $\mathcal{K} \in \mathcal{Y}_2$).

Subsequently, we will present several versions of the fixed point theorem based on the degree of nondensifiability in a strong $b$-Banach space $(\mathcal{E}, \|\cdot\|, s)$ for $s > 1$.

**Theorem 3.6.** *Let $F$ be a bounded closed convex subset of a strong $b$-Banach space $\mathcal{E}$ and $T : F \to F$ be a $k$-contraction$^{\Phi_s}$ continuous operator. Then, $T$ has at least one fixed point.*

*Proof.* Consider a sequence of sets $\{M_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ defined as follows

$$M_0 = F, \ M_{n+1} = \overline{\mathrm{Co}\, T(M_n)}, n = 0, 1, \ldots$$

It is clear that $M_{n+1} \subseteq M_n$ for $n = 0, 1, \ldots$, thereby creating a decreasing sequence of closed convex and bounded subsets of $F$.
Since $T$ is a $k$-contraction$^{\Phi_s}$, it follows that

$$\Phi_s(M_{n+1}) = \Phi_s\left(\overline{\mathrm{Co}\, T(M_n)}\right) \leq s\Phi_s(T(M_n)) \leq k\left(\Phi_s(M_n)\right). \tag{3.2}$$

By continuing this process, we obtain

$$\Phi_s(M_{n+1}) \leq k^n\Phi_s(F) \xrightarrow[n\to+\infty]{} 0.$$

According to property $(p_5)$ in Proposition 3.3, $M^* = \bigcap_{n\in\mathbb{N}} M_n \neq \emptyset$, and this set is both convex and compact. Additionally, due to the convexity of the sequence $\{M_n\}_{n\in\mathbb{N}}$ and the fact that $T(M_0) \subset M_0$, it follows that $T(M_n) \subseteq M_n$ for all $n \in \mathbb{N}$. Therefore,

$$T(M^*) = \bigcap_{n=0}^{\infty} T(M_n) \subset \bigcap_{n=0}^{\infty} M_n = M^*.$$

Therefore, by applying a Schauder-type fixed point theorem (in strong b-Banach space), the operator $T : M^* \to M^*$ has at least one fixed point. □

Now, we present the first generalization of Theorem 3.6.

**Theorem 3.7.** *Let $F$ be a closed, bounded and convex subset of a strong b-Banach space $\mathcal{E}$, and let $T : F \to F$ be a continuous operator. For any subset $\Omega$ of $F$, set*

$$T^1\Omega = T\Omega, \quad T^n\Omega = T\left(\overline{\mathrm{Co}\, T^{n-1}(\Omega)}\right), \quad n = 2, 3, \ldots \tag{3.3}$$

*Suppose there exists a positive integer $p_0$, such that $T^{p_0}$ be a $k$-contraction$^{\Phi_s}$ operator. Then, $\mathfrak{F}_T \neq \emptyset$.*

*Proof.* The proof is analogous, we simply need to use the sequence

$$\mathcal{W}_0 = F,\ \mathcal{W}_{n+1} = \overline{\mathrm{Co}\, T^n(\mathcal{W}_n)}, n = 0, 1, \ldots$$

and the fact that $T^{p_0}$ is $k$-contraction$^{\Phi_s}$ to ensure that $\Phi_s(\mathcal{W}_n) \to 0,\ as\ n \to +\infty$. □

**Lemma 3.8.** *Let $T : \mathcal{E} \to \mathcal{E}$ be a $k$-contraction$^{\Phi_s}$ operator, then for every $j > 0$, $T^j$ is $k^j$-contraction$^{\Phi_s}$ operator. (where $T^j$ is the $j$-th composition of the operator $T$ verify expression (3.3)).*

*Proof.* Put $\Phi_s(T) = k$, then for a given $\varepsilon > 0$, we have $T$ is $(k+\varepsilon)$-contraction$^{\Phi_s}$. Therefore, we have

$$s\Phi_s(T(M)) \leq (k+\varepsilon)\,\Phi_s(M), \qquad \text{for all } M \in \mathcal{P}_b(\mathcal{E}).$$

