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SOME RESULTS ON FIXED POINTS ON THE PRODUCT OF b-METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove an existence smd uniqueness theorem for a solution of a system of nonlinear equations in the product of b-metric spaces. We obtain generalization of the classical results of Banach, Kannan and Reich in the product of b-metric spaces. Also, some variants of the results of Czerwik, Bhaskar and Lakshmikantham are obtained. Finally, we give an application in support of our result.
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1. Introduction and preliminaries

Let $n \in \mathbb{N}$. We will use the notation $[n] = \{1, \ldots, n\}$. Let (E_i, d_i, s_i), $i \in [n]$ be complete b-metric spaces. For $e_i \in E_i$, $i \in [n]$, consider the system of equations

$$\begin{aligned} T_1(e_1, \ldots, e_n) &= e_1 \\ &\vdots \\ T_n(e_1, \ldots, e_n) &= e_n, \end{aligned} \tag{1.1}$$

where

$$T_i : E_1 \times \cdots \times E_n \to E_i, i \in [n]. \tag{1.2}$$

In this paper, we give some conditions under which the system (1.1) has a unique solution. Also, we give some applications of the obtained results.
Problem to solve such a system of equations (1.1) is of real interest. Some of relevant previous results for the same problem in the case of metric spaces and b-metric spaces (e.g., coupled fixed point results, tripled fixed point results, multiple fixed point results, and their generalizations) can be seen in [8, 29, 30, 31, 32, 33, 34, 38, 39, 40].
Actually, Czerwik [14] proposed the terms b-metric and b-metric space. Bakhtin [4] called them "almost metric spaces". But this kind of spaces were earlier considered under various names (see the Introduction to [12]). In [12] one says that Bakhtin proposed the term "quasi-metric", but the exact translation from Russian is that of "almost metric". Also, according to the historical notes

in the recent paper [7], it appears that the concept of b-metric space (under the name "quasimetric space") was introduced before Bakhtin and Czerwik, by Vulpe et al. [41]. The theory of fixed points in b-metric spaces has expanded in the past ten years (see [1, 2, 3, 9, 10, 12, 15, 16, 18, 21, 23, 25, 24, 27, 28, 35, 37]).
Here are the well-known definitions.

Definition 1.1. Let E be a nonempty set and $s \in [1, +\infty)$. A mapping $d : E \times E \to [0, +\infty)$ is called a b-metric if:

(1_b) $d(e_1, e_2) = 0$ if and only if $e_1 = e_2$,
(2_b) $d(e_1, e_2) = d(e_2, e_1)$,
(3_b) $d(e_1, e_3) \leq s[d(e_1, e_2) + d(e_2, e_3)]$,

for all $e_1, e_2, e_3 \in E$. In this case (E, d, s) is called a b-metric space.

Remark 1.2. If $s = 1$ then the b-metric space is a metric space. The notions of convergent sequence, Cauchy sequence and completeness in b-metric spaces are defined as in metric spaces.

Remark 1.3. The space $l^p = \{\{x_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{+\infty} |x_n|^p < +\infty\}, p \in (0, 1)$, together with the function $d_p : l^p \times l^p \to \mathbb{R}$, defined by

$$d_p(x, y) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = \{x_n\}, y = \{y_n\} \in l^p$, is not a metric space (the function d_p do not satisfy the triangle inequality), but (l^p, d_p, s) is a b-metric space with $s = 2^{\frac{1}{p}-1}$, [17, 22].

Definition 1.4. Let (E, d, s) be a b-metric space, $\{e_n\}$ a sequence in E and $e \in E$.
(a) The sequence $\{e_n\}$ is convergent and converges to e, if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(e_n, e) < \epsilon$ for all $n > n_\epsilon$. We denote this by $\lim\limits_{n \to +\infty} e_n = e$ or $e_n \to e$ as $n \to +\infty$.
(b) The sequence $\{e_n\}$ is called Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(e_n, e_m) < \epsilon$ for all $n, m > n_\epsilon$.
(c) If every Cauchy sequence in E converges to some $e \in E$ then (E, d, s) is called a complete b-metric space.

Recently, Miculescu and Mihail [23] (Lemma 2.2) and Suzuki [37] (Lemma 6) obtained a nice result that is very useful for examining convergence in b-metric spaces.

