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UNIFORMLY ELLIPTIC EQUATIONS WITH CONCAVE GROWTH IN THE GRADIENT AND MEASURES

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Abstract. We deal with quasilinear elliptic problems with measure data:

$$\begin{cases} Lw = H(x, w, \nabla w) + \mu & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $Lw := -\mathrm{div}(A(x)\nabla w)$ with $A = A(x)$ a bounded, coercive, and symmetric matrix field, the Hamiltonian H has at most q-growth in the gradient for $0 < q < 1$, and μ is any Radon measure. We employ the compactness of the Green operator associated to L from the space of measures to $W_0^{1,p}(\Omega)$ for all $p \in [1, N/(N-1))$ together with fixed point arguments to solve problem (0.1) for any measure μ. Moreover, we provide explicit estimates of the solution in terms of the data. As an application, stability results are given. We also give conditions for the existence of $W_0^{1,2}$-solutions through the classic theory of monotone and coercive operators. In any case, we do not impose any size restriction on μ and any sign condition on H.

Key Words and Phrases: Quasilinear elliptic equations, fixed point, Green's functions, weak solutions, uniqueness.
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1. Introduction

In this paper, we consider second-order quasilinear elliptic problems of the form

$$\begin{cases} Lw = H(x, w, \nabla w) + \mu & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, L is a linear and uniformly elliptic operator $Lw = -\mathrm{div}(A(x)\nabla w)$, with $A = A(x)$ a bounded, coercive, and symmetric matrix field. Also, the Hamiltonian H is a continuous function satisfying a q-growth condition in the gradient with $0 < q < 1$, and μ is a Radon measure. Further details in the assumptions will be given in Section 2.

Problems with first order terms arise naturally in the study of stationary models of growing interfaces, like the Kardar-Parisi-Zhang model [22], and in stochastic control problems, see for instance [23].

The main assumption on H will be the following growth condition

$$|H(x,r,\xi)| \leq c_0 h(x)|r| + b(x)|\xi|^q + |g(x)| \quad \text{for all } r \in \mathbb{R}, \xi \in \mathbb{R}^N, \text{ and a. e. } x \in \Omega, \tag{1.2}$$

for $q \in (0,1)$, $h, b \geq 0$, and $g \in L^1(\Omega)$. The model problem reads as

$$\begin{cases} -\Delta w + b(x)|\nabla w|^q = c_0 h(x) w + \mu & \text{in } \Omega \\ w = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Our main contributions are the following. We prove that if b and h in (1.2) satisfy certain variational conditions (see (2.3) and (2.4)) and c_0 is small enough, then a solution u of (1.1) exists for any measure μ. Here, we do not impose any size condition on μ and any sign condition on H. Furthermore, we state the regularity $u \in W_0^{1,p}(\Omega)$ for all $p \in [1, N/(N-1))$, and we give a control on the norm of u in terms of the data. As a consequence of these estimates, and among other stability results, we may recover solutions of equations like $-\Delta u + |\nabla u|^q = \mu$ as limits of solutions u_ε as $\varepsilon \to 0^+$ of perturbed problems $-\Delta u_\varepsilon + |\nabla u_\varepsilon|^q = \varepsilon u_\varepsilon + \mu_\varepsilon$. Moreover, in the case $h \equiv 1$, and when $|r|$ is replaced by $|r|^l$ in (1.2), with $l \in (0,1)$, we show that no size condition on c_0 is requested to solve (1.1). Finally, we show that $W_0^{1,2}(\Omega)$-regularity may be expected for solutions when the source μ belongs to $\mathcal{M}(\Omega) \cap W^{-1,2}(\Omega)$ and c_0 is small.

There are several references considering linear problems like (1.1). We first mention the works [25] and [28], where they introduced a notion of solution by duality that we recover in this paper. They also proved existence and uniqueness of solutions, and stated that when the right-hand side is a measure or an L^1-function, the solution belongs to $W^{1,p}(\Omega)$, for all $p \in [1, N/(N-1))$. In [28], Stampacchia also provided a further hierarchy of regularity, depending on the regularity of the source terms. We remark that coercivity of the elliptic operator is essential in his results.

We now discuss about noncoercivity of the linear operators. We first mention the work [6], where it is proved that the same results of existence and regularity of [28] may be obtained for noncoercive linear operators, using techniques from nonlinear problems. In [8], the authors studied noncoercive linear problems with discontinuous coefficients and singular drifts in L^N. By duality methods and a nonlinear approach, the Spampacchia theory is recovered when the source is sufficiently integrable, and, when the integrability is lower, the Calderón-Zygmund theory of distributional solutions is obtained. In the second case, there is a parallelism with [11] when there is no drift term. Weak maximum principles in this framework can be found in [9]. For less regular drift terms, we quote the works [12] and [7].

We recall that for problems with first order terms having sub-quadratic growth in the gradient, a size condition or higher regularity on the source are needed to obtain existence of solutions (see for instance [1], [14], [16], [15], [21], and the references therein). We also cite the surveys [27] and [17] where nice presentations of elliptic problems with first order terms are given. Finally, a discussion of existence of weak solutions for structures with q-power in the norm of the gradient (for any $q > 1$) is provided in [20] and, for more general growths, in [4] and [1, Remark 2.5], among many other references.

Existence of solutions to boundary value problems with a q-power of the gradient, where $q \in (0,1)$, was treated in [5] and [26]. The problem (1.1) with $A = I$, $\mu = 0$, $H(x,\xi) = -|\xi|^q$, and non homogeneous boundary data, has been treated in [26], where existence of very weak solutions is obtained. More general, in [5], the authors established existence of renormalized solutions for the Laplace operator and first order structures with sub-linear growth in the gradient, without any size condition on the data. Much more general structures are considered in the book [29]. However, the results contained in Chapters 5 and 6 of [29], like Theorem 6.2.3, for renormalized solutions do not apply to our case since the growth or the sign assumptions imposed there on the source and the absorption terms H are not valid for our case.

Here, we do not follow the renormalized-solution approach. The existence of Sobolev weak solutions to (1.1) for any measure is achieved by using compactness arguments in the Green operator $\mathbb{G}$ (see [28] and [13] for related arguments). Here, and as a consequence of an L^p-estimate of solutions, we will provide a proof of the compactness of $\mathbb{G}$. Moreover, our solution belongs to $W^{1,p}(\Omega)$ for all $p \in [1, N/(N-1))$ and hence we recover the well-known regularity results of solutions to linear uniformly elliptic problems with measure data ([25], [28], see also [10] and [11]). Finally, for any data in $W^{-1,2}$, we provide the existence of weak solutions in $W_0^{1,2}$. This is achieved by applying the classical theory of coercive and monotone operators of Leray-Lions type.

The paper is organized as follows. In Section 2, we give notation, definitions, and assumptions. In Section 3, we prove the compactness of the Green operator. In Section 4, we provide the main result of the paper and we also give some stability results. Finally, in Section 5, we state the solvability of problem (1.1) for $\mu \in W^{-1,2}$.

2. Notation and preliminaries

2.1. **Basic notation.** For a given real Banach space $\mathcal{B}$, we let $\langle \cdot, \cdot \rangle$ for the usual pairing between $\mathcal{B}$ and its dual. The underlying norm in $\mathcal{B}$ will be denoted by $\|\cdot\|$ or $\|\cdot\|_{\mathcal{B}}$ when is needed for clarity. When $\mathcal{B} = \mathbb{R}^N$ for some $N \geq 1$, we denote the Euclidean norm by $|\cdot|$.

