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FIXED POINT PROPERTY OF FULL HILBERT C*-MODULES OVER UNITAL C*-ALGEBRAS

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Abstract. We show that full Hilbert C*-modules over a commutative unital C*-algebra $\mathcal{A}$ have fixed point property for nonexpansive mappings if and only if $\mathcal{A}$ is finite dimensional. We also show that the same is true for every Hilbert C*-module with unit vectors over an arbitrary unital C*-algebra. In particular, a classification of full Hilbert C*-modules with unit vectors over unital C*-algebras is given via fixed point property for nonexpansive mappings.
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1. Introduction and Preliminaries

A self-mapping T on a nonempty subset C of a Banach space X is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for every $x, y \in C$. The Banach space X is said to have fixed point property (FPP for short) if for every nonempty closed, convex and bounded subset C of X, every nonexpansive mapping $T : C \to C$ has a fixed point. The FPP of Banach spaces has been extensively investigated for quite a while, and as known so far, all classical reflexive Banach spaces have FPP. More information on this issue can be found, for instance, in monographs [3, 4, 8].

An important generalization of Hilbert spaces are Hilbert C*-modules which play a crucial role in KK-theory, Morita equivalence of C*-algebras and noncommutative geometry. The first investigation of the FPP of Hilbert C*-modules has been recently carried out by Gloabi and the second author in [5], where it was proved that Hilbert C*-modules over finite dimensional C*-algebras are reflexive as well as they have FPP.

In this paper, we investigate the FPP of full Hilbert C*-modules over unital C*-algebras. We first show, in Theorem 2.2, that a Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$ has FPP if and only if $\mathcal{A}$ is reflexive, or equivalently, finite dimensional. For this purpose, we use two known facts: one is the commutative case of the Gelfand-Naimark theorem for commutative unital C*-algebras and the other one is the Serre-Swan theorem which states that Hilbert C*-modules over commutative unital C*-algebras may be alternatively described as continuous fields of Hilbert spaces over some compact Hausdorff space. In Theorem 2.3, we show that the same is true for every Hilbert C*-module with unit vectors over an arbitrary unital C*-algebra. Finally, in Theorem 2.4, we give a classification of full Hilbert C*-modules with unit vectors over unital C*-algebras via FPP for nonexpansive mappings. In particular, Theorem 2.4 gives an affirmative answer to the open problems stated in [5] in the case of full Hilbert C*-modules with unit vectors over unital C*-algebras.

We begin with some definitions and basic facts which we will use in the sequel.

A C*-algebra $\mathcal{A}$ is a Banach algebra with an involution $a \to a^*$ such that $(a^*)^* = a$, $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$, $(ab)^* = b^* a^*$ and satisfies the C*-identity $\|aa^*\| = \|a\|^2$, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A detailed information about C*-algebras can be found, for instance, in [2] and [12]. Let us also give the following definition:

Definition 1.1. Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{M}$ be a left $\mathcal{A}$-module such that $\lambda(ax) = (\lambda a)x = a(\lambda x)$ whenever $\lambda \in \mathbb{C}$, $x \in \mathcal{M}$ and $a \in \mathcal{A}$. An $\mathcal{A}$-valued inner product on $\mathcal{M}$ is a mapping $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ which satisfies the following properties for all $x, y, z \in \mathcal{M}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:
(1) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$;
(2) $\langle x, x \rangle \geq 0$;
(3) $\langle x, x \rangle = 0$ if and only if $x = 0$;
(4) $\langle x, y \rangle^* = \langle y, x \rangle$;
(5) $\langle ax, y \rangle = a \langle x, y \rangle$.
If $\mathcal{M}$ is complete with respect to the norm $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$, then $\mathcal{M}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module (module) over $\mathcal{A}$.

A Hilbert $\mathcal{A}$-module $\mathcal{M}$ is said to be full if $\langle \mathcal{M}, \mathcal{M} \rangle$, the closure of the linear span of $\{\langle x, y \rangle : x, y \in \mathcal{M}\}$, is equal to $\mathcal{A}$. We refer the reader to [9] and [11] for more details about Hilbert C*-modules.

Hereafter, we assume that Z is a compact Hausdorff space and $C(Z)$ is the commutative unital C*-algebra consisting of all complex-valued continuous functions on Z, equipped with pointwise operations and the supremum norm $\|\cdot\|_\infty$. By the Gelfand-Naimark theorem, any commutative unital C*-algebra is isometrically $*$-isomorphic to $C(Z)$, for some compact Hausdorff space Z. Hence, one can consider Hilbert $C(Z)$-modules instead of Hilbert C*-modules over commutative unital C*-algebras. On the other hand, a generalization of the Serre-Swan theorem [14] asserts that the category of Hilbert $C(Z)$-modules is equivalent to the category of continuous fields of Hilbert spaces over Z. Therefore, such objects themselves can be regarded as Hilbert $C(Z)$-modules as well. This opens an approach toward investigating the FPP of Hilbert $C(Z)$-modules. Let us recall the notion of continuous fields of Hilbert spaces and some related facts; see [2] and [7, Remark 4.4, Proposition 4.8] for useful details on the continuous field of Banach spaces.

