Fixed Point Theory, 27(2026), No. 1, 45-66
DOI: 10.24193/fpt-ro.2026.1.03
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A UNIFIED APPROACH TO EQUIEXPANSIVE λ-SET CONTRACTIONS AND THEIR APPLICATION TO FRACTIONAL DIFFERENTIAL SYSTEMS

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Abstract. This research explores equiexpansive mappings of the Hardy-Roger type, focusing on the conditions required for the existence and uniqueness of fixed points. By utilizing foundational results established by Krasnoselskii and Sadovskii, the study develops a robust theoretical framework to support these conditions. To enhance understanding, illustrative examples with graphical representations are provided, showcasing fixed points in mappings that satisfy specific criteria. Furthermore, the study demonstrates the practical relevance of equiexpansive mappings by addressing a fractional differential system governed by the semi-linear Atangana-Baleanu-Caputo model. This application serves to validate the theoretical findings while emphasizing the significance of these mappings in solving complex differential systems. In summary, the work enriches the theoretical landscape and underscores its practical utility, paving the way for further advancements and applications in this domain.

Key Words and Phrases: Krasnoselskii fixed point theorem, expansive mapping, Hardy-Roger's type equiexpansive mapping, λ-set contraction.
2020 Mathematics Subject Classification: 47H10, 54H25, 34A08.

1. Introduction

The study of fixed point theory has become a cornerstone in exploring the existence and stability of various linear and non-linear problems. This powerful framework has been instrumental in addressing complex mathematical challenges and uncovering deeper insights into their structures. Its applications extend far beyond theoretical

mathematics, influencing areas such as functional analysis, differential equations, optimization, and applied sciences. This widespread relevance highlights the pivotal role of fixed point theory in modern research and analytical methodologies. There are two most well-known results, namely Schauder's fixed point result [29] and Banach contraction principle [10] based on fixed point theory. Schauder's finding is still more practical, despite the elegance of the Banach principle in determining the originality and stability of any solution. Both results are significant and relevant in their fields. There are many generalizations by the authors in the direction of Banach (see [6, 8, 13, 14, 3, 24, 35]) and Schauder (see [12, 16, 32, 34]) individually.

In the work of Krasnoselskii [23], a generalization of the findings by Schauder and Banach was presented, specifically about the identification of fixed points of the mappings that are both compact and contraction. The fixed point problem for a subset Y of a Banach space B, given by the equation

$$Cw + Ew = w, w \in Y,$$

was examined by Krasnoselskii in 1958 through the use of the Schauder theorem and the Banach contraction theorem. The operator equation $Cw + Ew = w$ has been addressed by several researchers, who have made use of different assumptions regarding C, E, and Y, keeping in mind the relevance of these kinds of issues.

In 2003, Barroso [11] conducted a study on Krasnoselskii's fixed point theorem, focusing on its application to weakly continuous maps. The investigation encompassed the resolution of several mathematical problems, including the Dirichlet problem with non-linearities, the eigenvalue problem, and the integral equation. In 2007, Park [25] made further contributions to the existing theory. Garcia-Falset et al. [19] extended the existing theory for weakly sequentially continuous mappings to investigate the solution of a non-linear integral equation that is formulated in a Banach space. By introducing convex-power condensing, Ezzinbi et al. [18] investigated Sadovskii-Krasnoselskii fixed point results. Combining the concept of Meir-Keeler mapping and expansive Krasnoselskii fixed point theorem, Pourhadi et al. [26] presented some results in this field. Additional results have been shown in previous studies [9, 22] about expansive mappings.

Within this framework, we introduce several significant extensions of Krasnoselskii's fixed point theorem, broadening its applicability and theoretical depth. Furthermore, we explore a new class of extended equiexpansive mappings, which generalizes existing concepts by incorporating λ-set contractive mappings as a subset. A central focus of this study is the examination of a specialized category of differential problems, specifically the semi-linear Atangana-Baleanu-Caputo fractional differential system. Through the resolution of this problem, we seek to advance theoretical insights into fractional systems and provide meaningful contributions to the broader field of differential equations. Our findings not only enhance the understanding of such systems but also lay a foundation for future research and practical applications in related mathematical and scientific domains.

The subsequent sections of this article are organized in the following manner. In Section 2, we present the terminology and preliminary results that will be used throughout the rest of the paper. We present equiexpansive mapping of the Hardy-Roger type in Section 3, and by generalizing Krasnoselskii's and Sadovski's fixed point theorems, we get adequate conditions for the existence of fixed points. To back up our findings, we present several examples. Using the fixed point technique presented in Section 3, we derive a solution to a semi-linear Atangana-Baleanu-Caputo fractional differential system, which is the primary objective of the controllability investigation.

2. Preliminaries

We now present some known formulations of the Krasnoselskii theorem consisting of continuous, contraction, compact, and expansive mappings.

Theorem 2.1 ([31]). *Assume Y is a subset of any Banach space B that is convex, closed and not empty. When Y is mapped into B by the mappings E and C, that ensures*

(i) *$Cv + Ew \in Y$ for all $v, w \in Y$;*
(ii) *C is continuous and compact;*
(iii) *E is a contraction.*

Then, $w \in Y$ exists such that $Cw + Ew = w$.

The Banach principle is used in the following theorem, established by Xiang and Yuan ([37]) for expansive mapping.

Definition 2.1 ([37]). Let Y be a non-empty subset of a metric space (B, d). When Y is mapped into B by the mapping E, that ensures

$$d(Ev, Ew) \geq hd(v, w),$$

for all $v, w \in Y$ and $h > 1$. Then E claims to be expansive mapping.

Theorem 2.2 ([37, 36]). *Let Y be a non-empty and closed subset of a complete metric space (B, d). When Y is mapped into B by the expansive mapping E with $Y \subseteq E(Y)$. Then, $w \in Y$ exists uniquely such that $Ew = w$.*

The following result is another variant of the Krasnoselskii result that includes one expansive mapping.

Theorem 2.3 ([37]). *Let Y be a non-empty, closed and convex subset of a metric space (B, d). When Y is mapped into B by the mappings C and E, that ensures*

(i) *$Y \subseteq v + E(Y)$ for all $v \in C(Y)$;*
(ii) *for some compact $K \subseteq B$, $C(Y) \subseteq K$ and C is continuous;*
(iii) *E is expansive.*

Then, $w \in Y$ exists such that $Cw + Ew = w$.

Definition 2.2 ([1]). Let Y be a non-empty subset of a metric space (B, d). When $Y \times Y$ is mapped into B by the mapping H, that ensures

$$d\left(H(v, w'), H(v, w'')\right) \geq hd(w', w''),$$

for all $(v, w'), (v, w'') \in Y \times Y$ and $h > 1$. Then H claims to be equiexpansive mapping.

In 1973, Hardy and Roger's [20] introduced Hardy-Rogers contraction and proved the following.

Theorem 2.4 ([20]). *Let (B, d) be any complete metric space and suppose $C : B \longrightarrow B$ is a self-mapping that satisfies*

$$d(Cv, Cw) \leq \varrho_1 d(v, w) + \varrho_2 d(Cv, v) + \varrho_3 d(Cw, w) + \varrho_4 d(Cv, w) + \varrho_5 d(Cw, v), \quad (1)$$

for all $v, w \in B$ and for $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0$ with $\varrho_1 + \varrho_2 + \varrho_3 + \varrho_4 + \varrho_5 < 1$. Then, $w \in B$ exists uniquely such that $Cw = w$.

The following two definitions about the measure of non-compactness are important in our context.

Definition 2.3 ([33]). Let $\tilde{Y}$ be the collection of all subsets of any Banach Space (B, d) which are bounded. A map $\mu : Y \longrightarrow B$ is said to be a measure of non-compactness if for $Y_1, Y_2 \in \tilde{Y}$ it satisfies

(a) $\mu(Y) = 0 \Longleftrightarrow Y$ is pre-compact;
(b) $\mu(Y) = \mu(\overline{Y})$;
(c) $\mu(Y_1 \cup Y_2) = \max\{\mu(Y_1), \mu(Y_2)\}$.