Lemma 1.5. *Let (E, d, s) be a b-metric space and sequence $\{e_n\} \subseteq E$. If there exists $\kappa \in [0, 1)$ such that*

$$d(e_{n+1}, e_n) \leq \kappa d(e_n, e_{n-1}), \tag{1.3}$$

for all $n \in \mathbb{N}$, then $\{e_n\}$ is Cauchy.

Remark 1.6. From Lemma 1.5 we obtain that $\{e_n\}$ is Cauchy if there exist $a \in [0, +\infty)$ and $\kappa \in [0, 1)$ such that $d(e_{n+1}, e_n) \leq \kappa^n a$ for any $n \in \mathbb{N}$.

Lemma 1.5 is essential for the proof of our main result and can be used to obtain a simpler proof of the result of Czerwik ([13], Theorem on page 137). We provide a single-valued mappings version of Czerwik in b-metric spaces.

2. Main result

Now we shall prove the main result.

Theorem 2.1. *Let* (E_i, d_i, s_i)*, be complete* b*-metric spaces,* $i \in [n]$ *and* $T_i : E_1 \times \cdots \times E_n \to E_i, i \in [n]$ *be given mappings. Assume that the following conditions hold*

$$d_i(T_i(e_1, \ldots, e_n), T_i(f_1, \ldots, f_n)) \leq \sum_{j=1}^{n} a_{ij} d_j(e_j, f_j) \tag{2.1}$$

$$+\sum_{j=1}^{n} b_{ij} d_j(e_j, T_j(e_1, \ldots, e_n)) + c d_i(f_i, T_i(f_1, \ldots, f_n)),$$

for all $e_i, f_i \in E_i, i \in [n]$*, where* $a_{ij}, b_{ij}, c \in [0,1)$ *are such that*

$$c < \min\{1 - \max\{\sum_{j=1}^{n}(a_{ij} + b_{ij}) : i \in [n]\}, \min\{1/s_i : i \in [n]\}\}. \tag{2.2}$$

Then the system (1.1) has a unique solution.

Proof. Let $x_i^0 \in E_i$ and $x_i^{k+1} = T_i(x_1^k, \ldots, x_n^k)$ for $i \in [n]$ and $k \in \mathbb{N}$. Put $\delta = \max\{d_i(x_i^0, x_i^1) : i \in [n]\}$. From condition (2.1) we obtain

$$\begin{aligned} d_i(x_i^1, x_i^2) &= d_i(T_i(x_1^0, \ldots, x_n^0), T_i(x_1^1, \ldots, x_n^1)) \\ &\leq \sum_{j=1}^{n} a_{ij} d_j(x_j^0, x_j^1) + \sum_{j=1}^{n} b_{ij} d_j(x_j^0, x_j^1) + c d_i(x_i^1, x_i^2). \end{aligned}$$

From the above inequality we get

$$d_i(x_i^1, x_i^2) \leq \lambda\delta, \tag{2.3}$$

for all $i \in [n]$, where

$$\lambda = \frac{\max\{\sum_{j=1}^{n}(a_{ij} + b_{ij}) : i \in [n]\}}{1 - c}.$$

From condition (2.2) we conclude that $\lambda \in [0, 1)$. Again, using (2.1) and (2.3) we get

$$\begin{aligned} d_i(x_i^2, x_i^3) &= d_i(T_i(x_1^1, \ldots, x_n^1), T_i(x_1^2, \ldots, x_n^2)) \\ &\leq \sum_{j=1}^{n} a_{ij} d_j(x_j^1, x_j^2) + \sum_{j=1}^{n} b_{ij} d_j(x_j^1, x_j^2) + c d_i(x_i^2, x_i^3) \\ &\leq \sum_{j=1}^{n}(a_{ij} + b_{ij})\lambda\delta + c d_i(x_i^2, x_i^3). \end{aligned}$$