Then, we get

$$\begin{aligned} s\Phi_s\left(T\left(\overline{\mathrm{Co}\, T^{j-1}(M)}\right)\right) &\leq (k+\varepsilon)\,\Phi_s\left(\overline{\mathrm{Co}\, T^{j-1}(M)}\right) \\ &\leq (k+\varepsilon)\, s\Phi_s\left(T^{j-1}(M)\right) \\ &\leq (k+\varepsilon)^2\,\Phi_s\left(\overline{\mathrm{Co}\, T^{j-2}(M)}\right) \\ &\vdots \\ &\leq (k+\varepsilon)^j\,\Phi_s(M). \end{aligned}$$

Since $\varepsilon$ is arbitrary, then $T^j$ is $k^j$-contraction$^{\Phi_s}$ operator. □

**Remark 3.9.** If $T$ is $k$-contraction$^{\Phi_s}$ operator, then by Lemma 3.8 for every $j > 0$, $T^j$ is $k^j$-contraction$^{\Phi_s}$ operator. Given that $k \in [0, 1)$, then for all $j \in \mathbb{N}^*$, we have $k^j < 1$. Thus by Theorem 3.7, $T$ has at least one fixed point.

The next theorem provides another generalization of Theorem 3.6, utilizing the concept of $\mathcal{K}$-contraction$^{\Phi_s}$.

**Theorem 3.10.** *Let $F$ be a non-empty, closed, convex and bounded subset of a strong b-Banach space $\mathcal{E}$ and let $T : F \to F$ be a continuous nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$. Then $\mathfrak{F}_T \neq \emptyset$.*

*Proof.* The proof follows a similar approach to that of Theorem 3.6, with the difference lying in inequality (3.2). Here, for all $n = 0, 1, 2, \ldots$, we obtain

$$\Phi_s(M_{n+1}) \leq \mathcal{K}^n(\Phi_s(F)) \to 0,\ as\ n \to +\infty.$$

Consequently, we reach the same result by applying the same reasoning. □

**Remark 3.11.**

- If $p_0 = 1$ in Theorem 3.7, or in Theorem 3.10 if the operator $T$ is linear $\mathcal{K}$-contraction$^{\Phi_s}$ on $\mathcal{E}$, then then we recover Theorem 3.10 (Darbo-type fixed point theorem) in strong $b$-Banach space.
- If Schauder's fixed theorem in strong $b$-Banach spaces is applicable, then the Theorem 3.10 is applicable.
- In the case when $s = 1$, the theorems 3.6, 3.7 and 3.10 remain applicable, which corresponds to the case of Banach spaces.
- From Remark 3.9 Theorem 3.6 implies Theorem 3.7.
- In Theorem 3.10, we can take the following examples of functions $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+$:
$$\mathcal{K}(r) = \arctan(r),\ \mathcal{K}(r) = \ln(r+1),\ \mathcal{K}(r) = \frac{r}{r+1}.$$
- In Theorem 3.10 since $\mathcal{K}(v) < v$ for $v > 0$, then we get
$$\Phi_s(T(F)) < \Phi_s(F),$$
in this case, we say that $T$ is $\Phi_s$-condensing operator, and we recover Sadovskii's theorem in strong $b$-Banach spaces. However, we will not provide the proof, as it is similar to the usual case (see, [11]).

We will now introduce a version of Krasnoselskii's type fixed point theorem on strong $b$-Banach space based on DND$^s$.