Lemma 2.1. Let $\mu : Y \longrightarrow B$ be a measure of non-compactness. Then for $Y_1, Y_2 \in \tilde{Y}$

(i) $Y_1 \subseteq Y_2 \Longrightarrow \mu(Y_1) \leq \mu(Y_2)$;
(ii) $\mu(Y_1 + Y_2) \leq \mu(Y_1) + \mu(Y_2)$.

Definition 2.4 ([37]). A map $C : Y \longrightarrow B$, where Y is a subset of any Banach space B that is not empty, is known as λ-set contractive, for $0 \leq \lambda < 1$, if

(i) $\mu(C(Y_1)) \leq \lambda\mu(Y_1)$, for all bounded subset Y_1 of Y, and
(ii) C is bounded and continuous.

Again, if the inequality in (i) is strict with $\mu(Y_1) \neq 0$, then C claims to be strictly λ-set contraction, and if C is 1-set contractive then C is said to be a condensing map.

Sadovskii presented the following result encompassing λ-set contractive mapping [28].

Theorem 2.5 ([28]). *Consider Y to be a subset of any Banach space B that is bounded, convex, closed and not empty and $C : Y \longrightarrow B$ is a λ-set contractive. Then C has a fixed point.*

3. Main results

We will use the definitions of Hardy-Roger contraction and expansive mapping to define Hardy-Roger's equiexpansive mapping, a new class of generalized equiexpansive mapping. Also, we will state and prove the result for mappings that satisfy such expansive conditions must have a unique fixed point. Consequently, we will conclude some corollaries which are generalizations of Ćirić [17], Chatterjea [15], Reich [27], and Kannan [21] type contractions in the equiexpansive form.

We now define expansive mapping of Hardy-Roger's type.

Definition 3.1. Let Y be a non empty subset of any metric space (B, d). When Y is mapped into B by the mapping E such that

$$\varrho_1 d(Ev, Ew) + \varrho_2 d(Ev, v) + \varrho_3 d(Ew, w) + \varrho_4 d(Ev, w) + \varrho_5 d(Ew, v) \geq h d(v, w), \quad (2)$$

for all $v, w \in Y$, where $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0; \varrho_1 + \varrho_2 + \varrho_3 + \varrho_4 + \varrho_5 < h$. Then E claims to be Hardy-Roger's type expansive mapping.

The following theorem is for Hardy-Roger's type expansive mapping.

Theorem 3.3 *Let Y be a closed and not empty subset of any complete metric space (B, d) and E be a Hardy-Roger's type expansive, one to one mapping of Y into B such that $Y \subseteq E(Y)$. Then, $w \in Y$ exists uniquely such that $Ew = w$.*

Proof. Since the mapping E is a Hardy-Roger's type expansive, we have

$$\varrho_1 d(Ev, Ew) + \varrho_2 d(Ev, v) + \varrho_3 d(Ew, w) + \varrho_4 d(Ev, w) + \varrho_5 d(Ew, v) \geq h d(v, w),$$

for all $v, w \in Y$ and for $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0$ with $\varrho_1 + \varrho_2 + \varrho_3 + \varrho_4 + \varrho_5 < h$.

Again the mapping $E : Y \longrightarrow B$ is one to one, then $E : Y \longrightarrow E(Y)$ is invertible and $E^{-1} : E(Y) \longrightarrow Y$ exists. So, for every, $v, w \in Y$ we will have $x, y \in E(Y)$ such that $E^{-1}x = v$ and $E^{-1}y = w$. So the last inequality reduced to

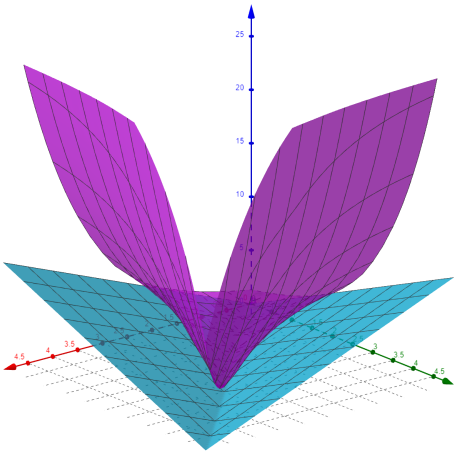
$$\begin{aligned} &\varrho_1 d(x, y) + \varrho_2 d(x, E^{-1}x) + \varrho_3 d(y, E^{-1}y) + \varrho_4 d(x, E^{-1}y) + \varrho_5 d(y, E^{-1}x) \\ &\qquad \geq h d(E^{-1}x, E^{-1}y) \\ &\Rightarrow d(E^{-1}x, E^{-1}y) \\ &\quad \leq \tfrac{\varrho_1}{h} d(x, y) + \tfrac{\varrho_2}{h} d(x, E^{-1}x) + \tfrac{\varrho_3}{h} d(y, E^{-1}y) + \tfrac{\varrho_4}{h} d(x, E^{-1}y) + \tfrac{\varrho_5}{h} d(y, E^{-1}x), \end{aligned}$$

where $\frac{\varrho_1}{h} + \frac{\varrho_2}{h} + \frac{\varrho_3}{h} + \frac{\varrho_4}{h} + \frac{\varrho_5}{h} < 1$. Since $Y \subseteq E(Y)$, we see that $E^{-1} : Y \longrightarrow Y$ becomes a contraction mapping of the Hardy-Roger's type (1). Hence, by Theorem 2.1 there must be $w \in Y$ such that, $E^{-1}w = w$ i.e., $Ew = w$.

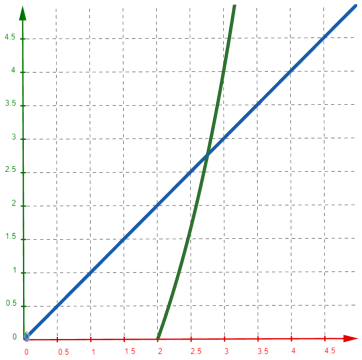
For uniqueness, on the contrary, suppose there are two distinct $w^*, w^{**} \in Y$ such that $Ew^* = w^*$ and $Ew^{**} = w^{**}$. Then in (1), putting $v = w^*, w = w^{**}$ we have

$$(\varrho_1 + \varrho_4 + \varrho_5) d(w^*, w^{**}) \geq h d(w^*, w^{**}),$$

which implies $\varrho_1 + \varrho_4 + \varrho_5 \geq h$, a contradiction. So $w \in Y$ is unique such that $Ew = w$.



(A) Graph of left and right side terms in inequality (3).



(B) Existence of unique fixed point.

FIGURE 1. Graphical representation for Example 3.1.

The example below illustrates our result.

Example 3.1. Consider the subset $[2,5]$ of the standard complete metric space $(\mathbb{R}, d)$ and the mapping $E : [2,5] \longrightarrow \mathbb{R}$ defined as

$$E(w) = 2^w - 4, \ \forall \ w \in [2,5].$$

Now,

$$\begin{aligned} 2d(v,w) &= 2|v-w| \\ &\leq |2^v - 2^w| + \frac{1}{7}|2^v - 4 - v| + \frac{1}{11}|2^w - 4 - w| + \frac{1}{13}|2^v - 4 - w| \\ &\quad + \frac{1}{8}|2^w - 4 - v| \qquad (3) \\ &\leq d(Ev, Ew) + \frac{1}{7}d(Ev, v) + \frac{1}{11}d(Ew, w) + \frac{1}{13}d(Ev, w) + \frac{1}{8}d(Ew, v), \end{aligned}$$

for all $u, w \in [2,5]$ so that $1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{8} < 2$. The graphical representation of the inequality and the existence of the fixed point can be seen from Figure 1.

Thus, mapping E is a Hardy-Roger's type expansive mapping. Also, the interval $[2,5]$ is closed in $\mathbb{R}$ and E is one to one with $[2,5] \subseteq E([2,5])$. That is, E satisfy Theorem 3.1 having unique $w \approx 2.7562153$ so that $Ew = w$.

We now have a consequent corollary.

Corollary 3.1. *Let Y be a closed and not empty subset of any complete metric space (B, d). When Y is mapped into B by the mapping E such that $Y \subseteq E(Y)$ and for all*

$v, w \in Y$ satisfies

$$d(Ev, Ew) \geq h_0 d(v, w), h_0 > 1.$$

Then, $w \in Y$ exists uniquely such that $Ew = w$.