So,

$$d_i(x_i^2, x_i^3) \leq \frac{\sum_{j=1}^{n}(a_{ij} + b_{ij})\lambda\delta}{1 - c} = \lambda^2\delta, \tag{2.4}$$

for all $i \in [n]$. Using the induction we conclude that

$$d_i(x_i^k, x_i^{k+1}) \leq \lambda^k\delta, \tag{2.5}$$

for all $i \in [n], k \in \mathbb{N}$. Now, using Lemma 1.5 we obtain that $\{x_i^k\}$ is Cauchy for all $i \in [n]$. Let $\lim_{k\to+\infty} x_i^k = e_i^*$ for all $i \in [n]$. Using inequality (3_b) for b-metric space (E_i, d_i, s_i) and condition (2.1) we have

$$\begin{aligned} d_i(e_i^*, T_i(e_1^*, \ldots, e_n^*)) &\leq s_i[d_i(e_i^*, x_i^{k+1}) + d_i(x_i^{k+1}, T_i(e_1^*, \ldots, e_n^*))] \\ &= s_i[d_i(e_i^*, x_i^{k+1}) + d_i(T_i(x_1^k, \ldots, x_n^k), T_i(e_1^*, \ldots, e_n^*))] \\ &\leq s_i[d_i(e_i^*, x_i^{k+1}) + \sum_{j=1}^{n} a_{ij} d_j(x_j^k, e_j^*) \\ &\quad + \sum_{j=1}^{n} b_{ij} d_j(x_j^k, x_j^{k+1})) + c d_i(e_i^*, T_i(e_1^*, \ldots, e_n^*))]. \end{aligned}$$

Therefore,

$$d_i(e_i^*, T_i(e_1^*, \dots, e_n^*)) \le s_i c d_i(e_i^*, T_i(e_1^*, \dots, e_n^*)), \tag{2.6}$$

for all $i \in [n]$. Now, from condition (2.2) we conclude that

$$d_i(e_i^*, T_i(e_1^*, \dots, e_n^*)) = 0,$$

for all $i \in [n]$. So, $(e_1^*, \dots, e_n^*)$ is a solution of the system (1.1). Uniqueness follows directly from the conditions (2.1) and (2.2). □

Corollary 2.2. *Let (E_i, d_i, s_i), be complete b-metric space, $i \in [n]$ and the mappings T_i, $i \in [n]$ be as given in (1.2). Assume that the following conditions hold*

$$d_i(T_i(e_1, \dots, e_n), T_i(f_1, \dots, f_n)) \le \sum_{j=1}^{n} a_{ij} d_j(e_j, f_j))), \tag{2.7}$$

for all $e_i, f_i \in E_i, i \in [n]$, where $a_{ij} \in [0,1), i, j \in [n]$ are such that

$$\max\{\sum_{j=1}^{n} a_{ij} : i \in [n]\} < 1. \tag{2.8}$$

Then the system (1.1) has a unique solution.

Remark 2.3. Note that if $d_1, \dots, d_n$ are metrics then the condition (2.2) becomes

$$\max\{\sum_{j=1}^{n} (a_{ij} + b_{ij}) : i \in [n]\} + c < 1. \tag{2.9}$$

Example 2.4. Let a, b, c and d be non-negative real numbers such that $a + b < 1$, $c + d < 1$, $2(9a^2 + b^2) <$ and $8(81c^4 + d^4) < 1$ then the system

$$\begin{array}{rcl} ax_1^3 + b\sin x_2 & = & x_1 \\ cx_2^3 + d\sin x_1 & = & x_2 \end{array} \tag{2.10}$$

has a unique solution on interval $[-1, 1]$.
Let (X_1, d_1, s_1) and (X_2, d_2, s_2) be b-metric spaces, where

$$X_1 = X_2 = [-1, 1],\ d_1(x, y) = d_2(x, y) = (x - y)^2.$$

Using inequality

$$(z + t)^2 \le 2(z^2 + t^2), z, t \in \mathbb{R},$$

we can put $s_1 = 2, s_2 = 2$.
Put

$$T_1(x_1, x_2) = ax_1^3 + b\sin x_2$$

and

$$T_2(x_1, x_2) = cx_2^3 + d\sin x_1.$$

Then we have

$$|T_1(x_1, x_2)| = |ax_1^3 + b\sin x_2| \le a|x_1^3| + b|x_2| \le a + b < 1,$$

we get similar $|T_2(x_1, x_2)| \le c + d < 1$. So, $T_i : X_1 \times X_2 \to X_i, i = 1, 2$.

