Fixed Point Theory, 25(2024), No. 2, 601-610
DOI: 10.24193/fpt-ro.2024.2.10
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

INTEGRAL EQUATION WITH MAXIMA VIA FIBRE CONTRACTION PRINCIPLE

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Abstract. The aim of this paper is to emphasize the role of the fibre contraction principle in the study of the solution of integral equations with maxima in connection with the weakly Picard operator technique. The results complement and extend some known results given in the paper: I.A. Rus, Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle, Advances in the Theory of Nonlinear Analysis and its Applications, 3(2019), no. 3, 111-120. The last section is devoted to Gronwall lemma type results and comparison theorems.

Key Words and Phrases: Integral equation with maxima, existence and uniqueness, fixed point, weakly Picard operator, fibre contraction principle, Gronwall lemma, comparison lemma.
2020 Mathematics Subject Classification: 47H10, 47H09, 45J05.

1. Introduction

In 2019 Burton gives the first result on existence and uniqueness for the solution of an integral equation in the context of progressive contraction. One year later, I.A. Rus formalized this notion (see [23]), with "step by step" instead of "progressive", and gave a variant of the step by step contraction principle. Since then, many other generalizations of these results were proved for problems involving functional differential equations with maxima, Volterra integral equations, Fredholm-Volterra integral equations in two variables (see [9, 10, 11], [16]).

Motivated by the above-mentioned papers, in this paper we discuss the existence of solutions of the following functional integral equation with maxima

$$x(t) = \int_a^t K(t, s, x(s), \max_{a \le \xi \le s} x(\xi))ds + f(t, x(t)),\ t \in [a, b], \tag{1.1}$$

where $K \in C([a,b] \times [a,b] \times \mathbb{R}^2, \mathbb{R})$ and $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$. To prove our results, we shall use step by step contraction principle and a new variant of fibre contraction principle given in [23] and [17].

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results. Using the weakly Picard operator theory, in the last sections we give Gronwall lemma type results and comparison theorems.

2. Preliminaries

2.1. **Weakly Picard operators.** In the sequel, the following results are useful for some of the proofs in the paper (see [18, 19]).

Let $(X, \to)$ be an L-space, where X is a nonempty space and $\to$ is a convergence structure in the sense of Fréchet, defined on X. An operator $A : X \to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $(A^n(x))_{n\in\mathbb{N}}$, converges in $(X, \to)$ for all $x \in X$ and its limit (which generally depend on x) is a fixed point of A. If an operator A is WPO with a unique fixed point, that is, $F_A = \{x^*\}$, then, by definition, A is called a Picard operator (PO).

If $A : X \to X$ is a WPO, we can define the operator $A^\infty : X \to X$, by $A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$.

In our next considerations, we consider the case of an ordered L-space, i.e., an L-space endowed with a partial ordering "$\leq$".

Abstract Gronwall lemma. *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:*

(i) *A is a WPO with respect to $\to$;*
(ii) *A is increasing with respect to $\leq$.*

Then:

(a) $x \leq A(x) \Longrightarrow x \leq A^\infty(x)$;
(b) $x \geq A(x) \Longrightarrow x \geq A^\infty(x)$.

Abstract comparison lemma. *Let $(X, \to, \leq)$ be an ordered L-space and $A, B, C : X \to X$ three operators having the following properties:*

(i) *$A \leq B \leq C$;*
(ii) *The operators A, B and C are WPO with respect to $\to$;*
(iii) *the operator B is increasing with respect to $\leq$.*

Then:

$$x \leq y \leq z \;\Rightarrow\; A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

For other details and results concerning the abstract Gronwall lemma and the abstract comparison principle see [18, 19], [21, 22] and [12, 15, 13, 14, 16].

2.2. **Step by step contraction.** Let $(X, \to)$ be an L-space and $G \subset X \times X$ be a nonempty set. An operator $A : X \to X$ is a G-contraction if there exists $l \in (0, 1)$ such that,

$$d(A(x), A(y)) \leq l d(x, y),\ \forall (x, y) \in G.$$

For other applications of G-contraction, see [23] and [20].