Proof. According to the statement for all $v, w \in Y$ satisfies

$$d(Ev, Ew) \geq h_0 d(v, w), h_0 > 1.$$

Suppose $v \neq w$ then $d(v, w) > 0$ also $h_0 > 1$ implies $d(Ev, Ew) > 0$. Thus E is one-to-one. Again taking $\varrho_2 = \varrho_3 = \varrho_4 = \varrho_5 = 0$ and $h_0 = \frac{h}{\varrho_1} > 1$ we see E satisfies all the conditions of Theorem 3.1. Thus we have $w \in Y$ unique such that $Ew = w$.

Remark 3.1. Note that, the Corollary 3.1 is another representation of Theorem 2.3 in [36, 37].

Now we define our new generalized Hardy-Roger's type equiexpansive mapping.

Definition 3.2. Let Y be a not empty subset of any metric space (B, d) and $h : Y \times Y \longrightarrow B$ such that

$$\varrho_1 d\Big(H(v, w'), H(v, w'')\Big) + \varrho_2 d\Big(H(v, w'), w'\Big) + \varrho_3 d\Big(H(v, w''), w''\Big) + \varrho_4 d\Big(H(v, w'), w''\Big) + \varrho_5 d\Big(H(v, w''), w'\Big) \geq h d(w', w''), \tag{4}$$

where $\varrho_1, h > 0$; $\varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0$, $\varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$ with $\varrho_i^* = 2\varrho_i, i = 1, 4, 5$ and for all $(v, w'), (v, w'') \in Y \times Y$. Then h claims to be Hardy-Roger's type generalized equiexpansive mapping.

We are now ready to present our main theorem concerning generalized Hardy-Roger's type equiexpansive mapping.

Theorem 3.2. *Let Y be a closed and not empty subset of any complete metric space (B, d) and $H : Y \times Y \longrightarrow B$ be a Hardy-Roger's type generalized equiexpansive one to one mapping concerning the second variable such that $Y \subseteq H(v, Y)$, for all $v \in Y$ and satisfies*

$$d\Big(H(v', w), H(v'', w)\Big) \leq d(v', v''), \text{ for all } (v', w), (v'', w) \in Y \times Y. \tag{5}$$

Then, $w \in Y$ exists uniquely such that $H(w, w) = w$.

Proof. For any fixed $v \in Y$, let us define $H_2(w) = H(v, w)$, for all $w \in Y$. Then $H_2 : Y \longrightarrow B$ such that $Y \subseteq H(v, Y) = H_2(Y)$. Again, since the function $H : Y \times Y \longrightarrow B$ is a Hardy-Roger's type generalized equiexpansive one-to-one mapping concerning the second variable. Then, for all $w', w'' \in Y$, $H(v, w') = H(v, w'') \Rightarrow w' = w''$, and $H_2(w') = H_2(w'') \Rightarrow w' = w''$. So, H_2 is one-to-one. Also, we have from (4)

$$\varrho_1 d\Big(H_2(w'), H_2(w'')\Big) + \varrho_2 d\Big(H_2(w'), w'\Big) + \varrho_3 d\Big(H_2(w''), w''\Big) + \varrho_4 d\Big(H_2(w'), w''\Big) + \varrho_5 d\Big(H_2(w''), w'\Big) \geq h d(w', w''),$$

for all $w', w'' \in Y$, where $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0$ and

$$\varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$$

$$\Rightarrow \varrho_1 + \varrho_2 + \varrho_3 + \varrho_4 + \varrho_5 < h.$$

Then H_2 satisfies all the conditions of Theorem 3.1. Thus, we have $w \in Y$ is unique such that $H_2 w = w$ i.e., for any fixed $v \in Y$, we have $w \in Y$ is unique such that $H_2 w = w = H(v, w)$. So, we can define a function $H_1 : Y \longrightarrow B$ such that for all $v \in Y$, $H_1 v = w$ i.e., $H_1 v = H(v, H_1 v)$. Now for distinct $v', v'' \in Y$, we have the following two cases:

CASE 1:

$$\begin{aligned}
&d(H_1 v', H_1 v'') \\
&= d\Big(H(v', H_1 v'), H(v'', H_1 v'')\Big) \\
&\geq d\Big(H(v'', H_1 v'), H(v'', H_1 v'')\Big) - d\Big(H(v', H_1 v'), H(v'', H_1 v')\Big) \\
&\geq \frac{h}{\varrho_1} d(H_1 v', H_1 v'') - \frac{\varrho_2}{\varrho_1} d\Big(H(v'', H_1 v'), H_1 v'\Big) - \frac{\varrho_3}{\varrho_1} d\Big(H(v'', H_1 v''), H_1 v''\Big) \\
&- \frac{\varrho_4}{\varrho_1} d\Big(H(v'', H_1 v'), H_1 v''\Big) - \frac{\varrho_5}{\varrho_1} d\Big(H(v'', H_1 v''), H_1 v'\Big) - d(v', v''), \\
&\text{by inequality (3) and (4)} \\
&= \frac{h}{\varrho_1} d(H_1 v', H_1 v'') - \frac{\varrho_2}{\varrho_1} d\Big(H(v'', H_1 v'), d(v', H_1 v')\Big) - \frac{\varrho_3}{\varrho_1} d(H_1 v'', H_1 v'') \\
&- \frac{\varrho_4}{\varrho_1} d\Big(H(v'', H_1 v'), d(v'', H_1 v'')\Big) - \frac{\varrho_5}{\varrho_1} d(H_1 v'', H_1 v') - d(v', v'') \\
&\geq \frac{h}{\varrho_1} d(H_1 v', H_1 v'') - \frac{\varrho_2}{\varrho_1} d\Big(H(v'', H_1 v'), d(v', H_1 v')\Big) - \frac{\varrho_4}{\varrho_1} \Big(d(H_1 v', H_1 v'') \\
&+ d(H(v', H_1 v'), H(v'', H_1 v')\Big) - \frac{\varrho_5}{\varrho_1} d(H_1 v'', H_1 v') - d(v', v'') \\
&\geq \frac{h}{\varrho_1} d(H_1 v', H_1 v'') - \frac{\varrho_2}{\varrho_1} d(v', v'') - \frac{\varrho_4}{\varrho_1} \Big(d(H_1 v', H_1 v'') + d(v', v'')\Big) \\
&- \frac{\varrho_5}{\varrho_1} d(H_1 v'', H_1 v') - d(v', v'') \text{ by inequality (4).}
\end{aligned}$$

So, it implies

$$\Big(1 + \frac{\varrho_2}{\varrho_1} + \frac{\varrho_4}{\varrho_1}\Big) d(v', v'') \geq \Big(\frac{h}{\varrho_1} - \frac{\varrho_4}{\varrho_1} - \frac{\varrho_5}{\varrho_1} - 1\Big) d(H_1 v', H_1 v'').$$

After rewriting, we get

$$d(H_1 v', H_1 v'') \leq \frac{\varrho_1 + \varrho_2 + \varrho_4}{h - \varrho_1 - \varrho_4 - \varrho_5} d(v', v'').$$

As $\varrho_1^* + \varrho_2 + \varrho_4^* + \varrho_5 < \varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$, we get $\dfrac{\varrho_1 + \varrho_2 + \varrho_4}{h - \varrho_1 - \varrho_4 - \varrho_5} < 1$.