$$\begin{array}{rl} & (T_1(x_1, x_2) - T_1(y_1, y_2))^2 = [a(x_1^3 - y_1^3) + b(\sin x_2 - \sin y_2)]^2 \\ \le & 2[a^2(x_1 - y_1)^2(x_1^2 + x_1y_1 + y_1^2)^2 + b^2(2\sin\dfrac{x_2 - y_2}{2}\cos\dfrac{x_2 + y_2}{2})]^2 \\ \le & 2[9a^2(x_1 - y_1)^2 + b^2(x_2 - y_2)^2], \end{array}$$

we get similar

$$(T_2(x_1, x_2) - T_2(y_1, y_2))^4 \le 2[9c^2(x_2 - y_2)^2 + d^2(x_1 - y_1)^2].$$

Now based on the Corollary 2.2 we get that the system (2.10) has a unique solution on $[-1, 1]$.

Example 2.5. For $p \in (\frac{1}{2}, 1)$, consider the b-metric for l^p given in Remark 1.3 and let $T_i : l^p \times l^p \to l^p, i = 1, 2$ be defined by

$$T_1(x,y) = (\frac{x_1 - y_1}{4}, 0, \frac{x_2 - y_2}{4^2}, 0, \frac{x_3 - y_3}{4^3}, 0, \ldots),$$

$$T_2(x,y) = (0, \frac{x_1 - y_1}{4}, 0, \frac{x_2 - y_2}{2^4}, 0, \frac{x_3 - y_3}{4^3}, 0, \ldots),$$

where $x = \{x_n\}, y = \{y_n\} \in l^p$. Then the system

$$\begin{array}{rcl} T_1(e_1, e_2) & = & e_1 \\ T_2(e_1, e_2) & = & e_2. \end{array}$$

has a unique solution. Let $k \in \{1, 2\}$, we have

$$\begin{aligned} d_p(T_k(e_1, e_2), T_k(f_1, f_2)) &= \left(\sum_{i=1}^{+\infty} \left| \frac{x_i - z_i - (y_i - t_i)}{4^i} \right|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{+\infty} \frac{|x_i - z_i - (y_i - t_i)|^p}{4^p} \right)^{\frac{1}{p}} \\ &= \frac{1}{4} \left(\sum_{i=1}^{+\infty} |x_i - z_i - (y_i - t_i)|^p \right)^{\frac{1}{p}} \\ &\leq \frac{1}{4} 2^{\frac{1}{p} - 1} \left[\left(\sum_{i=1}^{+\infty} |x_i - z_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{+\infty} |y_i - t_i|^p \right)^{\frac{1}{p}} \right] \\ &= \frac{2^{\frac{1}{p}}}{8} \left[d_p(e_1, f_1) + d_p(e_2, f_2) \right], \end{aligned}$$

where $e_1 = \{x_n\}, e_2 = \{y_n\}, f_1 = \{z_n\}, f_2 = \{t_n\} \in l^p$. If $p \in (\frac{1}{2}, 1)$ then we have that $\frac{2^{\frac{1}{p}}}{4} < 1$. So in this case, from Theorem 2.1 ($a_{ij} = \frac{2^{\frac{1}{p}}}{8}, b_{ij} = 0, i, j \in \{1, 2\}, c = 0$, the condition (2.2) is fulfilled), we obtain that there exists $(e_1^*, e_2^*) \in l^p \times l^p$ ($e_1^* = e_2^* = (0, 0, 0, \ldots)$) such that

$$\begin{array}{rcl} T_1(e_1^*, e_2^*) & = & e_1^* \\ T_2(e_1^*, e_2^*) & = & e_2^*. \end{array}$$

3. Some results in b-metric spaces

If in Theorem 2.1 we put $n = 1$, we obtain a result of Reich [36] in b-metric spaces.

Theorem 3.1. *Let (E, d, s) be a complete b-metric space and $T : E \to E$ a mapping. Assume that the following condition holds*

$$d(Te, Tf) \leq a_{11} d(e, f) + b_{11} d(e, Te) + c d(f, Tf), \tag{3.1}$$

for all $e, f \in E$, where $a_{11}, b_{11}, c \in [0, 1)$ are such that

$$c < \min\{1 - (a_{11} + b_{11}), 1/s\}. \tag{3.2}$$

Then the mapping T has a unique fixed point.