**Theorem 3.12.** *Let $F$ be a closed, bounded and convex subset of a strong $b$-Banach space $\mathcal{E}$, and for $i = 1, 2$ lets $T_i : F \to \mathcal{E}$ be two operators such that*

($K_1$) *$T_1$ is continuous operator and an $k$-contraction$^{\Phi_s}$ operator, with $ks < 1$.*
($K_2$) *$T_2$ is continuous and compact operator.*
($K_3$) *$T_1u + T_2u \subseteq F$ for all $u \in F$.*

*Then the operator $T_1 + T_2$ has a fixed point $F$.*

*Proof.* Let $T : F \to \mathcal{E}$ be a mapping defined by
$$Tx = T_1x + T_2x.$$
According to assumption ($K_3$), we have $T_1u+T_2u \subseteq F$ for every $u \in F$. Consequently, $T(F)$ is bounded and $T(F) \subseteq F$. Given that $T_i$ are continuous, this leads to the conclusion that $T$ is continuous.
Based on assumption ($K_1$), there exists $k \in [0, 1)$ such that for every non-empty subset $M$ of $F$, we have
$$s\Phi_s(T_1(M)) \leq k\Phi_s(M).$$
It follows from conditions ($p_1$) and ($p_7^*$) in Proposition 3.3 that
$$s\Phi_s(T(M)) \leq k\Phi_s(M).$$
This shows that $T$ is an $k$-contraction$^{\Phi_s}$ operator on $F$, mapping into itself. As a result, $T$ satisfies all the conditions of Theorem 3.6, implying that $T$ has a fixed point. Therefore, the equation $x = T_1x + T_2x$ has a solution. □

**Remark 3.13.** In the previous theorem:

- We can generalize the condition $(K_1)$ by:

  $(K_1^*)$ $T_1$ is continuous operator and nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$.

- We can weaken the condition $(K_2)$ by:

  $(K_2^*)$ $T_1$ is continuous and $k^*$-contraction$^{\Phi_s}$ operator, with

$$\min\{sk + k^*, k + sk^*\} < 1.$$

**Theorem 3.14.** *Let $F \in \mathcal{P}_b(\mathcal{E})$ be a closed and convex set and $T,\ T_i : F \longrightarrow \mathcal{E}; i = 1, 2, 3$ continuous operators. Assume the conditions:*

$(K_4)$ *$T(x) = T_1(x)T_2(x) + T_3(x) \in F$, for each $x \in F$.*

$(K_5)$ *$T_i$ are $k_{T_i}$-contraction$^{\Phi_s}$ operators for $i = 1, 3$.*

$(K_6)$ *$T_2$ is compact.*

*Additionally, assume that*

$$\alpha_* = \min\{sk_{T_1}\|T_2\| + k_{T_3}, k_{T_1}\|T_2\| + sk_{T_3}\} < 1. \tag{3.4}$$

*Then the operator $T$ has a fixed point.*

*Proof.* By condition $(K_4)$, $T$ belongs to $C(F, F)$, and according to condition $(K_6)$, $\|T_2\|$ is well defined. Now, based on property $(p_7^*)$ and $(p_9^*)$ of Proposition 3.3, for any non-empty, convex subset $M \subset F$ with $\Phi_s(M) > 0$, we obtain

$$\Phi_s(T(M)) \leq \min\{sk_{T_1}\|T_2\| + k_{T_3}, k_{T_1}\|T_2\| + sk_{T_3}\}\,\Phi_s(M).$$

Given that $\alpha_* < 1$, it follows that $T$ is an $\alpha_*$-contraction$^{\Phi_s}$ operator. As a result, by Theorem 3.10, $T$ has a fixed point. □

**Remark 3.15.** We can generalize condition $(K_5)$ of Theorem 3.14 by:

$(K_5^*)$ and $T_1, T_3$ are continuous operators and nonlinear $\mathcal{K}_i$-contraction$^{\Phi_s}$ $(i = 1, 3)$.