CASE 2:

$$
\begin{aligned}
&d(H_1v', H_1v'')\\
&= d\Big(H(v', H_1v'), H(v'', H_1v'')\Big)\\
&\geq d\Big(H(v', H_1v'), H(v', H_1v'')\Big) - d\Big(H(v', H_1v''), H(v'', H_1v'')\Big)\\
&\geq \frac{h}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_2}{\varrho_1}d\Big(H(v', H_1v'), H_1v'\Big) - \frac{\varrho_3}{\varrho_1}d\Big(H(v', H_1v''), H_1v''\Big)\\
&- \frac{\varrho_4}{\varrho_1}d\Big(H(v', H_1v'), H_1v''\Big) - \frac{\varrho_5}{\varrho_1}d\Big(H(v', H_1v''), H_1v'\Big) - d(v', v''),\\
&\text{by inequality (3) and (4)}\\
&= \frac{h}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_2}{\varrho_1}d(H_1v', H_1v') - \frac{\varrho_3}{\varrho_1}d\Big(d(v', H_1v''), d(v'', H_1v'')\Big)\\
&- \frac{\varrho_4}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_5}{\varrho_1}d\Big(H(v', H_1v''), H(v', H_1v')\Big) - d(v', v'')\\
&\geq \frac{h}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_3}{\varrho_1}d\Big(H(v'', H_1v'), d(v', H_1v')\Big) - \frac{\varrho_4}{\varrho_1}d(H_1v', H_1v'')\\
&- \frac{\varrho_5}{\varrho_1}\Big(d(H_1v', H_1v'') + d(v', v'')\Big) - d(v', v'')\\
&\geq \frac{h}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_3}{\varrho_1}d(v', v'') - \frac{\varrho_4}{\varrho_1}d(H_1v', H_1v'') - \frac{\varrho_5}{\varrho_1}\Big(d(H_1v', H_1v'')\\
&+ d(v', v'')\Big) - d(v', v'') \text{ by inequality (4).}
\end{aligned}
$$

It implies

$$d(H_1v', H_1v'') \quad \leq \quad \frac{\varrho_1 + \varrho_3 + \varrho_5}{h - \varrho_1 - \varrho_4 - \varrho_5}d(v', v'').$$

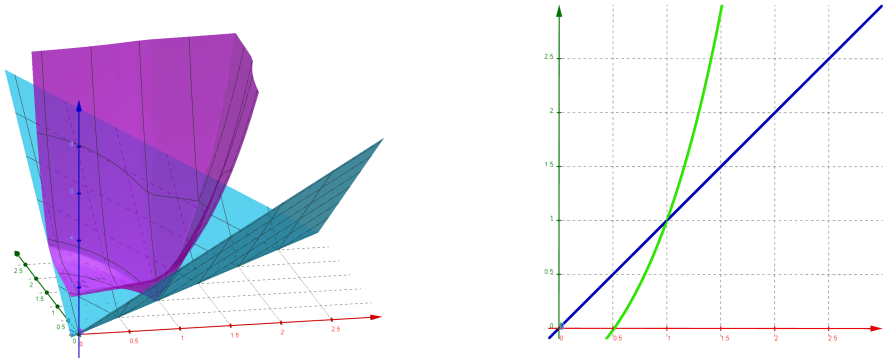
As $\varrho_1^* + \varrho_3 + \varrho_4 + \varrho_5^* < \varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$, we get $\frac{\varrho_1+\varrho_3+\varrho_5}{h-\varrho_1-\varrho_4-\varrho_5} < 1$.

In either case, we have $H_1 : Y \longrightarrow B$ as a contraction. Thus, $w \in Y$ exists uniquely such that $H_1w = w$ i.e., $H(w, H_1w) = w$ i.e., $H(w, w) = w$.

The example below illustrates our main result.

Example 3.2. Let $Y = [0.5, 2], B = \mathbb{R}$ and the mapping $H : Y \times Y \longrightarrow \mathbb{R}$ be defined as

$$H(v, w) = \frac{1 - v}{2} + 4^w - 3, \text{ for all } (v, w) \in [0.5, 2]^2.$$



(A) Graph of inequality (6). (B) Existence of unique fixed point.

FIGURE 2. Graphical representation for Example 3.2.

Now,

$$\begin{aligned}
\frac{5}{2}d(w',w'') &= \frac{3}{2}|w'-w''| \\
&\leq \frac{1}{2}|4^{w'}-4^{w''}|+\frac{1}{4}\Big|\frac{1-v}{2}+4^{w'}-3-w'\Big|+\frac{1}{3}\Big|\frac{1-v}{2}+4^{w''}-3-w''\Big| \\
&\quad +\frac{1}{5}\Big|\frac{1-v}{2}+4^{w'}-3-w''\Big|+\frac{1}{6}\Big|\frac{1-v}{2}+4^{w''}-3-w'\Big| \qquad (6)\\
&\leq \frac{1}{2}d(H(v,w'),H(v,w''))+\frac{1}{4}d(H(v,w'),w')+\frac{1}{3}d(h(v,w''),w'') \\
&\quad +\frac{1}{5}d(H(v,w'),w'')+\frac{1}{6}d(H(v,w''),w'),
\end{aligned}$$

and

$$\begin{aligned}
d\Big(H(v',w),H(v'',w)\Big) &= \Big|\frac{1-v'}{2}+4^{w}-3-\frac{1-v''}{2}-4^{w}+3\Big| \\
&= \Big|\frac{v'}{2}-\frac{v''}{2}\Big| \\
&\leq d(v',v'')
\end{aligned}$$

for all $v,w',w'',v',v'' \in [0.5,2]$ so that $1+\frac{1}{4}+\frac{1}{3}+\frac{2}{5}+\frac{2}{6}<\frac{5}{2}$. The graphical representation of the inequality and the existence of the fixed point can be seen from Figure 2.

Thus h satisfy Theorem 3.2 for $\varrho_1=\frac{1}{2},\varrho_2=\frac{1}{4},\varrho_3=\frac{1}{3},\varrho_4=\frac{1}{5},\varrho_5=\frac{1}{6}$ and $h=\frac{3}{2}$ having unique $1\in[0.5,2]$ so that $H(1,1)=1$.

Subsequently, we have a corollary for Banach-type equiexpansive mapping.

Corollary 3.2. *Let Y be a closed and not empty subset of any complete metric space (B,d) and $H: Y\times Y \longrightarrow B$ such that $Y \subseteq H(v,Y)$ for all $v \in Y$ and satisfies*

$$\varrho_1 d\Big(H(v,w'),H(v,w'')\Big) \geq hd(w',w''), \tag{7}$$

for all $(v,w'),(v,w'') \in Y\times Y; \varrho_1, h>0; 2\varrho_1 < h$ and

$$d\Big(H(v',w),H(v'',w)\Big) \leq d(v',v''), \text{for all } \ (v',w),(v'',w) \in Y\times Y. \tag{8}$$

Then, there is unique $w \in Y$ such that $H(w,w)=w$.

Proof. From inequality (7), the function $H: Y\times Y \longrightarrow B$ satisfies the one to one condition. Also putting $\varrho_2=\varrho_3=\varrho_4=\varrho_5=0$, we see that h satisfies all the conditions of Theorem 3.2, hence the Corollary follows.

The following result is after our theorem involving two distinct mappings.

Corollary 3.3. *Let Y be a closed and not empty subset of any complete metric space (B,d) and $C,E: Y\longrightarrow B$ be two mappings such that $Y \subseteq Cv+E(Y)$, for all $v\in Y$ and E is one to one and satisfies*

$$\begin{aligned}\varrho_1 d(Ew',Ew'') + \varrho_2 d\Big(Cv,(I-E)w'\Big) + \varrho_3 d\Big(Cv,(I-E)w''\Big)\\ +\varrho_4 d(Cv+Ew',w'') + \varrho_5 d(Cv+Ew'',w') \geq hd(w',w''),\end{aligned} \tag{9}$$

for all $v,w',w'' \in Y$, where $\varrho_1, h>0; \varrho_2,\varrho_3,\varrho_4,\varrho_5 \geq 0$ and $\varrho_1^*+\varrho_2+\varrho_3+\varrho_4^*+\varrho_5^* < h$ with $\varrho_i^*=2\varrho_i, i=1,4,5$. Also,

$$d(Cv',Cv'') \leq d(v',v''), \text{for all } \ v',v'' \in Y. \tag{10}$$

Then, $w\in Y$ exists uniquely such that $Cw+Ew=w$.

Proof. Let us define $H(v,w)=Cv+Ew$. Then, by the conditions given, h satisfies the conditions of the Theorem 3.2. Thus, $w\in Y$ exists uniquely such that $H(w,w)=w$ i.e., $Cw+Ew=w$.

Remark 3.2. Notice that, If we define $Cv=\frac{1-v}{2}$ and $Ew=4^w-3$, then the mapping h defined in Example 3.2 is a sum of two mappings C,E. Consequently, in Example 3.2 we can apply Corollary 3.3 for the unique fixed point.