Example 3.2. Let $T : l^p \to l^p, p \in (0, 1)$ be a mapping defined by

$$T(x_1, x_2, x_3, x_4 \ldots) = (0, 0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^2}, \frac{x_4}{2^2}, \ldots).$$

a mapping T has a unique fixed point.
For $x, y \in l^p$ we have

$$\begin{aligned} d_p(Tx, Ty) &= \left(\frac{|x_1 - y_1|^p}{2^p} + \frac{|x_2 - y_2|^p}{2^{2p}} + \frac{|x_3 - y_3|^p}{2^{2p}} + \cdots\right)^{\frac{1}{p}} \\ &\leq \left[\frac{1}{2^p}\left(|x_1 - y_1|^p + |x_2 - y_2|^p + |x_3 - y_3|^p + \cdots\right)\right]^{\frac{1}{p}} \\ &\leq \frac{1}{2} d_p(x, y). \end{aligned}$$

From Theorem 3.1 ($a_{11} = \frac{1}{2}, b_{11} = 0, c = 0$) we conclude that a mapping T has a unique fixed point $x^* \in l_p$ ($x^* = (0, 0, \ldots)$).

Remark 3.3. From Theorem 3.1 the classical results of Banach [5] and Kannan [20] are obtained in b-metric spaces.

The following result on coupled fixed points generalizes and improves the result of Bhaskar and Lakshmikantham [8].

Theorem 3.4. *Let (E, d, s) be a complete b-metric space and a mapping $T : E \times E \to E$. Assume that the following condition holds*

$$\begin{aligned} d(T(e_1, e_2), T(f_1, f_2)) \leq \frac{\alpha}{2}[d(e_1, f_1) + d(e_2, f_2)] \\ + \frac{\beta}{2}[d(e_1, T(e_1, e_2)) + d(e_2, T(e_1, e_2))], \end{aligned} \tag{3.3}$$

for all $e_1, e_2, f_1, f_2 \in E$, where $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta < 1$. Then the mapping T has a unique coupled fixed point $(e^*, f^*) \in E \times E$, i.e $T(e^*, f^*) = e^*, T(f^*, e^*) = f^*$.

Proof. Put in Theorem 2.1 $n = 2$, $E_1 = E_2 = E$, $T_1(e_1, e_2) = T(e_1, e_2)$, $T_2(e_1, e_2) = T(e_2, e_1)$, $a_{ij} = \alpha, b_{ij} = \beta, i, j = 1, 2$ and $c = 0$. □

Remark 3.5. Note that Theorem 2.1 also includes the result of Czerwik [13] for a system of single-valued mappings in b-metric spaces.

Remark 3.6. Some other related or particular cases, see the references [29, 30, 31, 32, 33, 34, 38, 39, 40] can be connected with Theorem 2.1.

4. An application in metric spaces

Finally, using Theorem 2.1, we provide sufficient conditions for the existence and uniqueness of a solution of the following system

$$\begin{aligned} a_{11}x_1^r + \cdots + a_{1n}x_n^r + b_1 &= x_1 \\ &\vdots \\ a_{n1}x_1^r + \cdots + a_{nn}x_n^r + b_n &= x_n, \end{aligned} \tag{4.1}$$

where $a_{ij}, b_i \in \mathbb{R}, i, j \in [n]$, $r \in \mathbb{N}$, $x_i \in [-M, M], i \in [n]$ and $M > 0$.
We will use the following simple lemma.

Lemma 4.1. *Let $r \in \mathbb{N}, M > 0$ and $x, y \in [-M, M]$ then*

$$|x^r - y^r| \leq rM^{r-1}|x - y|.$$

Proof. Prove following from

$$\begin{aligned} |x^r - y^r| &= |x - y| \cdot |x^{r-1} + x^{r-2}y + \cdots + y^r| \\ &\leq |x - y|(|x|^{r-1} + |x|^{r-2}|y| + \cdots + |y|^r) \\ &\leq |x - y|(M^{r-1} + M^{r-1} + \cdots + M^{r-1}) \\ &= rM^{r-1}|x - y|. \end{aligned}$$

□

Theorem 4.2. *If*

$$rM^{r-1}\max\{\sum_{j=1}^{n}|a_{ij}|:i\in[n]\}<1 \tag{4.2}$$

and

$$\max\{M^{r}\sum_{j=1}^{n}|a_{ij}|+|b_i|:i\in[n]\}<M \tag{4.3}$$

then the system (4.1) has a unique solution.