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space and $C([a,b],\mathbb{B})$ be the Banach space of continuous mapping with max-norm, $\|\cdot\|$. In what follows, in all spaces of functions we consider max-norm. For $m \in \mathbb{N},\ m \geq 2$, let $t_0 := a,\ t_k := t_0 + k\frac{b-a}{m},\ k = \overline{1,m}$.

Let $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ be an operator. The operator V has the Volterra property (see [23]), i.e.,

$$t \in (a,b),\ x,y \in C[a,b],\ x|_{[a,t]} = y|_{[a,t]} \Rightarrow V(x)|_{[a,t]} = V(y)|_{[a,t]}\,.$$

We consider $V_k : C([t_0,t_k],\mathbb{B}) \to C([t_0,t_k],\mathbb{B}), k = \overline{1,m-1}$ the operator induced by V on $C([t_0,t_k],\mathbb{B})$. We also consider the following sets,

$$G_k := \{(x,y)|\ x,y \in C([t_0,t_{k+1}],\mathbb{B}),\ x|_{[t_0,t_k]} = y|_{[t_0,t_k]}\},\ k = \overline{1,m-1}.$$

For $x_k \in C([t_0,t_k],\mathbb{B}),\ k = \overline{1,m-1}$, we denote

$$X_{x_k} := \{y \in C([t_0,t_{k+1}],\mathbb{B}),\ y|_{[t_0,t_k]} = x_k\}.$$

The following result is given in [23].

Theorem 2.1. (Theorem of step by step contraction) *We suppose that:*

(1) $V : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$ *has the Volterra property;*
(2) V_1 *is a contraction;*
(3) V_k *is a* G_{k-1}*-contraction, for* $k = \overline{2,m}$.

Then:

(i) $F_V = \{x^*\}$;
(ii) *the following relations hold:*

$$\begin{aligned} x^*|_{[t_0,t_1]} &= V_1^\infty(x),\ \forall x \in C([t_0,t_1],\mathbb{R}),\\ x^*|_{[t_0,t_2]} &= V_2^\infty(x),\ \forall x \in X_{x^*|_{[t_0,t_1]}},\\ &\ \vdots\\ x^*|_{[t_0,t_{m-1}]} &= V_{m-1}^\infty(x),\ \forall x \in X_{x^*|_{[t_0,t_{m-2}]}}. \end{aligned}$$

(iii) $x^* = V^\infty(x), \quad \forall x \in X_{x^*|_{[t_0,t_{m-1}]}}$.

2.3. **Fibre contraction principle.** In [17] the authors obtained a new fibre contraction principle in the following settings:

Let (X_i, d_i) be metric spaces ($i \in \{1,...,m\}$, where $m \geq 2$) and $U_1 \subset X_1 \times X_2$, $U_2 \subset U_1 \times X_3, \ldots, U_{m-1} \subset U_{m-2} \times X_m$, be nonempty subsets.

For $x \in X_1$, we define

$$U_{1x} := \{x_2 \in X_2 \mid (x,x_2) \in U_1\},$$

for $x \in U_1$, we define

$$U_{2x} := \{x_3 \in X_3 \mid (x,x_3) \in U_2\}, \ldots,$$

and for $x \in U_{m-2}$, we define

$$U_{m-1x} := \{x_m \in X_m \mid (x,x_m) \in U_{m-1}\}.$$

We suppose that $U_{1x}, U_{2x}, \ldots, U_{m-1x}$ are nonempty.

If $T_1 : X_1 \to X_1$, $T_2 : U_1 \to X_2, \ldots$, $T_m : U_{m-1} \to X_m$, then we consider the operator

$$T : U_{m-1} \to X_1 \times X_2 \times \ldots \times X_m,$$

defined by

$$T(x_1, \ldots, x_m) := (T_1(x_1), T_2(x_1, x_2), \ldots, T_m(x_1, x_2, \ldots, x_m)).$$

The result is the following.