Let $E \subset \mathbb{R}^N$ be a non empty set. The distance function to the complementary E^c of E is $\delta(x) := \operatorname{dist}(x, E^c), \quad x \in \mathbb{R}^N$. We now recall some well-known notation for function spaces. We let $\mathcal{M}(E)$ be the set of all signed Radon measures ν in E such that E is ν-measurable and the total variation norm $\|\nu\|_{\mathcal{M}(E)} := \int_E d|\nu|$ is finite. By $\mathcal{M}_+(E)$ we denote the positive cone of $\mathcal{M}(E)$. For any real-valued function φ, we set $\varphi^+ := \max\{0,\varphi\}$, and $\varphi^- := -\min\{0,\varphi\}$.

For $E \subset \mathbb{R}^N$ open, we denote by $C_c(E)$ the set of continuous functions with compact support in E, and by $C_0(\overline{E})$ the closure of $C_c(E)$ in $C(\overline{E})$, that is $C_0(\overline{E}) = \left\{\varphi \in C(\overline{E}) : \varphi = 0 \text{ on } \partial E\right\}$. By $C_c^\infty(E)$ we denote the space of infinitely differentiable functions which belong to $C_c(E)$.

We use the standard notation for Lebesgue and Sobolev spaces. Moreover, for $p \in (1,\infty)$, p' denotes its conjugate and for $p \in [1,N)$, p^* denotes the critical exponent in the Sobolev imbedding. Finally, in long calculations, by C we denote a positive universal constant, which may differ from line to line.

2.2. **Set of assumptions.** We now give the main assumptions that we will use in the paper. Further hypothesis may be specified in some particular results.

(H1) The subset $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded, open and C^2 domain;
(H2) The matrix field $A(x) = [a_{ij}(x)]$ is symmetric, measurable, bounded and uniformly elliptic, i.e., there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \text{ for all } \xi \in \mathbb{R}^N \text{ and a. e. } x \in \Omega; \tag{2.1}$$

(H3) The structural term $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$|H(x, r, \xi)| \leq c_0 h(x)|r| + b(x)|\xi|^q + |g(x)| \text{ for all } r \in \mathbb{R}, \xi \in \mathbb{R}^N \text{ and a. e. } x \in \Omega, \tag{2.2}$$

where $c_0 > 0$, $0 < q < 1$, $g \in L^1(\Omega)$ and the non-negative functions b and h, not zero in a set of positive measure, satisfy

$$[E_q(b)]^{-1} := \inf_{\phi\neq 0,\, \phi\in L^1(\Omega)} \frac{\int_\Omega |\phi|\, dx}{\left(\int_\Omega b|\phi|^q\, dx\right)^{1/q}} > 0, \tag{2.3}$$

and

$$[E(h)]^{-1} := \inf_{\phi\neq 0,\, \phi\in W_0^{1,1}(\Omega)} \frac{\int_\Omega |\nabla\phi|\, dx}{\int_\Omega h|\phi|\, dx} > 0. \tag{2.4}$$

(H4) The measure μ belongs to $\mathcal{M}(\Omega)$.

We say that 'assumption (H) holds' if the hypothesis $(H1) - (H4)$ are valid.

Observe that since $0 < q < 1$, $b \equiv 1$ satisfies (2.3). More generally, if $b \in L^{1/(1-q)}(\Omega)$, $b \neq 0$, then Hölder's inequality gives that $E(b) > 0$.

Also, we point out that condition (2.4) is analogous to the variational assumption impose to h in [2, Theorem 2.9] to obtain existence of solutions. Finally, observe that (2.4) implies that $h \in W^{-1,\infty}(\Omega)$.

2.3. **Notions of solutions.** We now give the meaning of solutions that we use in the paper. We first introduce the notion of weak solutions to problem (1.1). Afterwards, for the linear problem, we provide the connection between weak solutions and the notion of very weak solutions presented in [30]. Throughout the section, assume that (H) holds. From now on we let $Lu := -\mathrm{div}\,(A(x)\nabla u)$.

Definition 2.1. We say that $u \in W_0^{1,1}(\Omega)$ is a weak subsolution of problem (1.1) if

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\, dx \leq \int_\Omega H(x, u, \nabla u)\varphi\, dx + \int_\Omega \varphi\, d\mu \tag{2.5}$$

for all non-negative $\varphi \in C_c^\infty(\Omega)$. A similar definition is given for weak supersolutions. A weak solution is a weak sub- and supersolution.

Now we establish the definition of very weak solution from [30] for the linear case

$$\begin{cases} Lw = \mu & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Definition 2.2. We say that $u \in L^1(\Omega)$ is a very weak subsolution of problem (2.6) if

$$\int_\Omega u\psi \, dx \leq \int_\Omega \varphi \, d\mu,$$

for all non-negative $\varphi \in C_0^1(\overline{\Omega})$ solving (2.6) with $\mu = \psi \in L^\infty(\Omega)$ in the following sense

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, dx = \int_\Omega \psi\phi \, dx \quad \text{for all } \phi \in W_0^{1,2}(\Omega). \tag{2.7}$$

A similar definition is given for very weak supersolutions. A very weak solution is a very weak sub- and supersolution.

In the sequel, we will write $L\varphi$ instead of ψ. The following proposition states the equivalence between weak and very weak solutions in the linear problem.

Proposition 2.1. *Suppose that* $(H1)$, $(H2)$ *and* $(H4)$ *hold, and that* $A = [a_{ij}]$ *is Lipschitz. Let* $u \in W_0^{1,1}(\Omega)$. *Then* u *is a weak subsolution of problem* (2.6) *if and only if* u *is a very weak subsolution of* (2.6). *A similar result holds for supersolutions.*

Proof. Take $u \in W_0^{1,1}(\Omega)$ a very weak subsolution of (2.6) and $\varphi \in C_c^\infty(\Omega)$, non-negative. Then, there is a sequence $\{u_n\} \subset C_c^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,1}(\Omega)$. Since the coefficients a_{ij} are Lipschitz, $L\varphi \in L^\infty(\Omega)$ and (2.7) holds for $\psi = L\varphi$.

Now, since $u_n \to u$ in $W_0^{1,1}(\Omega)$ and φ is a test function for Definition 2.2 we have

$$\int_\Omega uL\varphi \, dx = \lim_{n\to\infty} \int_\Omega u_n L\varphi \, dx = \lim_{n\to\infty} \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, dx. \tag{2.8}$$

Therefore, since u is a very weak subsolution of (2.6), by (2.8) we get

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, dx = \lim_{n\to\infty} \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, dx \leq \int_\Omega \varphi \, d\mu. \tag{2.9}$$

Conversely, if $u \in W_0^{1,1}(\Omega)$ satisfies Definition 2.1, then for non-negative $\varphi \in C_0^1(\overline{\Omega})$ with $L\varphi \in L^\infty(\Omega)$ we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi_n}{\partial x_j} \, dx \leq \int_\Omega \varphi_n \, d\mu$$

for any $\varphi_n \in C_c^\infty(\Omega)$ converging to φ in $C^1(\Omega)$ with $\varphi_n \geq 0$. Taking $n \to \infty$ and recalling (2.8), we obtain that u is a very weak subsolution. $\square$

2.4. **Preliminaries on Green operators.** The Green function $G = G(x, y)$ of the operator L in Ω is defined as the unique very weak solution of (2.6) with $\mu = \delta_y$, the Dirac measure at y. The following result can be found in [30, Sections 2.3 and 2.4].