## 4. Application to perturbed conformable differential Equations

In this section, we prove the existence of solution to the following problem:

$$\begin{cases} \mathcal{D}_a^{\beta}\left(\varpi(t)\left[u(t) - \Psi_1(t, u(t))\Psi_2(t, u(t))\right]\right) \\ = \Psi_4\left(t, u(t)\right) + \mathcal{D}_a^{\beta}\left(\Psi_3(t, u(t))\right),\ t \in I := [a, b], \\ u(a) = u_0 \in \mathcal{X}, \end{cases} \tag{4.1}$$

where $a$ and $b$ are strictly positive constants with $a < b$, the functions $\Psi_i$ belong to the space $C(I \times \mathcal{X}, \mathcal{X})$ and $\varpi \in C(I, \mathbb{R}\backslash\{0\})$. $(\mathcal{X}, \|\cdot\|, \sqrt{2})$ is a strong b-Banach algebra space and $\mathcal{D}_a^{\beta}u(t)$ is the conformable derivative starting from the initial time $a$ and of order $\beta \in (0, 1)$. (the interested reader to this type of problems is referred to [1, 15, 16, 19] and the sources given therein).

Let $\mathcal{X}_a = C([a, b], \mathcal{X})$ be the space of all real continuous functions on $[a, b]$ with

$$\|u\|_* = \sup_{t\in[a,b]} \|u(t)\|.$$

Then, $(\mathcal{X}_a, \|\cdot\|_*, \sqrt{2})$ is a strong $b$-Banach space with the structure of $b$-Banach algebra.

Now, following the methodology outlined in the papers [1, 15], we derive the following lemma:

**Lemma 4.1.** *A function $u \in \mathcal{X}_a$ is a solution of problem (4.1), if and only if $u$ satisfies the following integral equation*

$$u(t) = \Psi_1^*(t)\Psi_2^*(t) + \varpi^+(t)\left(g_a + \Psi_3^*(t) + \int_a^t (\tau - a)^{\beta-1}\Psi_4(\tau, u(\tau))d\tau\right),\ t \in [a, b],$$

*where $g_a = \varpi(a)\left[u_0 - \Psi_1^*(a)\Psi_2^*(a)\right] - \Psi_3^*(a)$, $\varpi^+(t) = \frac{1}{\varpi(t)}$ and $\Psi_i^*(t) = \Psi_i(t, u(t))$.*

To provide an existence result for the solution of problem (4.1), we require the following hypotheses:

$Q_1$) There exists a constant $\varpi^* > 0$ such that:

$$|\varpi(t)| \geq \frac{1}{\varpi^*},\ \text{for each } t \in I.$$

$Q_2$) There exist continuous functions $g_i : I \to (0, +\infty)$, $(i = 1, 2, 3)$ and $g_4 \in L^1(I, \mathbb{R}^+)$ such that

$$\|\Psi_i(t, u)\| \leq g_i(t)\|u\|, \quad \forall t \in I, \forall u \in \mathcal{X};\ i = 1, 3, 4,$$

$$\|\Psi_2(t, u)\| \leq g_2(t), \quad \forall t \in I, \forall u \in \mathcal{X},$$

and $g_i^* = \sup_{t\in[a,b]} g_i(t)$.

$Q_3$) For any bounded set $M \in \mathcal{X}$ and $t \in I$, we have

$$\Phi_s(\Psi_i(t, M)) \leq g_i(t)\Phi_s(M);\ i = 1, 3, 4.$$

**Theorem 4.2.** *Assume that the conditions $(Q_1) - (Q_3)$ are satisfied, with*

$$\tau^* = \min\left\{g_1^*\|\Psi_2\|_\infty\sqrt{2} + \xi, g_1^*\|\Psi_2\|_\infty + \xi\sqrt{2}\right\} < 1. \tag{4.2}$$

*Then, the system (4.1) has at least one solution.*

**Remark 4.3.** $\theta$ and $\xi$ are constants that will be specified later.

Now, before presenting the proof of the previous theorem, we need the following lemma.