Theorem 2.2. ([17]) *In the above notations we suppose that:*

(1) (X_i, d_i), $i \in \{2, ..., m\}$ *are complete metric spaces and* U_i, $i \in \{1, ..., m-1\}$ *are closed subsets;*

(2) $(T_1, T_2, \ldots, T_{i+1})(U_i) \subset U_i$, $i \in \{1, ..., m-1\}$*;*

(3) T_1 *is a WPO;*

(4) *there exist* $L_i > 0$ *and* $0 < l_i < 1$, $i \in \{1, ..., m-1\}$ *such that*

$$d_{i+1}(T_{i+1}(x, y,), T_{i+1}(\widetilde{x}, \widetilde{y})) \leq L_i \widetilde{d_i}(x, \widetilde{x}) + l_i d_{i+1}(y, \widetilde{y}),$$

for all $(x, y), (\widetilde{x}, \widetilde{y}) \in U_i$, $i \in \{1, ..., m-1\}$, *where* $\widetilde{d_i}$ *is a metric induced by* $d_1, \ldots, d_i$ *on* $X_1 \times \cdots \times X_i$, *defined by* $\tilde{d_i} := \max\{d_1, \ldots, d_i\}$.

Then T *is WPO. If* T_1 *is PO, then* T *is a PO too.*

For other results concerning the fibre contraction theorem, its generalization and applications, see also [8, 9, 10, 11, 15, 12, 13, 14], [18, 19, 20, 21, 22, 23].

3. Main result

In this section, we establish some new results on the existence and uniqueness of the solution of the integral equation with maxima (1.1).

The equation (1.1), $x \in C([a, b], \mathbb{R})$ is equivalent with the fixed point equation

$$x(t) = V(x)(t) \tag{3.1}$$

where the operator $V : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ is defined by

$$V(x)(t) := \int_a^t K(t, s, x(s), \max_{a \leq \xi \leq s} x(\xi))ds + f(t, x(t)),\ t \in [a, b] \tag{3.2}$$

We remark that the operator V has the Volterra property, i.e.,

$$t \in (a, b),\ x, y \in C[a, b],\ x|_{[a,t]} = y|_{[a,t]} \Rightarrow V(x)|_{[a,t]} = V(y)|_{[a,t]}.$$

This implies that the operator V induced, for each c with $a < c < b$ and, the operator $V_c : C[a, c] \to C[a, c]$, defined by, $V_c(x)(t) := V(\widetilde{x})$, where $\widetilde{x} \in C[a, b]$ is such that, $\widetilde{x}|_{[a,c]} = x$.

In what follows we consider the notations from Section 2.3 with m suitable chosen.

Theorem 3.1. *Assume that the following hypotheses are satisfied:*

(C1) *There exists* $L > 0$, *such that*

$$|K(t, s, u_1, u_2) - K(t, s, v_1, v_2)| \leq L \max(|u_1 - v_1|, |u_2 - v_2|),$$

for all $t, s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2.$

(C2) *There exists* $0 < l < 1$, *such that*

$$|f(t,u) - f(t,v)| \leq l\,|u - v|\,,$$

for all $t \in [a,b], u \in \mathbb{R}$.

Then, choosing $m \in \mathbb{N}^*$ *such that*

$$l + \frac{L(b-a)}{m} < 1, \tag{3.3}$$

we have

(i) $F_V = \{x^*\}$, *i.e., the equation (3.1) has a unique solution.*

(ii) *the following relations hold:*

$$\begin{aligned} x^*|_{[t_0,t_1]} &= V_1^\infty(x),\ \forall x \in C[t_0,t_1],\\ x^*|_{[t_0,t_2]} &= V_2^\infty(x),\ \forall x \in X_{x^*}\\ &\vdots\\ x^*|_{[t_0,t_{m-1}]} &= V_{m-1}^\infty(x),\ \forall x \in\ X_{x^*}|_{[t_0,t_{m-1}]}\,. \end{aligned}$$

(iii) $x^* = V^\infty(x),\ \forall x \in\ X_{x^*}|_{[t_0,t_{m-1}]}\,.$

Proof. We shall prove that in the conditions (C1) and (C2), we are in the conditions of Theorem of step by step contractions, with $\mathbb{B} := \mathbb{R}$.