Theorem 2.1. (i) *For* $A = [a_{ij}]$ *Lipschitz and for any* $\mu \in \mathcal{M}(\Omega)$, *the unique very weak (or weak) solution* u *of* (2.6) *can be represented, for a.e.* x *in* Ω, *as*

$$u(x) = \int_\Omega G(x, y) d\mu(y), \tag{2.10}$$

and $u \in W_0^{1,p}(\Omega)$ *for all* $p \in [1, N/(N-1))$. *(ii) For all* $p \in [1, N/(N-1))$, *the mapping* $\mathbb{G} : \mathcal{M}(\Omega) \to W_0^{1,p}(\Omega)$ *which assigns to each* $\mu \in \mathcal{M}(\Omega)$ *the solution* (2.10) *is called the Green operator and it is continuous, that is, there is* $C'_p > 0$ *depending on* N, p *and* Ω *so that*

$$\|\mathbb{G}(\mu)\|_{W_0^{1,p}(\Omega)} \leq C'_p \|\mu\|_{\mathcal{M}(\Omega)}, \quad \textit{for all } \mu \in \mathcal{M}(\Omega). \tag{2.11}$$

Remark 2.1. It is worth mentioning that, when $A = [a_{ij}]$ is Lipschitz, very weak solutions $u \in L^1(\Omega)$ of (2.6) are also duality solutions in the sense of [25] since very weak solutions are unique and by Theorem 2.1 satisfy the representation (2.10). Hence, the estimate in Theorem 2.1 also follows from [25, Sections 5 and 6].

Remark 2.2. The representation (2.10) gives that $\mu(\{x\}) = 0$ for a.e. x in Ω. Indeed, take x for which (2.10) holds and is finite. Since $G \geq 0$ and $\lim_{y \to x} G(x, y) = +\infty$, for each positive integer n, there is $r > 0$ such that $G(x, y) \geq n$ for all $y \in B(x, r)$. Hence, $u(x) \geq n\mu(B(x, r)) \geq n\mu(\{x\})$. Letting $n \to \infty$, we prove the claim.

For convenience of the reader, we also provide some estimates for the Green function G and its gradient of uniformly elliptic operators L with Lipschitz coefficients. From [18] and [30, Theorem 2.11], it follows that

$$C^{-1} \min\{\delta(x), \delta(y)\} |x-y|^{1-N} \leq G(x, y) \leq C \min\{\delta(x), \delta(y)\} |x-y|^{1-N} \tag{2.12}$$

for some $C > 0$ and all $x \neq y$ in Ω. Finally, by [19, Theorem 3.3], we have the next upper bound for the norm of the gradient of G,

$$|\nabla_x G(x, y)| \leq C \min\left\{|x-y|^{1-N}, \delta(y)|x-y|^{-N}\right\}, \ C > 0, \, x \neq y. \tag{2.13}$$

3. The linear case: $H \equiv 0$

Throughout this section, we consider uniformly elliptic operators L of the form $L = -\mathrm{div}\,(A(x)\nabla u)$ with associate Green operator $\mathbb{G}$.

The aim of this section is to provide an L^p-estimate for weak subsolutions in the linear case. Afterwards, we apply the estimate to prove that the Green operator $\mathbb{G}$ of L with Lipschitz coefficients is compact from $\mathcal{M}(\Omega)$ into $W_0^{1,p}(\Omega)$ for any $p \in [1, N/(N-1))$ (some related results can be found in [11]).

Theorem 3.1. *Assume (H1), (H2) and that* $A(x) = [a_{ij}(x)]$ *is Lipschitz. Let* $\mu \in \mathcal{M}_+(\Omega)$ *and let* $u \in W_0^{1,1}(\Omega)$, $u \gneqq 0$, *be a weak subsolution to* (2.6). *Then,* $u \in W_0^{1,p}(\Omega)$ *for any* $p \in [1, N/(N-1))$, *and for any such* p *there is* $r_0 \in [1, N/(N-2))$ *such that*

$$\int_\Omega |\nabla u(x)|^p \, dx \leq C \left[\int_\Omega \frac{u^p(x)}{\delta^p(x)} \, dx + \|u\|^{p-1}_{L^{r_0}(\Omega)} \|\mu\|_{\mathcal{M}(\Omega)} \right]. \tag{3.1}$$

Proof. For $\varphi \in C_c^\infty(\Omega)$, define

$$F(\varphi) := \int_\Omega \varphi \, d\mu - \int_\Omega u L\varphi.$$

Then, by Proposition 2.1, $F(\varphi) \geq 0$ for $\varphi \geq 0$. Thus, by the Riesz representation Theorem, there is $\nu \in \mathcal{M}_+(\Omega)$ so that

$$Lu = \mu - \nu \tag{3.2}$$

in the sense that $\int_\Omega uL\varphi\,dx = \int_\Omega \varphi d(\mu-\nu)$ for any $\varphi \in C_c^\infty(\Omega)$. By Proposition 2.1, (3.2) also holds in the very weak sense. By comparison principle ([30, Theorem 2.9]), we have $Lu \geq 0$ weakly, because otherwise we would obtain $u \leq 0$. In particular,

$$\int_\Omega \varphi d(\mu-\nu) \geq 0 \tag{3.3}$$

for any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. We now prove that (3.3) implies $\mu \geq \nu$. Indeed, let $B \subset\subset \Omega$ be open, with $\overline{B} \subset \Omega$, and take $\varphi_n \to \chi_B$ locally uniformly as $n \to \infty$ with $\varphi_n \in C_c^\infty(\Omega)$, $\varphi_n \geq 0$, and supp $\varphi_n \subset K \subset \Omega$ for some fixed compact K. Then $\int_\Omega \varphi_n\, d(\mu-\nu) \geq 0$ for all n. Taking $n \to \infty$, we get $\mu(B) \geq \nu(B)$. Now, suppose there is a Borel set A such that $\mu(A) < \nu(A)$. For each j, take open sets B_j containing A so that $\mu(B_j) < \mu(A) + \frac{1}{j}$. Hence $\mu(A) < \nu(A) \leq \nu(B_j) \leq \mu(B_j) < \mu(A) + \frac{1}{j}$ which gives a contradiction when $j \to \infty$.

By Theorem 2.1, we have $u \in W_0^{1,p}(\Omega)$ for any $p \in [1, N/(N-1))$ and $u(x) = \int_\Omega G(x,y)\, d(\mu-\nu)(y)$ for a.e. x in Ω. Observe that $(\mu-\nu)(\{x\}) = 0$ for a. e. x by Remark 2.2. Thus, for a.e. x,

$$|\nabla u(x)| \leq \int_\Omega |\nabla_x G(x,y)|\, d(\mu-\nu)(y) = \int_\Omega \frac{|\nabla_x G(x,y)|}{G(x,y)} G(x,y)\, d(\mu-\nu)(y).$$

Fix $p \in [1, N/(N-1))$. Then

$$|\nabla u(x)|^p \leq \left(\int_\Omega \frac{|\nabla_x G(x,y)|}{G(x,y)} G(x,y)\, d(\mu-\nu)(y) \right)^p.$$

By the estimates (2.12) and (2.13), it follows that

$$\frac{|\nabla_x G(x,y)|}{G(x,y)} \leq C \max\left\{ \frac{1}{\delta(x)}, \frac{1}{|x-y|} \right\}$$

for all $x \neq y$ in Ω. Hence, we have

$$\begin{aligned} |\nabla u(x)|^p \leq C \Bigg[& \left(\frac{1}{\delta(x)} \int_{\delta(x) \leq |x-y|} G(x,y)\, d(\mu-\nu)(y) \right)^p \\ & + \left(\int_{\delta(x) > |x-y|} \frac{G(x,y)}{|x-y|}\, d(\mu-\nu)(y) \right)^p \Bigg] = C\left(I_1(x) + I_2(x)\right). \end{aligned}$$

For I_1 we have that $I_1(x) \leq \frac{u^p(x)}{\delta^p(x)}$. Hence

$$\int_\Omega I_1(x)\, dx \leq \int_\Omega \frac{u^p(x)}{\delta^p(x)}\, dx. \tag{3.4}$$