We now demonstrate the following result using our main theorem encompassing λ-set contractive condition.

Theorem 3.3. *Let Y be a bounded, convex, closed, and not empty subset of any Banach space (B,d) and $H: C(Y)\times Y \longrightarrow B$ be a one to one mapping concerning second variable with $Y \subseteq H(Cv,Y)$ for all $Cv \in C(Y)$, satisfies*

$$\begin{aligned}\varrho_1 d\Big(H(Cv,w'),H(Cv,w'')\Big) + \varrho_2 d\Big(H(Cv,w'),w'\Big) + \varrho_3 d\Big(H(Cv,w''),w''\Big)\\ +\ \varrho_4 d\Big(H(Cv,w'),w''\Big) + \varrho_5 d\Big(H(Cv,w''),w'\Big) \geq hd(w',w''),\end{aligned} \tag{11}$$

for all $(Cv, w'), (Cv, w'') \in C(Y) \times Y$ *and*

$$d\Big(H(Cv', w), H(Cv'', w)\Big) \leq d(Cv', Cv''), \text{for all } (Cv', w), (Cv'', w) \in C(Y) \times Y, \tag{12}$$

where $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0; \varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$ *with* $\varrho_i^* = 2\varrho_i, i = 1, 4, 5$ *and* $C : Y \longrightarrow B$ *is* λ*-set contractive with*

$$\lambda < \min\Big\{\frac{h - \varrho_1 - \varrho_4 - \varrho_5}{\varrho_1 + \varrho_2 + \varrho_4}, \frac{h - \varrho_1 - \varrho_4 - \varrho_5}{\varrho_1 + \varrho_3 + \varrho_5}\Big\}.$$

Then, $w \in Y$ *exists uniquely such that* $H(Cw, w) = w$.

Proof. Proceeding similarly as Theorem 3.2, for fixed $Cv \in C(Y)$ the map $H_2 : Y \longrightarrow B$ given by $H_2(w) = H(Cv, w), \forall\ w \in Y$ has a unique $w \in Y$ such that $H_2 w = w$ i.e., for fixed $Cv \in C(Y)$, we have $w \in Y$ is unique such that $H_2 w = w = H(Cv, w)$. So we can define a function $H_1 : C(Y) \longrightarrow B$ such that for all $Cv \in C(Y)$, $H_1(Cv) = w$ i.e., $H_1(Cv) = H(Cv, H_1(Cv))$. Also for distinct $Cv', Cv'' \in C(Y)$ following $\underline{CASE\ 1}, \underline{CASE\ 2}$ of Theorem 3.2, we have

$$d(H_1(Cv'), H_1(Cv'')) \leq \frac{\varrho_1 + \varrho_2 + \varrho_4}{h - \varrho_1 - \varrho_4 - \varrho_5} d(Cv', Cv''), \tag{13}$$

$$d(H_1(Cv'), H_1(Cv'')) \leq \frac{\varrho_1 + \varrho_3 + \varrho_5}{h - \varrho_1 - \varrho_4 - \varrho_5} d(Cv', Cv''). \tag{14}$$

This shows that $H_1 \circ C$ is continuous and bounded. Again using λ-set contraction of C we have

$$\begin{aligned}
\mu(H_1 \circ C(K)) &= \mu(H_1(C(K))) \\
&\leq \max\Big\{\frac{\varrho_1 + \varrho_2 + \varrho_4}{h - \varrho_1 - \varrho_4 - \varrho_5}, \frac{\varrho_1 + \varrho_3 + \varrho_5}{h - \varrho_1 - \varrho_4 - \varrho_5}\Big\}\mu(C(K)) \\
&\leq \lambda \max\Big\{\frac{\varrho_1 + \varrho_2 + \varrho_4}{h - \varrho_1 - \varrho_4 - \varrho_5}, \frac{\varrho_1 + \varrho_3 + \varrho_5}{h - \varrho_1 - \varrho_4 - \varrho_5}\Big\}\mu(K) \\
&< \mu(K),
\end{aligned}$$

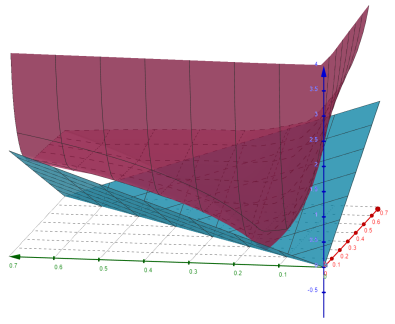
for all $K \subseteq Y$. Thus $H_1 \circ C$ becomes λ-set contractive map and by Theorem 2.5 there is $w \in Y$ such that $H_1(C(w)) = w$ i.e., $H(Cw, w) = w$.

The example below illustrates Theorem 3.3.

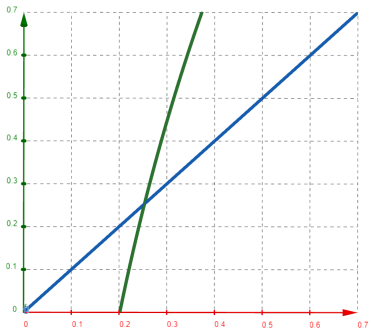
Example 3.3. Let $Y = [0, 0.4], B = \mathbb{R}$ and the mapping $H : C(Y) \times Y \longrightarrow \mathbb{R}$ be defined as

$$H(v, w) = \frac{3v + 3}{2} + \ln w, \text{ for all } (v, w) \in C([0, 0.4]) \times [0, 0.4].$$

Where λ-set contractive map $C : Y \longrightarrow \mathbb{R}$ is given by $Cv = \frac{v}{3}, \forall\ v \in [0, 0.4]$ for $\lambda = \frac{1}{3}$.



(A) Graph of left and right side terms in inequality (15).



(B) Existence of unique fixed point.

FIGURE 3. Graphical representation for Example 3.3.

Now,

$$\begin{aligned} 3d(w',w'') &= 3|w'-w''| \\ &\le \frac{1}{2}|\ln w' - \ln w''| + \frac{3}{10}\Big|\frac{v+3}{2} + \ln w' - w'\Big| + \frac{1}{2}\Big|\frac{v+3}{2} + \ln w'' - w''\Big| \\ &+ \frac{1}{3}\Big|\frac{v+3}{2} + \ln w' - w''\Big| + \frac{1}{4}\Big|\frac{v+3}{2} + \ln w'' - w'\Big| \qquad (15) \\ &\le \frac{1}{2}d(H(Cv,w'),H(Cv,w'')) + \frac{1}{4}d(H(Cv,w'),w') + \frac{1}{3}d(h(Cv,w''),w'') \\ &+ \frac{1}{5}d(H(Cv,w'),w'') + \frac{1}{6}d(H(Cv,w''),w'), \end{aligned}$$

and

$$\begin{aligned} d\Big(H(Cv',w),H(Cv'',w)\Big) &= \Big|\frac{v'+1}{2} + \ln w - \frac{v''+1}{2} - \ln w\Big| \\ &= \Big|\frac{v'}{2} - \frac{v''}{2}\Big| \\ &\le d(v',v'') \end{aligned}$$

for all $v, w', w'', v', v'' \in [0, 0.4]$ so that $1 + \frac{3}{10} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} < 3$. The graphical representation of the inequality and the existence of the fixed point can be seen from Figure 3.

Thus h satisfy Theorem 3.3 for $\varrho_1 = \frac{1}{2}, \varrho_2 = \frac{3}{10}, \varrho_3 = \frac{1}{2}, \varrho_4 = \frac{1}{3}, \varrho_5 = \frac{1}{4}, h = 3$ and

$$\frac{1}{3} = \lambda < \min\left\{\frac{h-\varrho_1-\varrho_4-\varrho_5}{\varrho_1+\varrho_2+\varrho_4}, \frac{h-\varrho_1-\varrho_4-\varrho_5}{\varrho_1+\varrho_3+\varrho_5}\right\} = \left\{\frac{115}{68}, \frac{23}{15}\right\},$$

having unique $0.253 \in [0, 0.4]$ (correct up to 3 decimal places) so that $H(C(0.253), 0.253) = 0.253$.

Afterward, we have consequences of Theorem 3.3.