First we prove that V_1 is a contraction.

We have:

$$\begin{aligned} |V_1(x)(t) - V_1(y)(t)| &\leq \left|\int_a^t K(t,s,x(s),\max_{a\leq\xi\leq s} x(\xi))ds - \int_a^t K(t,s,y(s),\max_{a\leq\xi\leq s} y(\xi))ds\right| + \\ &\quad + |f(t,x(t)) - f(t,y(t))| \\ &\leq L\int_a^t \max\left(|x(s)-y(s)|\,, \left|\max_{a\leq\xi\leq s} x(\xi) - \max_{a\leq\xi\leq s} y(\xi)\right|\right) ds + \\ &\quad + l\,|x(s)-y(s)| \\ &\leq \left(l + \frac{L(b-a)}{m}\right)\max_{t_0\leq t\leq t_1}|x(t)-y(t)|\,. \end{aligned}$$

From

$$\max_{t_0\leq t\leq t_1}|V_1(x)(t) - V_1(y)(t)| \leq \left(l + \frac{L(b-a)}{m}\right)\max_{t_0\leq t\leq t_1}|x(t)-y(t)|\,.$$

and condition (3.3), it follows that V_1 is a contraction.

Let us prove now that V_2 is a G_1-contraction. First we remark that, for $t \in [t_0,t_1]$

$$V_2(x)(t) = V_2(y)(t),\ \forall x,y \in G_1.$$

$$
\begin{aligned}
|V_2(x)(t) - V_2(y)(t)| &= \left| \int_a^{t_1} \left[K(t,s,x(s), \max_{a\le\xi\le s} x(\xi))ds - K(t,s,y(s), \max_{a\le\xi\le s} y(\xi)) \right] ds \right. \\
&\quad \left. + \int_{t_1}^{t} \left[K(t,s,x(s), \max_{a\le\xi\le s} x(\xi)) - K(t,s,y(s), \max_{a\le\xi\le s} y(\xi)) \right] ds \right| + \\
&\quad + |f(t,x(t)) - f(t,y(t))| \\
&= \left| \int_{t_1}^{t} \left[K(t,s,x(s), \max_{a\le\xi\le s} x(\xi))ds - K(t,s,y(s), \max_{a\le\xi\le s} y(\xi)) \right] ds \right| + \\
&\quad + |f(t,x(t)) - f(t,y(t))| \\
&\le \left(l + \frac{L(b-a)}{m} \right) \max_{t_0\le t\le t_2} |x(t) - y(t)|.
\end{aligned}
$$

Analogously, we prove that $V_3, \dots, V_m$ are $G_2, \dots, G_{m-1}$ contractions. The conclusion will follow by applying Theorem of step by step contraction. □

Now we establish a new iterative algorithm for (1.1). We apply the new variant of fibre contraction principle, Theorem 2.2, with $X_k := C[a, t_k]$.

We consider the spaces of continuous functions with the max-norms. We need the following subsets:

$$
U_i = \{(x_1, \dots, x_i) \in \prod_{k=1}^{i} X_k | \ x_k(t_k) = x_{k+1}(t_k), \ k = \overline{1, m-1}\}, \ i = \overline{1, m}.
$$

For $x \in X_1$, $U_{1x} := \{x_2 \in X_2 | \ (x, x_2) \in U_1\}$, for $x \in X_{i-2}$, $U_{i-1x} := \{x_i \in X_i | \ (x, x_i) \in U_{i-1}\}$, $i = \overline{2, m}$.

We remark that, U_i, U_{ix}, $i = \overline{1, m-1}$ are nonempty closed subsets.

We also need the following operators:

$$
R_i : C[a, t_i] \to \prod_{k=1}^{i} X_k, \ R_i(x) = \left(x|_{[t_0, t_1]}, \dots, x|_{[t_{i-1}, t_i]} \right), \ i = \overline{1, m-1}.
$$

It is clear that, $R_i\left(C[a, t_i]\right) = U_i$ and $R_i : C[a, t_i] \to U_i$ is an increasing homeomorphism.