Regarding $I_2(x)$, using Hölder's inequality and estimate (2.12), we get

$$\begin{aligned} I_2(x) &\leq \left(\int_\Omega G(x,y)^{1-1/p} \frac{G(x,y)^{1/p}}{|x-y|}\, d(\mu-\nu)(y) \right)^p \\ &\leq \left(\int_\Omega G(x,y)\, d\mu(y) \right)^{p-1} \left(\int_\Omega \frac{G(x,y)}{|x-y|^p}\, d\mu(y) \right) \quad \text{(recall } \mu-\nu \leq \mu\text{)} \\ &\leq Cu(x)^{p-1} \left(\int_\Omega \frac{1}{|x-y|^{N-2+p}}\, d\mu(y) \right). \end{aligned}$$

Then

$$\int_\Omega I_2(x)\,dx \le C \int_\Omega \left(\int_\Omega \frac{u(x)^{p-1}}{|x-y|^{N-2+p}}\,dx \right) d\mu(y).$$

Next, choose $r_0 \ge 1$ such that

$$N\left(\frac{p-1}{2-p}\right) < r_0 < \frac{N}{N-2}. \tag{3.5}$$

The choice is possible since the quantity $N(p-1)/(2-p)$ is increasing with p and $N\left(\frac{p-1}{2-p}\right) \nearrow \frac{N}{N-2}$ as $p \to \frac{N}{N-1}$. By Hölder's inequality,

$$\begin{aligned}&\int_\Omega I_2(x)\,dx \\ &\le C \int_\Omega \left[\left(\int_\Omega u(x)^{r_0}\,dx \right)^{\frac{p-1}{r_0}} \left(\int_\Omega \frac{dx}{|x-y|^{(N-2+p)\frac{r_0}{r_0-(p-1)}}} \right)^{\frac{r_0-(p-1)}{r_0}} \right] d\mu(y).\end{aligned}$$

Now, by the fact $p < N/(N-1)$ and (3.5), we have

$$1 < N-1 < (N-2+p)\frac{r_0}{r_0-(p-1)} < N,$$

and so we obtain that

$$\int_\Omega I_2(x)\,dx \le C\|u\|_{L^{r_0}}^{p-1}\|\mu\|_{\mathcal{M}(\Omega)}. \tag{3.6}$$

Therefore, by (3.4) and (3.6) we get (3.1). $\square$

Proposition 3.1. *Suppose that $(H1)$, $(H2)$ and $(H4)$ hold, and that $A = [a_{ij}]$ is Lipschitz. Let u be the solution of* (2.6)*. Then u^+ and u^- solve respectively*

$$\begin{cases} Lw \le \mu^+ & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad \textit{and} \quad \begin{cases} Lw \le \mu^- & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Proof. Let u be the solution of (2.6) and take $\{\mu_n\} \subset C^\infty(\mathbb{R}^N)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in the sense

$$\int_\Omega \mu_n \varphi\,dx \to \int_\Omega \varphi\,d\mu \quad \text{for all } \varphi \in C_0^1(\overline{\Omega}). \tag{3.8}$$

Consider u_n the solution of $Lw = \mu_n$ in Ω, with $w = 0$ on $\partial\Omega$. Then, by Theorem 2.1, we have for $1 \le p < N/(N-1)$ that $\|u_n\|_{W_0^{1,p}(\Omega)} \le C\|\mu_n\|_{L^1(\Omega)} \le C$. Hence there is $v \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$.
Since u_n solves $Lw = \mu_n$, we have for all $\varphi \in C_0^1(\overline{\Omega})$ that $L\varphi \in L^\infty(\Omega)$ and

$$\int_\Omega u_n L\varphi\,dx = \int_\Omega \mu_n \varphi\,dx.$$

Then, by the weak convergence of u_n to v in $W_0^{1,p}(\Omega)$ and (3.8) we get

$$\int_\Omega v L\varphi\,dx = \int_\Omega \varphi\,d\mu \quad \text{for all } \varphi \in C_0^1(\overline{\Omega}),\ L\varphi \in L^\infty(\Omega).$$

Therefore, the uniqueness of solutions to problem (2.6) implies $v = u$. Hence $u_n(x) \to u(x)$ a.e. in Ω. Now we can write $[u_n(x)]^+ = \frac{u_n(x)}{2} + \frac{|u_n(x)|}{2}$. Thus

$$[u_n(x)]^+ \to [u(x)]^+ \quad \text{a.e. in } \Omega. \tag{3.9}$$

On the other hand, $\|u_n^+\|_{W_0^{1,p}(\Omega)} \le \|u_n\|_{W_0^{1,p}(\Omega)} \le C$. Then, up to a subsequence $u_n^+ \rightharpoonup w$ in $W_0^{1,p}(\Omega)$ for some $w \in W_0^{1,p}(\Omega)$. Hence, by (3.9) we get $w = u^+$.
Now, applying Kato's inequality (see for instance Theorem 2.4 in [30]) we have

$$\int_\Omega u_n^+ L\varphi \, dx \le \int_\Omega \mu_n \varphi \, dx.$$

Therefore, by the weak convergence of u_n^+ to u^+ in $W_0^{1,p}(\Omega)$ and (3.8) we obtain

$$\int_\Omega u^+ L\varphi \, dx \le \int_\Omega \varphi \, d\mu \le \int_\Omega \varphi \, d\mu^+,$$

for all nonnegative function $\varphi \in C_0^1(\overline{\Omega})$. Hence u^+ solves the first problem in (3.7). Finally, we may use the same argument with $-u$ to get the conclusion for u^-. ☐

Theorem 3.2 (Compactness of the Green operator)**.** *Assume (H1), (H2) and that $A(x) = [a_{ij}(x)]$ is Lipschitz. Then, the Green operator $\mathbb{G} : \mathcal{M}(\Omega) \mapsto W_0^{1,p}(\Omega)$ is compact for any $p \in [1, N/(N-1))$.*

Proof. Take a sequence $\{\mu_i\} \subset \mathcal{M}(\Omega)$ such that $\|\mu_i\|_{\mathcal{M}(\Omega)} \le C$ for all i. Hence, up to a subsequence which we do not relabel, $\mu_i \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ for some $\mu \in \mathcal{M}(\Omega)$.

Let u_i be the weak solution of (2.6) with $\mu = \mu_i$, and let u the weak solution of (2.6) with the limiting μ. Then, $L(u_i - u) = \mu_i - \mu$, and by Proposition 2.1 and Theorem 2.1, $u_i - u \in W_0^{1,p}(\Omega)$ and $\|u_i - u\|_{W^{1,p}(\Omega)} \le C$ for all $p \in [1, N/(N-1))$. In particular, up to a subsequence, $u_i - u \to \tilde{u}$ in $L^p(\Omega)$, for some $\tilde{u} \in W_0^{1,p}(\Omega)$. Moreover, applying Definitions 2.1 and 2.2 we get for all $\varphi \in C_c^\infty(\Omega)$ that

$$\int_\Omega (u_i - u) L\varphi \, dx = \int_\Omega \sum_{j,k=1}^N a_{jk}(x) \frac{\partial (u_i - u)}{\partial x_j} \frac{\partial \varphi}{\partial x_k} \, dx = \int_\Omega \varphi \, d(\mu_i - \mu) \to 0$$

as $i \to \infty$. Hence, $\int_\Omega \tilde{u} L\varphi \, dx = 0$ for all $\varphi \in C_c^\infty(\Omega)$. Consequently, $\tilde{u} \in W_0^{1,p}(\Omega)$ solves $L\tilde{u} = 0$ in Ω. By uniqueness, we deduce $\tilde{u} = 0$. Let v_i^+ (resp. v_i^-) be the positive (resp. negative) part of $u_i - u$. By Proposition 3.1, it follows that

$$\begin{cases} Lv_i^\pm & \le & (\mu_i - \mu)^\pm & \text{in } \Omega \\ v_i^\pm & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

in the weak sense. Now, for each i, by definition we have $v_i^\pm \in W_0^{1,r}(\Omega)$ for all $r < \frac{N}{N-1}$ and

$$\|\nabla v_i^\pm\|_{L^r(\Omega)} \le \|\nabla u_i - \nabla u\|_{L^r(\Omega)} \le C\|\mu_i - \mu\|_{\mathcal{M}(\Omega)} \le C. \tag{3.11}$$