Proof. Put

$$T_i(x_1,\dots,x_n)=a_{i1}x_1+\cdots+a_{in}x_n+b_i,$$
$$X_i=[-M,M],\ d_i(x,y)=|x-y|,\ i\in[n].$$

First, from condition (4.3) we obtain that $T_i : X_1\times\cdots\times X_n\to X_i$ for all $i\in[n]$. Now, the proof follows from Corollary 2.2, Lemma 4.1 and condition (4.2). □

Remark 4.3. (Some open questions:)
1. Compare Theorem 2.1 with the main result in [16] (Theorem 4);
2. Obtain a version of Theorem 2.1 in generalized b-metric spaces (see [16]);
3. Obtain versions of the results on fixed points in the sense of Berinde ([6], Theorem 4) and Ćirić ([11], Theorem 1) in the product of b-metric spaces;
4. Obtain a version of the result in the sense of Nadler ([26], Theorem 5) for multi-valued mappings in the product of b-metric spaces;
5. Obtain versions of the results on common fixed points in the sense of Jungck ([19], Theorem) in the product of b-metric spaces;
6. Can we replace in Theorem 2.1 the condition (2.1)

$$cd_i(f_i,T_i(f_1,\dots,f_n))$$

with

$$\sum_{j=1}^{n}c_{ij}d_j(f_j,T_j(f_1,\dots,f_n))?$$

Conclusion

We have given a result on the existence of a solution of a system of nonlinear equations in the product of b-metric spaces. We obtained a generalization of the classical results of Banach, Kannan and Reich in the product of b-metric spaces. In the technique we present, we have used the result of Romanian mathematicians Miculescu and Mihail. Some variants of the results of Czerwik, Bhaskar and Lakshmikantham are obtained. Also, we have given some applications in metric spaces. Finally, we list some possibilities for further research on this topic.

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References

[1] S. Aleksić, Z.D. Mitrović, S. Radenović, *S. Picard sequences in b-metric spaces*, Fixed Point Theory, **21**(2020), 35-46.
[2] H. Aydi, M.F. Bota, E. Karapınar, S. Mitrović, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory Appl., (2012), 2012:88.
[3] H. Aydi, S. Czerwik, *Fixed point theorems in generalized b-metric spaces*, Modern Discrete Mathematics and Analysis: With Applications in Cryptography, Information Systems and Modeling, Springer, (2018), 1-9.

[4] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Ulianowsk Gos. Ped. Inst., **30**(1989), 26-37.

[5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3**(1922), 133-181.

[6] V. Berinde, *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory, **4**(2003), 131-142.

[7] V. Berinde, M. Păcurar, *The early developments in fixed point theory on b-metric spaces: A brief survey and some important related aspects*, Carpathian J. Math., **38**(2022), 523-538.

[8] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379-1393.

[9] M. Bota, A. Molnár, C. Varga, *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory, **12**(2011), 21-28.

[10] J. Brzdęk, *Comments on fixed point results in classes of function with values in a b-metric space*, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM, **116**(2022), Paper No. 35, 17 p.

[11] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc., **45**(1974), 267-273.

[12] S. Cobzaş, S. Czerwik, *The completion of generalized b-metric spaces and fixed points*, Fixed Point Theory, **21**(2020), 133-150.

[13] S. Czerwik, *A fixed point theorem for a system of multivalued transformations*, Proc. Am. Math. Soc., **55**(1976), 136-139.

[14] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., **1**(1993), 5-11.

[15] S. Czerwik, *On b-metric spaces and Brouwer and Schauder fixed point principles*, Rassias, Th. M. (ed.), Approximation Theory and Analytic Inequalities. Cham: Springer, (2021), 71-86.

[16] S. Czerwik, M.T. Rassias, *Fixed point theorems for a system of mappings in generalized b-metric spaces*, Rassias, Th. M. (ed.) et al., Mathematical Analysis and Applications. Cham: Springer. Springer Optim. Appl., **154**(2019), 41-51.

[17] M.M. Day, *The spaces Lp with* $0 < p < 1$, Bull. Amer. Math. Soc., **46**(1940), 816-823.