Since the operator, $V : C[a, b] \to C[a, b]$ defined by equation (3.2), is a forward Volterra operator on $[a, b]$, it induces the following operators:

$$
\begin{aligned}
&T_1 : U_1 \to X_1, \\
&T_1(x_1)(t) := V(x_1)(t), \ t \in [a, t_1], \\
&T_2 : U_2 \to X_2, \\
&T_2(x_1, x_2)(t) := \int_a^t K(t, s, x_1(s), \max_{a\le\xi\le s} x_1(\xi))ds + \\
&\quad + \int_{t_1}^{t} K(t, s, (x_1, x_2)(s), \max_{a\le\xi\le s} R_1^{-1}(x_1, x_2)(\xi)))ds + f(t, (x_1, x_2)(t)), \ t \in [t_1, t_2],
\end{aligned}
$$

$$T_3 : U_3 \to X_3,$$

$$\begin{aligned}T_3(x_1,x_2,x_3)(t) := &\int_a^t K(t,s,x_1(s), \max_{a\le\xi\le s} x_1(\xi))ds+\\ &+\int_{t_1}^t K(t,s,(x_1,x_2)(s), \max_{a\le\xi\le s} R_1^{-1}(x_1,x_2)(\xi)))ds+\\ &+\int_{t_2}^t K(t,s,(x_1,x_2,x_3)(s), \max_{a\le\xi\le s} R_2^{-1}(x_1,x_2,x_3)(\xi)))ds+\\ &+f(t,(x_1,x_2,x_3)(t)),\ t\in[t_1,t_2],\\ &\dots\end{aligned}$$

$$T_m : U_m \to X_m,$$

$$\begin{aligned}T_m(x_1,\dots,x_m)(t) := &\int_a^t K(t,s,x_0(s), \max_{a\le\xi\le s} x_0(\xi))ds+\dots+\\ &+\int_{t_{m-1}}^t K(t,s,(x_1,\dots,x_m)(s), \max_{a\le\xi\le s} R_{m-1}^{-1}(x_1,\dots,x_m)(\xi)))ds+\\ &+f(t,(x_1,\dots,x_m)(t)),\ t\in[t_{m-1},b],\end{aligned}$$

Let

$$T := (T_1,\dots,T_m),$$
$$T(x_1,\dots,x_m) := (T_1(x_1),\dots,T_m(x_1,\dots,x_m)).$$

If on the cartesian product we consider max-norms, the operators R_i, $i=\overline{1,m-1}$ are isometries. From the above definitions, we remark that $T_1(U_1)\subset U_1$, $(T_1,\dots,T_m)(U_m)\subset U_m$.

In the conditions $(C1)-(C2)$ we have that: T_1 is $l+\frac{L(b-a)}{m}$-Lipschitz.

For a suitable choice of m we are in the conditions of Theorem 2.2 with $\widetilde{L}=l+\frac{L(b-a)}{m}$.

From this theorem we have that T is PO.

Since $V=R_{m-1}^{-1}TR_{m-1}$ and $V^n=R_{m-1}^{-1}T^nR_{m-1}$, it follows that V is PO.

Now we present the existence, uniqueness and approximation result for the equation (1.1).

Theorem 3.2. *We consider the equation (1.1) in the conditions* $(C1)-(C2)$. *We have that:*

(i) *The equation (1.1) has in $C[a,b]$ a unique solution, x^*.*

(ii) *The sequence, $(x_n)_{n\in\mathbb{N}}$, defined by*

$$x^0\in C[a,b],$$
$$x^{n+1}(t)=\int_a^t K(t,s,x^n(s), \max_{a\le\xi\le s} x^n(\xi))ds+f(t,x^n(t)),\ t\in[a,b],$$

converges to x^*, i.e., the operator V is PO.

Remark 3.3. For other types of saturated fibre contraction principle see [24].

Remark 3.4. For other applications of the fibre contraction principle to integro-differential equations with delays see [7], [15].