Thus, there are $v^{(\pm)} \in W_0^{1,p}(\Omega)$ such that, up to subsequences,

$$v_i^\pm \rightharpoonup v^{(\pm)} \text{ in } W_0^{1,p}(\Omega), \text{ and } v_i^\pm \to v^{(\pm)} \text{ in } L^\alpha(\Omega) \text{ for all } \alpha < \frac{N}{N-2}. \tag{3.12}$$

However, by the weak convergence of u_i to u in $W^{1,p}(\Omega)$, we get $v^{(\pm)} = 0$. On the other hand, (3.1) yields

$$\int_\Omega |\nabla v_i^\pm|^p \, dx \le C \left[\int_\Omega \frac{(v_i^\pm)^p(x)}{\delta^p(x)} \, dx + \|v_i^\pm\|_{L^{r_0}(\Omega)}^{p-1} \|\mu_i - \mu\|_{\mathcal{M}(\Omega)} \right]. \tag{3.13}$$

Since $\|\mu_i - \mu\|_{\mathcal{M}(\Omega)} \le C$ for all i and $v_i^\pm \to 0$ in $L^{r_0}(\Omega)$, the last term in (3.13) tends to 0 as $i \to \infty$.

For the first term observe the following: if $\{h_i\}$ satisfies $\int_\Omega |h_i|^r \, dx \le C$ for some $r > 1$ and for all i, then $\{h_i\}$ is uniformly integrable. Indeed, let $\varepsilon > 0$ and choose $\gamma > 0$ such that $\gamma^{\frac{1}{r'}} < \frac{\varepsilon}{C^{1/r}}$. Then, by Hölder's inequality, for any measurable set $A \subset \Omega$ with $|A| < \gamma$ we have

$$\int_A |h_i| \, dx = \int_\Omega |h_i| \mathcal{X}_A \, dx \le \|h_i\|_{L^r(\Omega)} |A|^{\frac{1}{r'}} < \varepsilon.$$

We can apply this fact to $h_i = \frac{(v_i^\pm)^p}{\delta^p}$, which is in $L^s(\Omega)$ for some $s > 1$ such that $ps < \frac{N}{N-1}$. Indeed, by Hardy's inequality and (3.11) with $r = ps$, we get

$$\int_\Omega \left(\frac{(v_i^\pm)^p}{\delta^p} \right)^s dx \le C \int_\Omega |\nabla v_i^\pm|^{ps} \, dx \le C, \quad \text{for all } i.$$

Thus, the sequence $\left\{ \frac{(v_i^\pm)^p}{\delta^p} \right\}$ is uniformly integrable in Ω.
Hence, by Vitali's convergence theorem, the first term in the right hand side of (3.13) goes to 0 as $i \to \infty$. Therefore we get the strong convergence of $v_i^\pm$ to 0 in $W_0^{1,p}(\Omega)$. Consequently, $u_i \to u$ strongly in $W_0^{1,p}(\Omega)$. □

4. The general case: existence of solutions for Lipschitz coefficients and any measure data

In this section, we deal with the existence of solutions to (1.1) for Lipschitz coefficients. Without loss of generality, we assume that $\|g\|_{L^1(\Omega)}$ in (2.2) is so that

$$C_p'(\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)}) \ge 1, \tag{4.1}$$

where C_p' is the constant from (2.11).

Theorem 4.1. *Assume that* (H) *holds, with* (4.1), *and that* $A(x) = [a_{ij}(x)]$ *is Lipschitz. Let* $1 \le p < \frac{N}{N-1}$. *Suppose that* c_0 *in* (2.2) *satisfies*

$$c_0 < \frac{1}{S_p C_p' E(h)} \tag{4.2}$$

where C_p' *and* $E(h)$ *come from* (2.11) *and* (2.4), *respectively, and* $S_p \ge 1$ *is a constant from the inclusion* $L^p(\Omega) \subset L^1(\Omega)$. *Then, problem* (1.1) *admits a weak solution* v *such that*

$$v \in \bigcap_{1 \le r < \frac{N}{N-1}} W_0^{1,r}(\Omega).$$

Moreover, the following estimate holds

$$\|v\|_{W_0^{1,p}(\Omega)} \le C\left[\frac{[E_q(b)]^q + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)}}{1 - S_p C_p' c_0 E(h)}\right]^{1/(1-q)}. \tag{4.3}$$

Remark 4.1. As illustrative examples, Theorem 4.1 may be applied to problems where the main operator is $-\Delta$ (i.e., $A(x) = I$), $-\mathrm{div}(a(x)\nabla u)$ for $0 < c \le a(x) \le M$ (i.e., $A(x) = a(x)I$), or $-\mathrm{div}(\mathrm{diag}[a_{ii}]\nabla u)$, for $0 < c \le a_i(x) \le M$, among many others. Regarding the lower order term H, we may take

$$H(x, u, \nabla u) = c_0 h(x) u + b(x)|\nabla u|^q + \mu,$$

where

- h and b are any L^∞-functions, $\|h\|_{L^\infty} \neq 0$. Then we may take $E(h) = C\|h\|_{L^\infty}$, for some $C > 0$ and c_0 satisfying (4.2).
- Similarly, we may have $b \in L^{1/(1-q)}(\Omega)$ (for instance, $b(x) = |x|^\alpha$, for any $\alpha \ge -N(1-q)$, or $b(x) = |x|^\alpha \log|x|$, $\alpha > -N(1-q)$) and h as in the above item.
- Finally, our theorem applies to the standard and largely studied problems: $-\Delta u \pm |\nabla u|^q = \mu$ in Ω, with $u = 0$ on $\partial\Omega$.

Proof of Theorem 4.1. Let $1 \le p < \frac{N}{N-1}$.
First consider sequences $\{H_n\} \subset C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ and $\{\mu_n\} \subset C_0^1(\mathbb{R}^N)$ such that

$$H_n(x, r, \xi) := \begin{cases} -n, & \text{if } H(x, r, \xi) \le -n, \\ H(x, r, \xi), & \text{if } -n \le H(x, r, \xi) \le n, \\ n, & \text{if } H(x, r, \xi) \ge n, \end{cases}$$

and

$$\lim_{n\to\infty} \int_{\overline{\Omega}} \xi \mu_n^{\pm}\, dx = \int_{\overline{\Omega}} \xi\, d\mu^{\pm} \quad \text{for all } \xi \in C_0(\overline{\Omega}). \tag{4.4}$$

Now, $\mu^{\pm}(\overline{\Omega}) \ge \limsup_{n\to\infty} \mu_n^{\pm}(\overline{\Omega})$. Then, for any $\alpha > 0$, there is n_α so that $n \ge n_\alpha$ implies

$$\|\mu_n\|_{\mathcal{M}(\overline{\Omega})} = \mu_n^+(\overline{\Omega}) + \mu_n^-(\overline{\Omega}) < \mu^+(\overline{\Omega}) + \mu^-(\overline{\Omega}) + \alpha = \|\mu\|_{\mathcal{M}(\overline{\Omega})} + \alpha. \tag{4.5}$$

From now on, we take $n \ge n_\alpha$.