**Lemma 4.4.** *Let $\Psi : I \times X \to X$ be a continuous nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$ functions and $g : I \times I \to \mathbb{R}$ be a continuous function ($X$ is a strong $b$-Banach space), then for any bounded set $M \subset X$ and $t_0,\ t \in I$ with $t_0 \leq t$, we have*

$$\begin{aligned} s\Phi_s\left(\int_{t_0}^t g(t, \tau)\Psi(\tau, M(\tau))d\tau\right) &\leq \int_{t_0}^t s\Phi_s\left(g(t, \tau)\Psi(\tau, M(\tau))\right)d\tau \\ &\leq \left(\int_{t_0}^t |g(t, \tau)|d\tau\right)\mathcal{K}\left(\Phi_d(M)\right). \end{aligned}$$

*Or more strictly*

$$s\Phi_s\left(\int_{t_0}^t g(t, \tau)\Psi(\tau, M(\tau))d\tau\right) \leq m_{g^*}(b - a)\mathcal{K}\left(\Phi_d(M)\right),$$

*where*

$$m_{g^*} = \sup\left\{|g(t, \tau)|;\ a \leq \tau \leq t \leq b\right\}$$

*Proof.* Let us start by define the operators $T : X \to X$, by

$$Tx(t) = \int_{t_0}^{t} g(t,\tau)\Psi(\tau, x(\tau))d\tau$$

Since $\Psi$ is nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$, then from Definition 3.5, there exist $\mathcal{K} \in \mathcal{Y}_1$, such that for almost every $\tau \in [a,b]$, the following holds:

$$s\Phi_s(\{\Psi(\tau, x(\tau)) : x \in M\}) \leq \mathcal{K}(\Delta_\tau),$$

where $\Delta_\tau := \Phi_s(\{x(\tau) : x \in M\})$.
For a given $\varepsilon > 0$ and almost every $\tau \in I$, there exists a continuous mapping $\gamma_\Psi : \mathcal{J}_1 \longrightarrow (E, \|\cdot\|)$, with $\gamma_\Psi(\mathcal{J}_1) \subset \{\Psi(\tau, x(\tau)) :,\ x \in M\}$, such that for each $x \in M$ there is $r \in \mathcal{J}_1$ satisfying

$$s\left\|\Psi(\tau, x(\tau)) - \gamma_\Psi(r)\right\| \leq \mathcal{K}(\Delta_\tau) + \varepsilon, \text{ for almost every } \tau \ \in [a,b].$$

Now, we define the mapping $\widetilde{\gamma_\Psi} : \mathcal{J}_1 \to (X, \|\cdot\|_X)$ as

$$r \in \mathcal{J}_1 \longmapsto \widetilde{\gamma_\Psi}(r,t) := \int_{t_0}^{t} g(t,\tau)\gamma_\Psi(r)d\tau, \quad \text{for all } t \in [t_0, b],$$

This mapping is clearly continuous, and it holds that $\widetilde{\gamma_\Psi}(\mathcal{J}_1) \subset T(M)$. Moreover, for each $x \in M$, there exists $r \in \mathcal{J}_1$ such that

$$\begin{aligned} s\|Tx(t) - \widetilde{\gamma_\Psi}(r,t)\| &\leq s\int_{t_0}^{t} |g(t,\tau)| \left\|\Psi(\tau, x(\tau)) - \gamma_\Psi(r)\right\| d\tau \\ &\leq \int_{t_0}^{t} |g(t,\tau)| \left(\mathcal{K}(\Delta_\tau) + \varepsilon\right) d\tau. \end{aligned}$$

Let $\Delta := \Phi_s(M)$, then we have $\Delta_\tau \leq \Delta$ for almost every $\tau \in [a,b]$. Based on the preceding inequality, we can deduce that

$$s\|Tx(t) - \widetilde{\gamma_\Psi}(r,t)\| \leq \int_{t_0}^{t} |g(t,\tau)| \left(\mathcal{K}(\Delta) + \varepsilon\right) d\tau.$$