Let $(H_z, \langle \cdot, \cdot \rangle_z)_{z \in Z}$ be a family of Hilbert spaces. A function x defined on Z such that $x(z) \in H_z$ for each $z \in Z$, is called a vector field over Z. Notice that each vector field is an element of $\prod_{z \in Z} H_z$.

Definition 1.2. A continuous field of Hilbert spaces over Z is a pair $((H_z)_{z\in Z},\Gamma)$, where $(H_z)_{z\in Z}$ is a family of Hilbert spaces and Γ is a complex linear subspace of

$$C(Z)-\prod_{z\in Z}H_z:=\Big\{x\in\prod_{z\in Z}H_z:[z\mapsto\|x(z)\|]\in C(Z)\Big\},$$

which satisfies the following properties:
(i) For every $z\in Z$, $\{x(z):x\in\Gamma\}=H_z$;
(ii) For any vector field $x\in\prod_{z\in Z}H_z$, if for every $z\in Z$ and every $\epsilon>0$, there is an $x'\in\Gamma$ such that $\|x(s)-x'(s)\|<\epsilon$, for all s in some neighborhood of z, then $x\in\Gamma$.

The linear subspace Γ in the continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$ can be considered as a Hilbert $C(Z)$-module when it is equipped with the point-wise multiplication

$$(f\cdot x)(z)=f(z)x(z),$$

and $C(Z)$-valued inner product

$$\langle x,y\rangle(z)=\langle x(z),y(z)\rangle,$$

for all $f\in C(Z)$, $x,y\in\Gamma$, and $z\in Z$. Notice that the function $z\mapsto\langle x,y\rangle(z)$ belongs to $C(Z)$ (see [2, 10.7.1]). Furthermore, Γ is a Banach space with the norm $\|x\|=\sup_{z\in Z}\|x(z)\|$. The fact that every Hilbert $C(Z)$-module $\mathcal{M}$ is isomorphic to some continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$ as Hilbert $C(Z)$-modules (see [7] and [15, Theorem 3.12]), is crucial for our first result. It is worth mentioning that a Hilbert $C(Z)$-module $\mathcal{M}$ is full if and only if each Hilbert space $H_z=\{x(z):x\in\Gamma\}$ in its isomorphic continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$ is nontrivial.

Besides the results and examples given in [5], the following examples will help us to have more insight to the FPP in Hilbert C*-modules. For this, in the following we assume that H is an infinite dimensional Hilbert space, $K(H)$ and $B(H)$ denote the C*-algebras consisting of all compact linear operators and all bounded linear operators on H, respectively. Notice that every C*-algebra $\mathcal{A}$ may be treated as a Hilbert $\mathcal{A}$-module equipped with the natural left multiplication $a\cdot x:=ax$ and $\mathcal{A}$-valued inner product $\langle x,y\rangle:=xy^*$, for all $a,x,y\in\mathcal{A}$.

Example 1.3. Given two vectors $x,y\in H$, let $x\otimes y$ denote the elementary operator on H, i.e., $x\otimes y:u\mapsto\langle u,y\rangle x$. Obviously, $\|x\otimes y\|=\|x\|\|y\|$ and every rank-one projection on H is of the form $x\otimes x$, for some unit vector $x\in H$ (i.e., $\langle x,x\rangle=1$). Moreover, the set of all finite rank linear operators on H is spanned by rank-one projections ([12, Theorem 2.4.6]). Now, H may be regarded as a $K(H)$-module if we define the $K(H)$-valued inner product by $(x,y)\mapsto x\otimes y$. The $K(H)$-module norm on H coincides with the original norm and since finite rank operators on H are dense in $K(H)$ ([12, Theorem 2.4.5]), H as a $K(H)$-module is full. In particular, the Hilbert $K(H)$-module H has FPP.

Example 1.4. The C*-algebras $K(H)$ and $B(H)$ are full as Hilbert $K(H)$-module and $B(H)$-module, respectively, and they fail to have FPP (see [1]). The first Hilbert C*-module is over a non-unital C*-algebra while the second one is over a unital C*-algebra. Note that $K(H)$ may also be regarded as a Hilbert $B(H)$-module which is not full.