In what follows, we will apply a fixed point argument to find a solution for each of the approximating problems

$$\begin{cases} Lw = H_n(x, w, \nabla w) + \mu_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

For that purpose, define the closed and convex set

$$\mathcal{O}_{\overline{\lambda}} := \{v \in W_0^{1,p}(\Omega) \,:\, \|\nabla v\|_{L^p(\Omega)} \le \overline{\lambda}\}$$

for some $\overline{\lambda} > 0$ to be determined. For $v \in \mathcal{O}_{\overline{\lambda}}$, let $v_n = T_n(v)$ be the weak solution to (4.6) with right-hand side $H_n(x, v, \nabla v) + \mu_n$. The existence of such v_n is guaranteed by Theorem 2.1 and Proposition 2.1 since $H_n(x, v, \nabla v) + \mu_n \in L^1(\Omega)$. In addition, by Theorem 2.1 again, we have the representation $v_n = T_n(v) = \mathbb{G}[H_n(\cdot, v, \nabla v) + \mu_n]$, and

$$\|\nabla v_n\|_{L^p(\Omega)} \leq C_p' \int_\Omega (|H_n(x, v, \nabla v)| + |\mu_n|)\, dx. \tag{4.7}$$

We will prove, for an appropriate $\bar{\lambda}$, that T_n maps $\mathcal{O}_{\overline{\lambda}}$ into itself. Take $v \in \mathcal{O}_{\overline{\lambda}}$. Then, (4.7), the fact that $|H_n| \leq |H|$, (2.2), and (4.5) yield

$$\|\nabla T_n(v)\|_{L^p(\Omega)} \leq C_p' \left[c_0 \int_\Omega h(x)|v|\, dx + \int_\Omega b(x)|\nabla v|^q\, dx + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha \right].$$

Thus, appealing to (2.3) and (2.4) we have

$$\begin{aligned}
&\|\nabla T_n(v)\|_{L^p(\Omega)} \\
&\leq C_p' \left[c_0 E(h) \|\nabla v\|_{L^1(\Omega)} + [E_q(b)]^q \|\nabla v\|^q_{L^1(\Omega)} + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha \right] \\
&\leq C_p' \left[c_0 E(h) S_p \|\nabla v\|_{L^p(\Omega)} + [E_q(b)]^q S_p^q \|\nabla v\|^q_{L^p(\Omega)} + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha \right] \\
&\leq S_p C_p' \left[c_0 E(h) \|\nabla v\|_{L^p(\Omega)} + [E_q(b)]^q \|\nabla v\|^q_{L^p(\Omega)} + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha \right].
\end{aligned}$$

Then, since $v \in \mathcal{O}_{\overline{\lambda}}$ we get

$$\|\nabla T_n(v)\|_{L^p(\Omega)} \leq S_p C_p' \left[c_0 E(h) \overline{\lambda} + [E_q(b)]^q \overline{\lambda}^q + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha \right]. \tag{4.8}$$

Now consider

$$\mathbb{F}(\lambda) = S_p C_p' \left[[E_q(b)]^q \lambda^{q-1} + \frac{\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha}{\lambda} + c_0 E(h) \right] - 1,$$

and recall by the assumption on c_0 that $S_p C_p' c_0 E(h) < 1$. Then, for

$$\lambda' = S_p C_p' (\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)}) \geq 1$$

(by (4.1)), we get $\mathbb{F}(\lambda') > 0$. On the other hand, taking

$$\lambda'' = \left[\frac{S_p C_p' ([E_q(b)]^q + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)})}{1 - S_p C_p' c_0 E(h)} \right]^{1/(1-q)}$$

and recalling that $S_p \geq 1 \geq \frac{1}{C_p'(\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)})}$, it follows that $\lambda'' > 1$, $\lambda' < \lambda''$ and $\mathbb{F}(\lambda'') < 0$ for all $\alpha < (\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)})[(\lambda'')^q - 1]$. Therefore there is $0 < \overline{\lambda} \leq \lambda''$ such that $\mathbb{F}(\overline{\lambda}) = 0$. Then, for such $\overline{\lambda}$ (4.8) implies

$$\|\nabla T_n(v)\|_{L^p(\Omega)} \leq \overline{\lambda} \leq \left[\frac{S_p C_p' ([E_q(b)]^q + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)})}{1 - S_p C_p' c_0 E(h)} \right]^{1/(1-q)} \tag{4.9}$$

and we get $T_n : \mathcal{O}_{\overline{\lambda}} \to \mathcal{O}_{\overline{\lambda}}$. Observe that $\bar{\lambda}$ does not depend on n (but on α).

Next we prove that T_n is continuous. So, take $u_k \to u$ in $W_0^{1,p}(\Omega)$. Then $H_n(x, u_k, \nabla u_k) \to H_n(x, u, \nabla u)$ in $L^1(\Omega)$ since $\{H_n\}$ is bounded in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and $(u_k, |\nabla u_k|) \to (u, |\nabla u|)$ a.e. in Ω. Thus, by continuity of $\mathbb{G}$ from L^1 into $W^{1,p}$, we obtain that T_n is a continuous operator.

On the other hand, if $\{v_k\}$ is a (bounded) sequence in $\mathcal{O}_{\overline{\lambda}}$, $\{H_n(x, v_k, \nabla v_k)\}$ is bounded in $L^1(\Omega)$ for each n. Hence, by Theorem 3.2 we can extract a convergent subsequence of $\{T_n(v_k)\}$ in $W_0^{1,p}(\Omega)$. Thus T_n is a compact operator.

Therefore, since $\mathcal{O}_{\overline{\lambda}}$ is a convex and closed subset of $W_0^{1,p}(\Omega)$, by Schauder's fixed point Theorem there is $v_n \in \mathcal{O}_{\overline{\lambda}}$ such that $T_n(v_n) = v_n$ and v_n solves (4.6), that is,

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial v_n}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega H_n(x, v_n, \nabla v_n)\varphi\, dx + \int_\Omega \mu_n \varphi\, dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{4.10}$$

To finish the proof, we will prove that up to a subsequence $\{v_n\}$ converges to some v and that v solves problem (1.1) by taking the limit in (4.10).

First, observe that $\{H_n(x, v_n, \nabla v_n)\}$ is bounded in $L^1(\Omega)$ since by (2.4) and (2.3):

$$\int_\Omega |H_n(x, v_n, \nabla v_n)|\, dx \le C(\bar\lambda + \bar\lambda^q) + \|g\|_{L^1(\Omega)}. \tag{4.11}$$

Then, we use Theorem 3.2 to extract a convergent subsequence $\{v_{n_k}\}$ from $v_n = \mathbb{G}[H_n(\cdot, v_n, \nabla v_n) + \mu_n]$ in $W_0^{1,p}(\Omega)$. In particular, there exists $v \in W_0^{1,p}(\Omega)$ such that $H_{n_k}(x, v_{n_k}, \nabla v_{n_k}) \to H(x, v, \nabla v)$, a.e. in Ω. Now we will prove that $\{H_{n_k}(x, v_{n_k}, \nabla v_{n_k})\}$ is uniformly integrable in Ω. Let $E \subset \Omega$ be a Borel subset. Then,

$$\begin{aligned}\int_E |H_{n_k}(x, v_{n_k}, \nabla v_{n_k})|\, dx &\le C\left[\|h(v_{n_k} - v)\|_{L^1(E)} + \|hv\|_{L^1(E)}\right] \\ &\quad + C\left[\|b|\nabla v_{n_k} - \nabla v|^q\|_{L^1(E)} + \|b|\nabla v|^q\|_{L^1(E)}\right] + \|g\|_{L^1(E)}.\end{aligned} \tag{4.12}$$