Now, as $\varepsilon \to 0$, we get

$$s\|Tx(t) - \widetilde{\gamma_\Psi}(r,t)\| \leq \left(\int_{t_0}^{t} |g(t,\tau)|d\tau\right) \mathcal{K}(\Delta).$$

This means that, due to the arbitrariness of $t \in [t_0, b]$, we have

$$s\Phi_d(T(M)) \leq \left(\int_{t_0}^{t} |g(t,\tau)|d\tau\right) \mathcal{K}(\Phi_d(M)).$$

Furthermore

$$s\Phi_d(T(M)) \leq m_{g^*}(b-a)\mathcal{K}(\Phi_d(M)).$$

□

**Remark 4.5.** In the previous lemma:

(1) If $\Psi$ is an $k^*$-contraction$^{\Phi_s}$ functions, then we will get

$$s\Phi_d(T(M)) \leq k^* m_{g^*}(b-a)\Phi_d(M).$$

(2) In the case when $\mathcal{K}(r) = l_\Psi(t)r$ where $l_\Psi \in L^1(I, \mathbb{R}^+)$ we get

$$s\Phi_d(T(M)) \leq m_{g^*}\|l_\Psi\|_{L^1}\Phi_d(M).$$

*Proof of Theorem 4.2.* Transform the problem (4.1) into a fixed point problem and define the operator $T : \mathcal{X}_0 \to \mathcal{X}_0$ by:

$$Tu(t) = \Psi_1^*(t)\Psi_2^*(t) + \varpi^+(t)\left(g_a + \Psi_3^*(t) + \int_a^t (\tau - a)^{\beta-1}\Psi_4(\tau, u(\tau))d\tau\right),\ t \in [a, b].$$

We will prove that the operator $T$ has at least one fixed point. So we define the operators $T_i : \mathcal{X}_a \to \mathcal{X}_a; i = 1, 2, 3$, such that

$$Tu(t) = T_1u(t)T_2u(t) + T_3u(t),$$

where

$$T_iu(t) = \Psi_i(t, u(t)),\ i = 1, 2.$$

$$T_3u(t) = \varpi^+(t)g_a + \varpi^+(t)\Psi_3(t, u(t)) + \varpi^+(t)\int_a^t (\tau - a)^{\beta-1}\Psi_4(\tau, u(\tau))d\tau,$$

and the set $M_\theta = \{u \in \mathcal{X}_a : \|u\|_{\mathcal{X}_a} \leq \theta\}$, such that

$$\frac{\varpi^*\|g_a\|\sqrt{2}}{1 - \sqrt{2}\left(g_1^*g_2^* + \varpi^*\left(g_3^* + (b-a)^{\beta-1}\|g_4\|_{L^1}\right)\right)} \leq \theta.$$

**Step 1**: $T_i$ are continuous.
It is clear that the functions $T_i; (i = 1, 2)$ are continuous operators.
Now, let $\{u_n\}_n$ be a sequence such that $u_n \to u^*$ on $M_\theta$. For each $t \in I$, we have

$$\begin{aligned}\|(T_3u_n)(t) - (T_3u^*)(t)\| \quad &\leq \quad \varpi^*\sqrt{2}\|\Psi_3(t, u_n(t)) - \Psi_3(t, u^*(t))\| \\ &\quad + \varpi^*\sqrt{2}\left[\int_a^t (\tau - a)^{\beta-1}\left\|\Psi_4\left(\tau, u_n(\tau)\right) - \Psi_4\left(\tau, u^*(\tau)\right)\right\| d\tau\right].\end{aligned}$$