2. Main results

The second example in [5] shows that a Hilbert $C(Z)$-module which is not full may have FPP. On the other hand, as the examples above show, full Hilbert C*-modules over infinite dimensional C*-algebras may or may not have FPP. On the other hand, any Hilbert C*-module $\mathcal{M}$ can be regarded full over the C*-algebra $\langle \mathcal{M}, \mathcal{M} \rangle$. This is why we consider only full Hilbert C*-modules.

In our first result, we show that the FPP of full Hilbert $C(Z)$-modules completely depends on the FPP of the C*-algebra $C(Z)$. We need the following fact for giving the related result:

Lemma 2.1. [10, Lemma 2.3] *Let Z be an infinite compact Hausdorff space. Then, there exists a continuous function $f : Z \to \mathbb{R}$ such that $f(Z)$ is infinite.*

Theorem 2.2. *Let $\mathcal{M}$ be a full Hilbert $C(Z)$-module. Then, $\mathcal{M}$ has FPP if and only if Z is finite.*

Proof. Since Z is finite, clearly $\mathcal{M}$ has FPP by [5, Theorem 2.4]. Conversely, let Z be infinite. There exists, by Lemma 2.1, a continuous real-valued function f on Z such that $f(Z)$ is infinite. Let $(f(z_n))$ be a sequence of distinct elements of $f(Z)$. Since Z is compact, we may assume, by passing to a subsequence of (z_n) if necessary, that $z_n \to z_0$ for some $z_0 \in Z$. Without loss of generality we may assume that $f(z_0) = 0$; otherwise we can consider $f - f(z_0)$, instead. Let $((H_z)_{z \in Z}, \Gamma)$ be the continuous field of Hilbert spaces which is isomorphic to $\mathcal{M}$. Since $\mathcal{M}$ is full, there exist $v_0 \neq 0$ in H_{z_0} and $y \in \Gamma$ such that $y(z_0) = v_0$. Now, consider the set

$$C = \{x \in \Gamma : x(z_0) = v_0 \text{ and } \|x\| \leq \|y\|\}.$$

It is easy to see that C is nonempty, convex and bounded. Furthermore, C is closed. Indeed, if (x_n) is a sequence in C converging to $x \in \Gamma$, then

$$\|v_0 - x(z_0)\|_{z_0} = \|x_n(z_0) - x(z_0)\|_{z_0} \leq \|x_n - x\|_\Gamma \to 0.$$

That is, $x \in C$. Define $g : Z \to \mathbb{R}$ by

$$g(z) := 1 - \frac{|f(z)|}{\|f\|_\infty} \qquad (z \in Z).$$

Obviously $g \in C(Z)$ and $0 \leq g \leq 1$. Now, the mapping T defined as $Tx = g \cdot x$, for all $x \in C$ is a self-mapping on C. Moreover, we have

$$\begin{aligned}
\|Tx - Ty\|_\Gamma &= \sup_{z \in Z} \|Tx(z) - Ty(z)\|_z \\
&= \sup_{z \in Z} \|g(z)x(z) - g(z)y(z)\|_z \\
&\leq \sup_{z \in Z} \|x(z) - y(z)\|_z \\
&= \|x - y\|_\Gamma,
\end{aligned}$$

for every $x, y \in \Gamma$ which means that T is nonexpansive. But T has no fixed points in C. In fact, suppose on the contrary that $u \in C$ is a fixed point for T. From Definition 1.2, since $z \mapsto \|u(z)\|_z$ is a continuous function, we have that

$$\|u(z_n)\|_{z_n} \to \|u(z_0)\|_{z_0} = \|v_0\|_{z_0} \neq 0.$$

Choose $n_0 \in \mathbb{N}$ so that $\|u(z_{n_0})\|_{z_{n_0}} > 0$ and $0 < g(z_{n_0}) < 1$. Since $u = g \cdot u$, we observe that

$$0 < \|u(z_{n_0})\|_{z_{n_0}} = \|g(z_{n_0})u(z_{n_0})\|_{z_{n_0}} < \|u(z_{n_0})\|_{z_{n_0}},$$

which is a contradiction. This completes the proof.

The second example in [5] gives a Hilbert $C(Z)$-module with Z infinite which has FPP. This, in fact, shows that why the assumption of "fullness" is necessary in Theorem 2.2.

An element w in a Hilbert C*-module over a unital C^*-algebra with unit $\mathbf{1}$ is said to be a unit vector if $\langle w, w\rangle = \mathbf{1}$. Every Hilbert C*-module with unit vectors is full but unit vectors do not necessarily exist in full Hilbert modules over arbitrary unital C^*-algebras (see [13, Example 3.3] and [6, Example 1]).

In our second result, we show that the FPP of a full Hilbert C*-module $\mathcal{M}$ with unit vectors is derived from the FPP of the C*-algebra over which $\mathcal{M}$ is constructed.