Observe that by (2.4) and (2.3), $hv_{n_k} \to hv$ and $b|\nabla v_{n_k}|^q \to b|\nabla v|^q$ in $L^1(\Omega)$, and hence in any measurable $E \subset \Omega$. Let $\eta > 0$. Then there exist $k_0, \gamma_0 > 0$ so that $k \ge k_0$ implies

$$\|hv_{n_k} - hv\|_{L^1(\Omega)} + \|b|\nabla v_{n_k} - \nabla v|^q\|_{L^1(\Omega)} < \frac{\eta}{4C} \tag{4.13}$$

and for any $|E| < \gamma_0$,

$$\max\{\, C\|hv\|_{L^1(E)},\, C\|b|\nabla v|^q\|_{L^1(E)},\, \|g\|_{L^1(E)}\} < \frac{\eta}{4}. \tag{4.14}$$

On the other hand, for each $k \in \{1, \dots, k_0 - 1\}$ there is $\gamma_k > 0$ such that

$$\|h(v_{n_k} - v)\|_{L^1(E)} + \|b|\nabla v_{n_k} - \nabla v|^q\|_{L^1(E)} < \frac{\eta}{4C} \tag{4.15}$$

for all E with $|E| < \gamma_k$. Choose $\gamma := \min\{\gamma_0, \gamma_1, \dots, \gamma_{k_0-1}\}$. Then for any $|E| < \gamma$ we have by (4.12)-(4.15) that $\int_E |H_{n_k}(x, v_{n_k}, \nabla v_{n_k})|\, dx < \eta$, for all k. Therefore we can apply Vitali's convergence theorem and get $H_{n_k}(x, v_{n_k}, \nabla v_{n_k}) \to H(x, v, \nabla v)$ in $L^1(\Omega)$.

Hence, letting $n = n_k$ in (4.10) and taking $k \to \infty$, we obtain that v is a weak solution to problem (1.1). Also, observe that $v \in W_0^{1,r}(\Omega)$ for all $r \in [1, N/(N-1))$. Indeed, $v = \mathbb{G}(H(\cdot, v, \nabla v) + \mu)$ and $H(x, v, \nabla v) \in L^1(\Omega)$, hence by Theorem 2.1 v has the desired regularity.

Finally, by (4.9)

$$\|v_{n_k}\|_{W_0^{1,p}(\Omega)} \leq C\overline{\lambda} \leq C\left[\frac{S_pC_p'([E_q(b)]^q + \|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)})}{1 - S_pC_p'c_0E(h)}\right]^{1/(1-q)}.$$

Then, letting $k \to \infty$ yield (4.3). □

Remark 4.2. We observe that the existence of solutions stated in Theorem 4.1 remains valid if H has sub-linear growth in u of the form:

$$|H(x,r,\xi)| \leq c_0|u|^l + b(x)|\xi|^q + |g(x)|,\ \ l,q \in (0,1),$$

and with no need of imposing a size condition in c_0. Indeed, the function $\mathbb{F}$ adopts the form

$$\mathbb{F}(\lambda) = S_pC_p'\left[[E_q(b)]^q\lambda^{q-1} + \frac{\|g\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega)} + \alpha}{\lambda} + c_0E(h)\lambda^{l-1}\right] - 1.$$

Hence, the conclusion $\mathbb{F}(\bar{\lambda}) = 0$ for some $\bar{\lambda} > 0$, follows by taking appropriate values of λ.

Remark 4.3. Theorem 4.1 may be compared to [5, Theorem 3.1] where it is proved the existence of renormalized solutions for the p-Laplacian operator, and to the results in the survey [29] and the references therein, where a sign condition in the lower order term H is imposed. Here, the approach is different, appealing to the compactness of the Green operator and fixed-point argument. Moreover, we provide estimates of the solutions in terms of the data, which are, as we shall see below, useful to get stability results. See also the Introduction for more related comments and references.

As a consequence of Theorem 4.1, we may establish the following stability results. For simplicity, we state it for the model problem.

Corollary 4.1. *Let*

$$\begin{cases} -\Delta w \pm \varepsilon_n|\nabla w|^q &= \varepsilon_n w + \mu_{\varepsilon_n} \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

for $\varepsilon_n < 1/S_pC_p'E(1)$ *and* $\mu_{\varepsilon_n} \overset{*}{\rightharpoonup} \mu$ *for some* $\mu \in \mathcal{M}(\Omega)$. *Let* u_{ε_n} *be the solution obtained in Theorem 4.1 of* (4.16). *Then, there is a function* $u \in W_0^{1,p}(\Omega)$ *such that* $u_{\varepsilon_n} \to u$ *in* $W_0^{1,p}(\Omega)$ *and* u *is the only weak solution of*

$$\begin{cases} -\Delta w &= \mu \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.17}$$

Proof. Let ε_{n_k} be a subsequence of $\{\varepsilon_n\}$. Now, taking $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \nabla u_{\varepsilon_{n_k}} \cdot \nabla\varphi\,dx \pm \varepsilon_{n_k}\int_\Omega |\nabla u_{\varepsilon_{n_k}}|^q\varphi\,dx = \varepsilon_{n_k}\int_\Omega u_{\varepsilon_{n_k}}\varphi\,dx + \int_\Omega \varphi\,d\mu_{\varepsilon_{n_k}} \tag{4.18}$$

and the following representation for $u_{\varepsilon_{n_k}}$

$$u_{\varepsilon_{n_k}} = \mathbb{G}\left[\pm\varepsilon_{n_k}|\nabla u_{\varepsilon_{n_k}}|^q + \varepsilon_{n_k}u_{\varepsilon_{n_k}} + \mu_{\varepsilon_{n_k}}\right]. \tag{4.19}$$

Now observe that $[E_q(\varepsilon_{n_k})]^q = [E_q(1)]^q \varepsilon_{n_k} \le C \quad$ for all k. Therefore, by (4.3)

$$\begin{aligned}\|\nabla u_{\varepsilon_{n_k}}\|_{L^p(\Omega)} &\le C \left[\frac{[E_q(\varepsilon_{n_k})]^q + \|\mu_{\varepsilon_{n_k}}\|_{\mathcal{M}(\Omega)}}{1 - S_p C_p' \varepsilon_{n_k} E(1)}\right]^{1/(1-q)} \\ &\le C \left[\frac{1}{1 - S_p C_p' \varepsilon_{n_k} E(1)}\right]^{1/(1-q)}.\end{aligned}$$

Thus, by Poincaré's inequality and the fact that $\|\mu_{\varepsilon_{n_k}}\|_{\mathcal{M}(\Omega)} \le C$, we get that $\left\{\pm\varepsilon_{n_k}|\nabla u_{\varepsilon_{n_k}}|^q + \varepsilon_{n_k} u_{\varepsilon_{n_k}} + \mu_{\varepsilon_{n_k}}\right\}$ is bounded in $\mathcal{M}(\Omega)$. Due to (4.19), Theorem 3.2 allows us to get a further subsequence, which we do not relabel, $u_{\varepsilon_{n_k}}$ and $u \in W_0^{1,p}(\Omega)$ such that $u_{\varepsilon_{n_k}} \to u$ in $W_0^{1,p}(\Omega)$. Reasoning as in the proof of Theorem 4.1 we may pass the limit in (4.18) and get u solves (4.17). Hence, every subsequence of u_{ε_n} has a further subsequence converging, by uniqueness, to the same limit u. Therefore, the whole sequence u_{ε_n} converges to u. □

Remark 4.4. Other stability results may also be obtained: let $\{\varepsilon_n\}$ be as in Corollary 4.1 and $\{u_{\varepsilon_n}\}$ the sequence of functions solving $-\Delta w \pm |\nabla w|^q = \varepsilon_n w + \mu_{\varepsilon_n}$ in Ω, $w = 0$ on $\partial\Omega$, with $\|\mu_{\varepsilon_n}\|_{\mathcal{M}(\Omega)} \le C$. Then, we can reproduce the proof of Corollary 4.1 in a simpler way applying (4.3) with $E_q(1)$ to get a subsequence $\{\varepsilon_{n_k}\}$ and $u \in W_0^{1,p}(\Omega)$ such that $u_{\varepsilon_{n_k}} \to u$ in $W_0^{1,p}(\Omega)$ and u satisfies $-\Delta u \pm |\nabla w|^q = \mu$ in Ω, with $u = 0$ on $\partial\Omega$, for μ such that $\mu_{\varepsilon_n} \overset{*}{\rightharpoonup} \mu$ up to a subsequence in $\mathcal{M}(\Omega)$. In general, the limit u depends on the subsequence $\{u_{n_k}\}$ since by [3], uniqueness does not hold for the considered problems.