[18] M.B. Jleli, B. Samet, C. Vetro, F. Vetro, *Fixed points for multivalued mappings in b-metric spaces*, Abstr. Appl. Anal., (2015), Article ID 718074, 7 pages.

[19] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83**(1976), 261-263.

[20] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**(1969), 71-76.

[21] E. Karapınar, Z.D. Mitrović, A. Öztürk, S. Radenović, *On a theorem of Ćirić in b-metric spaces*, Rend. Circ. Mat. Palermo, **70**(2021), 217-225.

[22] G. Köthe, *Topological Vector Spaces I*, Springer-Verlag, New York, 1969.

[23] R. Miculescu, A. Mihail, *New fixed point theorems for set-valued contractions in b-metric spaces*, J. Fixed Point Theory Appl., **19**(2017), 2153-2163.

[24] Z.D. Mitrović, *Fixed point results in b-metric space*, Fixed Point Theory, **20**(2019), 559-566.

[25] Z.D. Mitrović, A. Ahmed, J.N. Salunke, *A cone generalized b-metric like space over Banach algebra and contraction principle*, Thai J. Math., **19**(2021), 583-592.

[26] S.B. Jr. Nadler, *Multi-valued contraction mappings*, Pac. J. Math., **30**(1969), 475-488.

[27] H.K. Nashine, Z. Kadelburg, *Generalized JS-contractions in b-metric spaces with application to Urysohn integral equations*, Cho, Yeol Je (ed.) et al., Advances in Metric Fixed Point Theory and Applications. Singapore: Springer, (2021), 385-408.

[28] E.A. Petrov, R.R. Salimov, *Quasisymmetric mappings in b-metric spaces*, J. Math. Sci., New York, **256**(2021), 770-778 (2021); translation from Ukr. Mat. Visn., **18**(2021), no. 1, 60-70.

[29] A. Petruşel, G. Petruşel, *A study of a general system of operator equations in b-metric spaces via the vector approach in fixed point theory*, J. Fixed Point Theory Appl., **19**(2017), 1793-1814.

[30] A. Petruşel, G. Petruşel, *Coupled fixed points and coupled coincidence points via fixed point theory*, in: Mathematical Analysis and Applications: Selected Topics, M. Ruzhansky, H. Dutta, R.P. Agarwal (eds.), Wiley, (2018), 661-708.

[31] A. Petruşel, G. Petruşel, B. Samet, *A study of the coupled fixed point problem for operators satisfying a max-symmetric condition in b-metric spaces with applications to a boundary value problem*, Miskolc Math. Notes, **17**(2016), 501-516.
[32] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to a system of integral equations and a periodic boundary value problem*, Fixed Point Theory, **17**(2016), 459-478.
[33] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Scalar and vectorial approaches for multi-valued fixed point and multi-valued coupled fixed point problems in b-metric spaces*, J. Nonlinear Convex Anal., **17**(2016), 2049-2061.
[34] A. Petruşel, G. Petruşel, Y.-B. Xiao, J.-C. Yao, *Fixed point theorems for generalized contractions with applications to coupled fixed point theory*, J. Nonlinear Convex Anal., **19**(2018), 71-88.
[35] D. Rakić, A. Mukheimer, T. Došenović, Z.D. Mitrović, S. Radenović, *On some new fixed point results in fuzzy b-metric spaces*, J. Inequal. Appl., **99**(2020), Paper No. 99, 14 p.
[36] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14**(1971), 121-124.
[37] T. Suzuki, *Basic inequality on a b-metric space and its applications*, J. Inequal. Appl., (2017), Paper No. 256, 11 p.
[38] C. Urs, *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Math. Notes, **14**(2013), no. 1, 323-333.
[39] C. Urs, *Coupled fixed point theorems for mixed monotone operators and applications*, Stud. Univ. Babeş-Bolyai Math., **59**(2014), no. 1, 113-122.
[40] C. Urs, *Fixed point theorems for a system of operator equations with applications*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), no. 1, 87-94.
[41] I.M. Vulpe, D. Ostraikh, F. Khoiman, *The topological structure of a quasimetric space*, (Russian), Investigations in Functional Analysis and Differential Equations, "Shtiintsa", Kishinev, **137**(1981), 14-19.

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