Remark 3.5. For the fixed point techniques in the integral equation theory see, for example, the following works: [1, 2, 3, 4, 5, 6].

4. Gronwall lemma type result

Related to the equation (1.1)

$$x(t) = \int_a^t K(t,s,x(s), \max_{a\leq\xi\leq s} x(\xi))ds + f(t,x(t)),\ t\in[a,b]$$

we consider the inequalities:

$$x(t) \leq \int_a^t K(t,s,x(s), \max_{a\leq\xi\leq s} x(\xi))ds + f(t,x(t)),\ t\in[a,b] \tag{4.1}$$

and

$$x(t) \geq \int_a^t K(t,s,x(s), \max_{a\leq\xi\leq s} x(\xi))ds + f(t,x(t)),\ t\in[a,b]. \tag{4.2}$$

As an application of the Abstract Gronwall lemma we have

Theorem 4.1. *We consider the equation (1.1) under the hypotheses* $(C1)-(C2)$ *of the Theorem 3.2. In addition, we suppose that:*

$(C3)$ $K(t,s,\cdot,\cdot)$ *and* $f(t,\cdot)$ *are increasing.*

Then:

(a) $x\leq x^*$ *for any* x *solution of (4.1);*
(b) $x\geq x^*$ *for any* x *solution of (4.2);*

where x^* *is the unique solution of (1.1).*

Proof. By applying Theorem 3.2 it follows that the operator $V: C[a,b]\to C[a,b]$ defined by, $V(x)(t) :=$ second part of equation (1.1) is a PO and from $(C3)$ we have that V is an increasing operator. The conclusion is obtained from Abstract Gronwall lemma. □

5. Comparison theorems

Using the results from Section 3 and the Abstract Comparison lemma we can obtain a comparison theorem for the functional integral equations:

$$x_i(t) = \int_a^t K_i(t,s,x(s), \max_{a\leq\xi\leq s} x(\xi))ds + f_i(t,x(t)),\ t\in[a,b],\ i=\overline{1,3}, \tag{5.1}$$

where $K\in C([a,b]\times[a,b]\times\mathbb{R}^2,\mathbb{R})$ and $f\in C([a,b]\times\mathbb{R},\mathbb{R})$. We have the following result:

Theorem 5.1. *We suppose that:*

(i) K_i, f_i, $i=\overline{1,3}$ *satisfy the conditions* $(C1)-(C2)$*;*
(ii) $K_1\leq K_2\leq K_3$ *and* $f_1\leq f_2\leq f_3$*;*
(iii) $K_2(t,s,\cdot)$ *and* $f_2(t,s,\cdot)$ *are increasing.*

If $x_1(a) \leq x_2(a) \leq x_3(a)$ *then* $x_1^* \leq x_2^* \leq x_3^*$ *where* x_i^* *is the unique solution of (5.1),* $i = \overline{1,3}$.

Proof. From Theorem 3.2 we have that the operator $V_i : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by,

$$V_i(x)(t) := \int_a^t K_i(t,s,x(s), \max_{a\leq\xi\leq s} x(\xi))ds + f_i(t,x(t)),\ t \in [a,b]$$

is PO, $i = \overline{1,3}$. Let $F_{V_i} = \{x_i^*\}$, $i = \overline{1,3}$.

If $u \in \mathbb{R}$ then we denote by $\tilde{u}$ the constant function

$$\tilde{u} : [a,b] \to \mathbb{R}, \tilde{u}(t) = u.$$

It is clear that

$$V_i^\infty(\widetilde{x_i(a)}) = x_i^*,\ i = \overline{1,3},$$

and from (ii) we get that

$$V_1(x) \leq V_2(x) \leq V_3(x),\ \forall x \in C[a,b].$$

From condition (iii) we get that the operator V_2 is an increasing operator. The conclusion is obtained by applying the Abstract Comparison lemma. □

Acknowledgement. The authors would like to express their special thanks and gratitude to Professor Ioan A. Rus for the ideas and continuous support along the years.

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Received: November 10, 2023; Accepted: February 9, 2024.