5. Existence for data in $W^{-1,2}(\Omega)$

The following necessary and sufficient conditions for solutions in $W_0^{1,2}(\Omega)$ resembles the linear case ([25, Theorem 5.2]). Note that it is not necessary to impose the Lipschitz condition on A.

Theorem 5.1. *Assume* (H)*, where:*

(i) c_0 *satisfies* $c_0 < \nu . c_\Omega^{-1}$ *with* $c_\Omega > 0$ *so that* $\|v\|_{W^{1,2}(\Omega)} \le c_\Omega \|\nabla v\|_{L^2(\Omega)}$ *for all* $v \in W_0^{1,2}(\Omega)$*;*
(ii) $h, b \equiv 1$ *and* $g \in L^2(\Omega)$.

Then $\mu \in \mathcal{M}(\Omega) \cap W^{-1,2}(\Omega)$ *if and only if problem* (1.1) *has a weak solution* $u \in W_0^{1,2}(\Omega)$.

Proof. Assume first that $\mu \in \mathcal{M}(\Omega) \cap W^{-1,2}(\Omega)$. We shall apply Theorem 2 in [24]. By Rellich-Kondrachov's Theorem, assumption (1.1) in [24, Theorem 2] is satisfied. Next, we define the operator $A : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$, by

$$\langle A(u), v\rangle := \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx - \int_\Omega H(x, u, \nabla u) v \, dx,$$

for $u, v \in W_0^{1,2}(\Omega)$. To see that A is well-defined and that $A(u) \in W^{-1,2}(\Omega)$ for all $u \in W_0^{1,2}(\Omega)$, we first define $r \in (1, N)$ by $r = \frac{2N}{N+2-qN}$. Observe that

$$\frac{1}{2^*} + \frac{1}{2/q} + \frac{1}{r} = 1.$$

Moreover, $g \in [L^{2^*}(\Omega)]'$ since $L^2(\Omega) \subset [L^{2^*}(\Omega)]'$. Hence, Hölder's inequality and the Sobolev's imbedding theorem imply

$$\begin{aligned} |\langle A(u), v\rangle| &\leq C(\|u\|_{W_0^{1,2}(\Omega)}\|v\|_{W_0^{1,2}(\Omega)} \\ &\quad + |\Omega|^{1/r}\||\nabla u|^q\|_{L^{2/q}(\Omega)}\|v\|_{L^{2^*}(\Omega)} + \|g\|_{(L^{2^*}(\Omega))'}\|v\|_{L^{2^*}(\Omega)}) \\ &\leq C\|v\|_{W_0^{1,2}(\Omega)}. \end{aligned} \tag{5.1}$$

Therefore A is well-defined and $A(u) \in W^{-1,2}(\Omega)$ for all u. To mimic the decomposition of the main operator in [24], we write

$$\langle A(u), v\rangle = \sum_{j=1}^{N} \int_\Omega A_j(x, \nabla u)\frac{\partial v}{\partial x_j}\,dx + \int_\Omega A_0(x, u, \nabla u)v\,dx$$

where for $\xi \in \mathbb{R}^N$ and $x \in \Omega$, $A_j(x, \xi) = \sum_{i=1}^N a_{ij}(x)\xi_i$, $(j = 1, \dots, N)$, and $A_0(x, r, \xi) = -H(x, r, \xi)$. Now observe that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$,

$$|A_j(x, \xi)| \leq \sum_{i=1}^{N} |a_{ij}(x)||\xi_i| \leq \max_{1\leq i,j\leq N} \|a_{ij}\|_{L^\infty(\Omega)}|\xi|$$

and $|A_0(x, r, \xi)| \leq C(|r| + |\xi| + |g(x)| + 1)$, where $g \in L^2(\Omega)$. Hence assumptions (1.3), $(2.2)_1$ and $(2.2)_2$ from [24, Theorem 2] hold true. We finally check that A is coercive, that is, $\dfrac{\langle A(u), u\rangle}{||u||_{W_0^{1,2}(\Omega)}} \to \infty$ as $||u||_{W_0^{1,2}(\Omega)} \to \infty$. As in (5.1), we obtain

$$\left|\int_\Omega H(x, u, \nabla u)u\,dx\right| \leq c_0\|u\|^2_{W^{1,2}(\Omega)} + C\left(\|\nabla u\|^{q+1}_{L^2(\Omega)} + \|g\|_{(L^{2^*}(\Omega))'}\|\nabla u\|_{L^2(\Omega)}\right). \tag{5.2}$$

Hence, (2.1) and (5.2) yield

$$\frac{\langle A(u), u\rangle}{||u||_{W^{1,2}(\Omega)}} \geq (\nu.c_\Omega^{-1} - c_0)\|u\|_{W^{1,2}(\Omega)} - C\left(\|\nabla u\|^q_{L^2(\Omega)} - \|g\|_{(L^{2^*}(\Omega))'}\right) \to \infty,$$

as $||u||_{W_0^{1,2}(\Omega)} \to \infty$. In this way, by [24, Theorem 2], there is $u \in W_0^{1,2}(\Omega)$ so that

$$\langle A(u), v\rangle = \langle \mu, v\rangle\,, \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{5.3}$$

Thus, (5.3) holds for any $\varphi \in C_c^\infty(\Omega)$. Hence, u is a weak solution of (1.1).

Next, suppose that problem (1.1) has a weak solution u in $W_0^{1,2}(\Omega)$. By a density argument we have for any $\varphi \in W_0^{1,2}(\Omega)$

$$\left|\int_\Omega \varphi\,d\mu\right| = |\langle A(u), \varphi\rangle| \leq C\|\varphi\|_{W_0^{1,2}(\Omega)}.$$

Hence, $\mu \in \mathcal{M}(\Omega) \cap W^{-1,2}(\Omega)$ □

A measure is not the most general datum for (1.1). Indeed, Theorem 5.1 holds true for data in $W^{-1,2}(\Omega)$, and for solutions $u \in W_0^{1,2}(\Omega)$ of (1.1) in the sense of (5.3). Hence, we have the following result

Theorem 5.2. *Assume* (H) *and* (i) *and* (ii) *from Theorem 5.1. If* $F \in W^{-1,2}(\Omega)$, *then problem* (1.1) *has a solution* $u \in W_0^{1,2}(\Omega)$ *in the sense*

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx = \int_\Omega H(x,u,\nabla u) v\, dx + \langle F, v\rangle \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Remark 5.1. For diagonal $A = [a_{ij}]$ but depending also on u and ∇u, and $W^{-1,2}$ data, an existence result for non coercive problems having first order terms with sub-linear growth has been obtained for instance in [14].

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