Theorem 2.3. *Let $\mathcal{M}$ be a Hilbert C*-module with unit vectors over a unital C*-algebra $\mathcal{A}$. Then, $\mathcal{M}$ has FPP if and only if $\mathcal{A}$ is finite dimensional.*

Proof. The if part is a consequence of [5, Theorem 2.4]. Conversely, suppose that $\mathcal{A}$ is infinite dimensional and w is a unit vector for $\mathcal{M}$. Define a mapping $\varphi : \mathcal{A} \to \mathcal{M}$ by $\varphi(a) = a \cdot w$, for all $a \in A$. Clearly, φ is linear. Moreover, since

$$\|a \cdot w\|^2_{\mathcal{M}} = \|\langle a \cdot w, a \cdot w\rangle\|_{\mathcal{A}} = \|a\langle w, w\rangle a^*\|_{\mathcal{A}} = \|aa^*\|_{\mathcal{A}} = \|a\|^2_{\mathcal{A}}$$

φ is an isometry. In particular, $\varphi(\mathcal{A})$ is a Banach space. Now, since $\mathcal{A}$ cannot have FPP (see [1]), there exist a nonempty closed, convex and bounded subset C of $\mathcal{A}$ and a nonexpansive mapping $T : C \to C$ which does not have any fixed point. On the other hand, $\varphi(C)$ is a nonempty closed, convex and bounded subset of $\varphi(\mathcal{A})$. Consider the mapping

$$\tilde{T} : \varphi(C) \to \varphi(C)$$

defined by

$$\tilde{T}(a \cdot w) = T(a) \cdot w,$$

for any $a \in C$. We have

$$\begin{aligned} \|\tilde{T}(x \cdot w) - \tilde{T}(y \cdot w)\|_{\mathcal{M}} &= \|(T(x) - T(y)) \cdot w\|_{\mathcal{M}} \\ &= \|T(x) - T(y)\|_{\mathcal{A}} \\ &\le \|x - y\|_{\mathcal{A}} \\ &= \|x \cdot w - y \cdot w\|_{\mathcal{M}}, \end{aligned}$$

for all $x, y \in C$. Hence, $\tilde{T}$ is nonexpansive. Suppose that $\tilde{T}(u \cdot w) = u \cdot w$, for some $u \in C$. Therefore, $T(u) \cdot w = u \cdot w$ and we have

$$T(u) = \langle T(u) \cdot w, w\rangle = \langle u \cdot w, w\rangle = u$$

which is a contradiction. Hence, $\tilde{T}$ is a nonexpansive mapping on $\varphi(C)$ which fails to have fixed points. This completes the proof.

We may give the following classification of full Hilbert C*-modules with unit vectors over unital C*-algebras via FPP:

Theorem 2.4. *Let $\mathcal{M}$ be a Hilbert C*-module with unit vectors over a unital C*-algebra $\mathcal{A}$. Then, $\mathcal{M}$ has FPP if and only if $\mathcal{M}$ is a reflexive Banach space.*

Proof. Suppose that $\mathcal{M}$ has FPP. By Theorem 2.3, $\mathcal{A}$ is finite dimensional and therefore by [5, Theorem 2.4] the Hilbert module $\mathcal{M}$ is reflexive. Conversely, suppose that $\mathcal{M}$ is reflexive as a Banach space over unital C*-algebra $\mathcal{A}$. Then $\mathcal{A}$ cannot be infinite dimensional. Since otherwise, by the well-known fact that "a C*-algebra is reflexive as a Banach space if and only if it is finite dimensional", $\mathcal{A}$ is not reflexive, and therefore from the isometry φ given in the proof of Theorem 2.3, $\varphi(A)$ is a non-reflexive Banach subspace of $\mathcal{M}$. This implies that $\mathcal{M}$ is itself non-reflexive which is a contradiction. Now, Lemma 2.1 in [5] gives the desired result.

Theorem 2.4 gives an affirmative answer to the open problems posed in [5] in the case of full Hilbert C*-modules with unit vectors over unital C*-algebras. We close the paper with the following questions related to Theorems 2.3 and 2.4.

Question 2.5. Does every full Hilbert C*-module $\mathcal{M}$ over a unital C*-algebra $\mathcal{A}$ have FPP only if $\mathcal{A}$ is finite dimensional? We of course know, by Theorem 2.2, that this is true for any full Hilbert C*-module over a commutative unital C*-algebra.

And a weakened form of Problems 1 and 2 in [5]:

Question 2.6. Does every full Hilbert C*-module $\mathcal{M}$ over a unital C*-algebra have FPP if and only if $\mathcal{M}$ is a reflexive Banach space?

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