By condition $(Q_2)$ and Lebesgue dominated convergence theorem, we deduce that

$$\|T_3u_n(t) - T_3u^*(t)\| \to 0 \quad \text{as } n \to +\infty.$$

Hence, $T_3$ is also continuous.
**Step 2**: $T_1(u)T_2(u) + T_3(u) \in M_\theta$, for each $u \in M_\theta$.
Let $u \in M_\theta$ and $t \in I$, we have

$$\begin{aligned}\|(Tu)(t)\| &\leq \sqrt{2}\|\Psi_1^*(t)\Psi_2^*(t)\| \\ &\quad + \frac{\sqrt{2}}{|\varpi(t)|}\left(\|g_a\| + \sqrt{2}\|\Psi_3^*(t)\| + 2\int_a^t (\tau - a)^{\beta-1}\|\Psi_4(\tau, u(\tau))|d\tau\right) \\ &\leq \sqrt{2}g_1^*g_2^*\theta + \varpi^*\sqrt{2}\left[\|g_a\| + g_3^*\theta + (b-a)^{\beta-1}\|g_4\|_{L^1}\theta\right] \\ &\leq \theta.\end{aligned}$$

Hence, $\|T(u)\|_{\mathcal{X}_a} \leq \theta$. Consequently, $T(M_\theta) \subset M_\theta$.
**Step 3**: Consider a non-empty and convex subset $M \subset M_\theta$. According to conditions $(Q_i); i = 1, 2, 3$, Proposition 3.3 and Lemma 4.4, we get

$$\sqrt{2}\Phi_d(T_3(M)) \leq \xi\Phi_d(M),$$

where

$$\xi = \varpi^* \min\left\{\sqrt{2}g_3^* + (b-a)^{\beta-1}\|g_4\|_{L^1}, g_3^* + \sqrt{2}(b-a)^{\beta-1}\|g_4\|_{L^1}\right\}.$$

Therefore, $T_3$ is $\xi$-contraction$^{\Phi_s}$ operator. on the other hand we know that $T_1$ is an $g_1^*$-contraction$^{\Phi_s}$ operator and $T_2$ is compact. Thus, all the condition of Theorem 3.14 are satisfied, then we can conclude that the equation $T_1(u)T_2(u)+T_3(u) = u$ has a solution, then the operator $T$ has fixed point, which is a solution of the problem (4.1). □

**Remark 4.6.** As special cases, we have:

(1) If $\Psi_1$ or $\Psi_2$ is the null function, then Theorem 3.6 is applicable.
(2) If $\Psi_4$ is a nonlinear $\mathcal{K}$-contraction$^{\Phi_s}$ functions, then Theorem 3.10 is applicable.
(3) If $\Psi_2$ is the identity function, then Theorem 3.12 is applicable.

4.1. **An Example.** To conclude, we will offer an example to demonstrate our findings.
Let $\mathcal{X} = c_0$ the Banach algebra space of real sequences converging to zero:

$$c_0 = \left\{x = (x_n)_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n = 0\right\},$$

with the standard norm:

$$\|x\|_0 = \sup_{n\geq 1}|x_n|.$$

Then, $(C([\pi, 2\pi], c_0), \|\cdot\|_*, s)$ is a strong $b$-Banach space endowed with the structure of $b$-Banach algebra for all $s > 1$, in particular for $s = \sqrt{2}$.
Now, consider the following problem:

$$\begin{cases}\mathcal{D}_{\pi}^{1/2}\left(\varpi(t)\left[u(t) - \Psi_1^*(t)\Psi_2^*(t)\right]\right) = \Psi_4\left(t, u(t)\right) + \mathcal{D}_{\pi}^{1/2}\left(\Psi_3^*(t)\right), \ t \in I := [\pi, 2\pi], \\ u(\pi) = e^{-\pi^2},\end{cases} \tag{4.3}$$