Corollary 3.4. *Let Y be a bounded, convex, closed, and not empty subset of any Banach space (B, d) and $C, E : Y \longrightarrow B$ be two mappings such that $Y \subseteq Cv + E(Y)$, for all $v \in Y$ and E is one to one and satisfies*

$$\begin{aligned} \varrho_1 d(Ew', Ew'') + \varrho_2 d\Big(Cv, (I-E)w'\Big) + \varrho_3 d\Big(Cv, (I-E)w''\Big) \\ + \varrho_4 d(Cv + Ew', w'') + \varrho_5 d(Cv + Ew'', w') \geq h d(w', w''), \end{aligned} \tag{16}$$

for all $v, w', w'' \in Y$, where $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4, \varrho_5 \geq 0$ and $\varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* + \varrho_5^* < h$ with $\varrho_i^* = 2\varrho_i, i = 1, 4, 5$. Also, $C : Y \longrightarrow B$ is λ-set contractive with

$$\lambda < \min\Big\{\frac{h - \varrho_1 - \varrho_4 - \varrho_5}{\varrho_1 + \varrho_2 + \varrho_4}, \frac{h - \varrho_1 - \varrho_4 - \varrho_5}{\varrho_1 + \varrho_3 + \varrho_5}\Big\}.$$

Then, $w \in Y$ exists uniquely such that $Cw + Ew = w$.

Proof. Let us define $H(Cv, w) = Cv + Ew$. Then, by the conditions given, h satisfies the conditions of Theorem 3.3. Thus, $w \in Y$ exists uniquely such that $H(Cw, w) = w$ i.e., $Cw + Ew = w$.

Corollary 3.5. *Let Y be a bounded, convex, closed, and not empty subset of any Banach space (B, d) and $C, E : Y \longrightarrow B$ be two mappings such that $Y \subseteq Cv + E(Y)$, for all $v \in Y$ and E is one to one and satisfies*

$$\varrho_1 d(Ew', Ew'') \geq h d(w', w''),$$

for all $w', w'' \in Y$, where $\varrho_1, h > 0$ and $2\varrho_1 < h$. Also $C : Y \longrightarrow B$ is λ-set contractive with

$$\lambda < \frac{h - \varrho_1}{\varrho_1}.$$

Then, $w \in Y$ exists uniquely so that $Cw + Ew = w$.

Proof. Let us define $H(Cv, w) = Cv + Ew$. Then, by the conditions given, h satisfies the conditions of Theorem 3.3 for $\varrho_2 = \varrho_3 = \varrho_4 = \varrho_5 = 0$. Thus, $w \in Y$ exists uniquely such that $H(Cw, w) = w$ i.e., $Cw + Ew = w$.

Besides, we have some subsequent corollaries of Theorem 3.2 for Ćirić, Chatterjea, Reich, and Kannan type equiexpansive mappings.

Corollary 3.6. *Let Y be a closed and not empty subset of a complete metric space (B, d) and $H : Y \times Y \longrightarrow B$ be one to one mapping concerning second variable such that $Y \subseteq H(v, Y)$, for all $v \in Y$ and satisfies*

$$\begin{aligned} \varrho_1 d\Big(H(v, w'), H(v, w'')\Big) + \varrho_2 d\Big(H(v, w'), w'\Big) + \varrho_3 d\Big(H(v, w''), w''\Big) \\ + \varrho_4 \Big[d\Big(H(v, w'), w''\Big) + d\Big(H(v, w''), w'\Big)\Big] \geq h d(w', w''), \end{aligned} \tag{17}$$

where $\varrho_1, h > 0; \varrho_2, \varrho_3, \varrho_4 \geq 0$, $\varrho_1^* + \varrho_2 + \varrho_3 + \varrho_4^* < h$ *with* $\varrho_1^* = 2\varrho_1, \varrho_4^* = 3\varrho_4$ *and for all* $(v, w'), (v, w'') \in Y \times Y$. *Also,*

$$d\Big(H(v', w), H(v'', w)\Big) \leq d(v', v''), \forall\ (v', w), (v'', w) \in Y \times Y. \tag{18}$$

Then, $w \in Y$ *exists uniquely such that* $H(w, w) = w$.

Proof. Letting $\varrho_4 = \varrho_5$ in Theorem 3.2, the result follows.

Remark 3.3. The Corollary 3.6 is a generalization of Ćirić [17] type contraction in the equiexpansive form.

Corollary 3.7. *Let* Y *be a closed and not empty subset of any complete metric space* (B, d) *and* $H : Y \times Y \longrightarrow B$ *be one to one mapping concerning the second variable such that* $Y \subseteq H(v, Y)$, *for all* $v \in Y$ *and satisfies*

$$\varrho_1\Big[d\Big(H(v, w'), w''\Big) + d\Big(H(v, w''), w'\Big)\Big] \geq hd(w', w''),$$

for all $(v, w'), (v, w'') \in Y \times Y$, *where* $h > 0; 0 \leq \varrho_1 < 1$ *and* $\varrho_1^* < h$ *with* $\varrho_1^* = 3\varrho_1$. *Also,*

$$d\Big(H(v', w), H(v'', w)\Big) \leq d(v', v''), \forall\ (v', w), (v'', w) \in Y \times Y. \tag{19}$$

Then, $w \in Y$ *exists uniquely such that* $H(w, w) = w$.

Proof. Assuming $\varrho_1 = \varrho_2 = \varrho_3 = 0$ in Corollary 3.6, Corollary 3.7 is immediate.

Remark 3.4. In the form of equiexpansive mappings, Corollary 3.7 is a generalization of Chatterjea [15] type contraction.

Corollary 3.8. *Let* Y *be a closed and not empty subset of any complete metric space* (B, d) *and* $H : Y \times Y \longrightarrow B$ *be one to one mapping concerning the second variable such that* $Y \subseteq H(v, Y)$, *for all* $v \in Y$ *and satisfies*

$$\varrho_1 d\Big(H(v, w'), H(v, w'')\Big) + \varrho_2 d\Big(H(v, w'), w'\Big) + \varrho_3 d\Big(H(v, w''), w''\Big) \geq hd(w', w''),$$

for all $(v, w'), (v, w'') \in Y \times Y$, *where* $\varrho_1, h > 0; \varrho_2, \varrho_3 \geq 0$ *and* $\varrho_1^* + \varrho_2 + \varrho_3 < h$ *with* $\varrho_1^* = 2\varrho_1$. *Also,*

$$d\Big(H(v', w), H(v'', w)\Big) \leq d(v', v''), \forall\ (v', w), (v'', w) \in Y \times Y. \tag{20}$$

Then, $w \in Y$ *exists uniquely such that* $H(w, w) = w$.

Proof. If we put $\varrho_4 = 0$ in Corollary 3.6, we will have our result.

Remark 3.5. The corollary stated above in equiexpansive form is none but a generalized form of Reich [27] type contraction.

Corollary 3.9. *Let Y to be a closed and not empty subset of a complete metric space (B, d) and $H : Y \times Y \longrightarrow B$ be one to one mapping concerning the second variable such that $Y \subseteq H(v, Y)$, for all $v \in Y$ and satisfies*

$$\varrho_1\Big[d\Big(H(v, w'), w'\Big) + d\Big(H(v, w''), w''\Big)\Big] \geq hd(w', w''),$$

for all $(v, w'), (v, w'') \in Y \times Y$, where $h > 0; 0 \leq \varrho_1 \leq \frac{1}{2}$ and $\varrho_1 < h$. Also,

$$d\Big(H(v', w), H(v'', w)\Big) \leq d(v', v''), \forall\ (v', w), (v'', w) \in Y \times Y. \tag{21}$$

Then, $w \in Y$ exists uniquely such that $H(w, w) = w$.

Proof. In Corollary 3.8, if we put $\varrho_2 = \varrho_3, \varrho_1 = 0$ and $\frac{1}{2} < h$, then the result is immediate.

Remark 3.6. The above corollary is a generalization of Kannan [21] type contraction in equiexpansive form.

Remark 3.7.

(i) In the 3D graphs (a), (c), and (e), we plotted two functions, one is the left side function and the other is the right side function of the inequalities (3), (6) and (15), respectively. In each graph, since the plot of the right side function is on the upper side of the plot of the left side function, we can say that the corresponding inequality is true in the defined interval.