where

$$\begin{aligned}
\Psi_1(t,u) &= \left(\frac{u_n(\sin(t))}{7+e^{t+1}}\right)_{n>0}, \\
\Psi_2(t,u) &= \left(\int_{\pi}^{2\pi}\sin(t-\tau)\left(\frac{|\sin(|u_n|)|}{t^2+1}\tau+\pi\right)d\tau\right)_{n>0}, \\
\Psi_3(t,u) &= \left(\frac{e^{-t+\frac{1}{3}}|u_n\cos\left(\frac{t}{2}\right)|}{e^{7\pi}(1+|u_n|)}\right)_{n>0}, \\
\Psi_4(t,u) &= \left(\frac{e^{-t}\arctan(t+1)}{\pi^2}\log\left(1+\frac{|u_n|}{\pi}\right)\right)_{n>0}, \\
\varpi(t) &= \frac{\pi^3+|\cos(t)|}{\sqrt{\pi}}, \\
u_0 &= e^{-\pi^2}.
\end{aligned}$$

Then, it's not difficult to see that

$$|\varpi(t)| \geq \pi^{5/2}, \text{ for each } t \in [\pi, 2\pi].$$

$$\|\Psi_1(t,u)\|_0 \leq \frac{1}{7+e^{t+1}}\|u\|_0,$$

$$\|\Psi_2(t,u)\|_0 \leq \frac{5\pi^2}{2(t^2+1)}\|u\|_0,$$

$$\|\Psi_3(t,u)\|_0 \leq e^{-t-7\pi+\frac{1}{3}}\|u\|_0,$$

and

$$\|\Psi_4(t,u)\|_0 \leq \frac{e^{-t}\arctan(t+1)}{\pi^3}\|u\|_0.$$

Then

$$\varpi^* = \pi^{-\frac{5}{2}},\ g_1(t) = \frac{1}{7+e^{t+1}},\ g_2(t) = \frac{5\pi^2}{2(t^2+1)},$$

$$g_3(t) = e^{-t-7\pi+\frac{1}{3}},\ g_4(t) = \frac{e^{-t}\arctan(t+1)}{\pi^3}.$$

Now, let $\mathcal{M} \subset \mathcal{X}$ be a non-empty, bounded, and convex set. For a fixed $t \in \mathcal{J}_1$, assume $\gamma$ is a $\varphi_t$-dense curve within the set $\{\zeta(t) : \zeta \in \mathcal{M}\}$ for some $\varphi_t > 0$. Then, for any $\zeta \in \mathcal{M}$, there exists $\tau \in I$ such that

$$\|\zeta(t) - \gamma(\tau,t)\| \leq \varphi_t.$$

Then, we get

$$\|\Psi_1(t,\zeta(t)) - \Psi_1(t,\gamma(\tau,t))\|_0 \leq \frac{1}{7+e^{t+1}}\varphi_t,$$

$$\|\Psi_3(t,\zeta(t)) - \Psi_3(t,\gamma(\tau,t))\|_0 \leq e^{-t-7\pi+\frac{1}{3}}\varphi_t,$$

and

$$\|\Psi_4(t,\zeta(t)) - \Psi_4(t,\gamma(\tau,t))\|_0 \leq \frac{e^{-t}\arctan(t+1)}{\pi^3}\varphi_t.$$

Thus

$$\begin{aligned}\Phi_s(\Psi_1(t,\mathcal{M})) &\leq \frac{1}{7+e^{t+1}}\Phi_s(\mathcal{M}),\\ \Phi_s(\Psi_3(t,\mathcal{M})) &\leq e^{-t-7\pi+\frac{1}{3}}\Phi_s(\mathcal{M}),\end{aligned}$$

and

$$\Phi_s(\Psi_4(t,\mathcal{M})) \leq \frac{e^{-t}\arctan(t+1)}{\pi^3}\Phi_s(\mathcal{M}).$$

On the other hand, we have

$$\max\left\{\tau^*, \sqrt{2}\left(g_1^* g_2^* + \varpi^*\left(g_3^* + (b-a)^{\beta-1}\|g_4\|_{L^1}\right)\right)\right\} < 1,$$

As a result, for $1.571 \times 10^{-4} < \theta$, all the criteria outlined in Theorem 4.2 are fulfilled.

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