(ii) As we want to visualize the fixed point w so that $H(w, w) = w$, in the 2D graphs (b), (d) and (f), we plotted two curves, one is for the function $H(w, w)$ and another is for w in Examples 3.1, 3.2 and 3.3 respectively. Clearly, in each case, the intersecting point is a unique fixed point.

4. Application

Impulsive differential problems have emerged as a challenging and intriguing area of research within the broader domain of non-linear systems. These problems are characterized by sudden changes in state at specific moments, making them essential for modeling complex real-world phenomena. Their versatility has led to applications in a wide array of fields, including population dynamics, where they help model abrupt changes in species populations; economics, for capturing rapid shifts in market conditions; physics, for studying discontinuous physical processes; biology, for representing sudden physiological events; and engineering, for analyzing systems subject to instantaneous perturbations.

Simultaneously, fractional calculus, despite its origins dating back to the work of Newton and Leibniz, continues to push the boundaries of mathematical research and application. This advanced framework extends classical calculus by introducing derivatives and integrals of arbitrary orders, making it exceptionally powerful for capturing memory effects and hereditary properties of systems. Over recent years, fractional calculus has gained widespread recognition as a more effective alternative

to classical derivatives, particularly for modeling real-world systems where traditional approaches fall short. Its ability to provide more accurate and nuanced representations has made it a preferred choice across various scientific disciplines.

The study of impulsive differential problems in the context of fractional calculus has thus become a focal point of mathematical inquiry. Many researchers have addressed these problems using fixed point theory (see [4, 5, 7]), particularly contraction principles, to establish the existence and uniqueness of solutions. This approach has proven to be a robust mathematical tool for solving impulsive problems, offering a structured method to tackle their inherent complexities.

Building on these advancements, our study focuses on the semi-linear Atangana-Baleanu-Caputo fractional differential system (see [2]), a specific yet versatile framework within fractional calculus. Here, we thoroughly examine the necessary conditions for the existence and uniqueness of solutions, contributing to the ongoing development of mathematical theory and its applications in solving complex impulsive systems. Through this analysis, we aim to expand the understanding of these systems and provide a robust foundation for future investigations in both theoretical and applied contexts.

$$\begin{cases} {}^{ABC}D^{\rho}v(v) &= \mathcal{A}v(v) + \varphi(v)w(v) + g(v, v(\psi(v))), v \in [0, V], V > 0, 0 < \rho < 1, \\ v(0) & = v_0 \in B. \end{cases} \tag{22}$$

This system represents a recent development in non-local derivatives, characterized by a nonsingular kernel. We emphasize the importance of further investigations to establish comprehensive connections between this derivative and other areas of scientific research. The favorable properties of this derivative facilitate the advancement of researchers' scientific endeavors, enabling the creation of more sophisticated and active research studies.

The typical definition and results of the solution operator are widely understood, so we've excluded them here. Interested readers can find more details in the suggested references [2, 30]. Our main concern is exploring the existence of a solution to the problem (22), which requires us to present the subsequent mild solution.

$$\begin{aligned} v(v) = {} & \mathcal{PM}_{\rho}(v)v_0 + \frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)} \int_0^v (v-\eta)^{\rho-1} g(\eta, v(\psi(\eta)))d\eta \\ & + \frac{\rho\mathcal{P}^2}{B(\rho)} \int_0^v \mathcal{N}_{\rho}(v-\eta) g(\eta, v(\psi(\eta)))d\eta + \frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)} \int_0^v (v-\eta)^{\rho-1} \varphi(\eta)w(\eta)d\eta \\ & + \frac{\rho\mathcal{P}^2}{B(\rho)} \int_0^v \mathcal{N}_{\rho}(v-\eta)\varphi(\eta)w(\eta)d\eta, \end{aligned}$$

where,

${}^{ABC}D^{\rho}$	Atangana-Baleanu-Caputo derivative of fractional order ρ
$\mathcal{A} : D(\mathcal{A}) \subset B \longrightarrow B$	the infinitesimal generator of an ρ-resolvent family $(\mathcal{N}_p(v))_{v\geq 0}$, $(\mathcal{M}_p(v))_{v\geq 0}$ is solution operator defined on complex Banach space $(B, \|\cdot\|)$, $D(\mathcal{A})$ is domain of $\mathcal{A}$
$\varphi : B \longrightarrow B$	a bounded linear operator from a Banach space B into B,
$w \in L^2([0, V], B)$	a control function, $L^2([0, V], B)$ is the Banach space of square integrable functions from $[0, V]$ to B
$\mathcal{B}$	the Banach space $\{h : [0, V] \longrightarrow B : h$ is continuous except for finite number of points s at which $h(s^+)$ and $h(s^-)$ exist with $h(s^+) = h(s^-)\}$ with respect to usual 'sup' norm
$\psi \in C([0, V], [0, V])$	$C([0, V], [0, V])$ is the Banach space of all continuous functions from $[0, V]$ to $[0, V]$
$g \in C([0, V] \times B, B)$	$C([0, V] \times B, B)$ is the Banach space of all continuous functions from $[0, V] \times B$ to B
$\mathcal{P}$	the linear operator $\mu(\mu I - \mathcal{A})^{-1}$
$\mathcal{Q}$	the linear operator $-\mu\mathcal{A}(\mu I - \mathcal{A})^{-1}$
$\mathcal{M}_\rho(v)$	$= E_\rho(-\mathcal{Q}v^\rho) = \frac{1}{2\pi i}\int_\Theta e^{\eta v}\eta^{\rho-1}(\eta^\rho I - \mathcal{Q})^{-1}d\eta$
$\mathcal{N}_\rho(v)$	$= v^{\rho-1}E_\rho(-\mathcal{Q}v^\rho) = \frac{1}{2\pi i}\int_\Theta e^{\eta v}(\eta^\rho I - \mathcal{Q})^{-1}d\eta$
Θ	a certain path lying on $\{z \in \mathbb{C} : z \neq r, \lvert arg(z - r)\rvert < \beta\}$ for fixed $r \in \mathbb{R}, \beta \in [\frac{\pi}{2}, \pi]$
$B(\rho)$	$= \mu(1 - \rho)$

Let $h : Y \times Y \longrightarrow \mathcal{B}$, for any closed subset Y of $\mathcal{B}$, be given by

$$\begin{aligned} H(v(v), w(v)) &= \mathcal{P}\mathcal{M}_\rho(v)v_0 + \frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^v (v-\eta)^{\rho-1}g(\eta, v(\psi(\eta)))d\eta \\ &\quad + \frac{\rho\mathcal{P}^2}{B(\rho)}\int_0^v \mathcal{N}_\rho(v-\eta)g(\eta, v(\psi(\eta)))d\eta \\ &\quad + \frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^v (v-\eta)^{\rho-1}\varphi(\eta)w(\eta)d\eta \\ &\quad + \frac{\rho\mathcal{P}^2}{B(\rho)}\int_0^v \mathcal{N}_\rho(v-\eta)\varphi(\eta)w(\eta)d\eta. \end{aligned}$$

Then

$$\begin{aligned} &|H(v(v), w_1(v)) - H(v(v), w_2(v))| \\ &= \left| \begin{array}{l} \frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^v (v-\eta)^{\rho-1}\varphi(\eta)w_1(\eta)d\eta + \frac{\rho\mathcal{P}^2}{B(\rho)}\int_0^v \mathcal{N}_\rho(v-\eta)\varphi(\eta)w_1(\eta)d\eta \\ -\frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^v (v-\eta)^{\rho-1}\varphi(\eta)w_2(\eta)d\eta - \frac{\rho\mathcal{P}^2}{B(\rho)}\int_0^v \mathcal{N}_\rho(v-\eta)\varphi(\eta)w_2(\eta)d\eta \end{array} \right| \\ &= \left| \begin{array}{l} \frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^v (v-\eta)^{\rho-1}\varphi(\eta)(w_1(\eta) - w_2(\eta))d\eta \\ +\frac{\rho\mathcal{P}^2}{B(\rho)}\int_0^v \mathcal{N}_\rho(v-\eta)\varphi(\eta)(w_1(\eta) - w_2(\eta))d\eta \end{array} \right| \\ &\geq \left| \min\left\{\frac{\mathcal{Q}\mathcal{P}(1-\rho)}{B(\rho)\Gamma(\rho)}, \frac{\rho\mathcal{P}^2}{B(\rho)}\right\}\int_0^v \Big\{(v-\eta)^{\rho-1} + \mathcal{N}_\rho(v-\eta)\Big\}\varphi(\eta)(w_1(\eta) - w_2(\eta))d\eta \right| \end{aligned}$$

and

$$
\begin{aligned}
&|H(v_1(v), w(v)) - H(v_2(v), w(v))| \\
&= \frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)} \int_0^v (v-\eta)^{\rho-1} g(\eta, v_1(\psi(\eta)))d\eta + \frac{\rho\mathcal{P}^2}{B(\rho)} \int_0^v \mathcal{N}_\rho(v-\eta) g(\eta, v_1(\psi(\eta)))d\eta \\
&- \frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)} \int_0^v (v-\eta)^{\rho-1} g(\eta, v_2(\psi(\eta)))d\eta - \frac{\rho\mathcal{P}^2}{B(\rho)} \int_0^v \mathcal{N}_\rho(v-\eta) g(\eta, v_2(\psi(\eta)))d\eta \\
&= \frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)} \int_0^v (v-\eta)^{\rho-1} (g(\eta, v_1(\psi(\eta))) - g(\eta, v_2(\psi(\eta))))d\eta \\
&\frac{\rho\mathcal{P}^2}{B(\rho)} \int_0^v \mathcal{N}_\rho(v-\eta)(g(\eta, v_1(\psi(\eta))) - g(\eta, v_2(\psi(\eta))))d\eta.
\end{aligned}
$$

Now $\mathcal{A} \in \mathcal{A}^\rho(a_0, b_0)$ for $a_0 > 0, b_0 \in \mathbb{R}$ implies that (22) has ρ-resolvent family $\mathcal{N}_p(v)$ and solution operator $\mathcal{M}_p(v)$ such that $\| \mathcal{M}_p(v) \| \leq a_0 e^{bv}$ and for some constant M, $\| \mathcal{N}_p(v) \| \leq Me^{bv}(1 + v^{\rho-1})$ for every $v > 0, b > b_0$, see [30].

Theorem 4.1. *The problem defined in equation (22) will possess a unique solution if it meets the following requirements.*

(i) $\| \int_0^v \Big\{(v-\eta)^{\rho-1} + \mathcal{N}_\rho(v-\eta)\Big\}\varphi(\eta)(w_1(\eta) - w_2(\eta))d\eta \| \geq M_1 \| w_1 - w_2 \|$, *for some* $M_1 > 0$,

(ii) $\| g(\eta, v_1(\psi(\eta))) - g(\eta, v_2(\psi(\eta))) \| \leq M_2 \| v_1 - v_2 \|$, *for some* $M_2 > 0$,

(iii) $\Big(\frac{\|\mathcal{Q}\|}{B(\rho)} \| \frac{\mathcal{P}(1-\rho)}{\Gamma(\rho)} \| \frac{\|v^\rho\|}{\rho} + \| \frac{\rho\mathcal{P}^2}{B(\rho)} \| \| Mve^{bv}(1 + v^{\rho-1}) \| \Big) M_2 \leq 1$ *and* $M_1 \min\Big\{\frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)}, \frac{\rho\mathcal{P}^2}{B(\rho)}\Big\} > 2$.

Proof. We have

$$
\begin{aligned}
&\varrho \| H(v(v), w_1(v)) - H(v(v), w_2(v)) \| \\
&\qquad \geq \varrho M_1 \min\Big\{\frac{\mathcal{QP}(1-\rho)}{B(\rho)\Gamma(\rho)}, \frac{\rho\mathcal{P}^2}{B(\rho)}\Big\} \| w_1 - w_2 \| \\
&\qquad \geq 2\varrho \| w_1 - w_2 \|
\end{aligned}
$$

and

$$
\begin{aligned}
&\| H(v_1(v), w(v)) - H(v_2(v), w(v)) \| \\
&\leq \Big(\frac{\|\mathcal{Q}\|}{B(\rho)} \| \frac{\mathcal{P}(1-\rho)}{\Gamma(\rho)} \| \frac{\|v^\rho\|}{\rho} + \| \frac{\rho\mathcal{P}^2}{B(\rho)} \| \| Mve^{bv}(1 + v^{\rho-1}) \| \Big) M_2 \| v_1 - v_2 \| \\
&\leq \| v_1 - v_2 \|.
\end{aligned}
$$

So, h satisfy Corollary 3.2, i.e., there exists unique $w \in Y$ such that $H(w, w) = w$.

We now present an illustrative example for the model (22).

Example 4.1. Consider an Atangana-Baleanu-Caputo type semi-linear differential system of fractional order $\frac{1}{2}$

$$\begin{cases} \frac{\partial^{\frac{1}{2}} v}{\partial \theta^{\frac{1}{2}}}(v,\theta) & = \frac{\partial^2 v}{\partial \theta^2}(v,\theta) + \varphi(v)w(v) + \frac{e^v}{(4+e^v)^2}\tan^{-1}(v), (v,\theta) \in [0,1]\times[0,\pi], \\ v(v,0) & = v(v,\pi), \quad \frac{\partial v}{\partial v}(v,0) = \frac{\partial v}{\partial v}(v,\pi), \quad v(0,\theta) = v_0(\theta) \in L^2([0,\pi]). \end{cases} \tag{23}$$

Now comparing with equation (22), we have

$$\rho = \frac{1}{2},\ V = 1,\ B = L^2([0,\pi]),\ D(\mathcal{A}) = \left\{ v, \frac{\partial v}{\partial \theta}, \frac{\partial^2 v}{\partial \theta^2} \right\} \subset L^2([0,\pi]),$$

and

$$g(v, v(v,\theta)) = \frac{e^{-v}}{(4+e^v)^2}\tan^{-1}(v(v,\theta)), \quad \text{for all} \ \ (v,\theta) \in [0,1]\times[0,\pi].$$

Then $\mathcal{A}(v) = \frac{\partial^2 v}{\partial \theta^2}(v,\theta)$ is the infinitesimal generator of a $\frac{1}{2}$-resolvent family $(\mathcal{N}_{\frac{1}{2}}(v))_{v\geq 0}$ densely defined in $L^2([0,\pi])$. Also,

$$\| \frac{e^{-v}}{(4+e^v)^2}\tan^{-1}(v_1(v,\theta)) - \frac{e^{-v}}{(4+e^v)^2}\tan^{-1}(v_2(v,\theta)) \| \leq M_2 \| v_1 - v_2 \|,$$

for $M_2 = \frac{1}{4}$.
Again, we can consider $\varphi(v) = e^{-v},\ w(v) = e^{2v}$ satisfying condition (i) of Theorem 4.1 for $M_1 = 1$. So, equation (23) satisfies Theorem 4.1 to have a solution.

5. Conclusion

Motivated by the idea of defining Hardy-Roger type contraction and expansive mapping, we defined Hardy-Roger's type equiexpansive mapping, a new class of generalized equiexpansive mapping. Besides generalizing the concept of expansive mappings and results in [36], our consequent results generalized Ćirić [17], Chatterjea [15], Reich [27], and Kannan [21] in equiexpansive sense. Further, we investigated our main theorem when the inequality involved two distinct mappings and encompassed λ-set contractive conditions respectively. Finally, we presented the prerequisites for the existence of a singular solution to a semi-linear Atangana-Baleanu-Caputo fractional differential system that provides a useful framework for defining unexpected changes that occur at specific points in various stages of continuous development and helps us comprehend many real occurrences that occur in the context of particular conditions in applied research. In the future, the aforesaid analysis can be extended to a nonlinear coupled system involving a Mittag-Leffler type derivative.

Acknowledgement. The author Khairul Habib Alam thanks the IISER Berhampur, for financial support as a Post-Doctoral Research Fellowship through employment PF No. C.B0160. The author Mi Zhou is partially supported by Sanya City Science and Technology Innovation Special Project (Grant No. 2022KJCX22).

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Received: March 12, 2025; Accepted: August 28, 2025.