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# ON THE STABILITY, INTEGRABILITY AND BOUNDEDNESS ANALYSIS OF SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME-DELAY

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**Abstract.** In the paper "AIMS Math. 5 (2020), no. 6, 6448-6456", Tian et al. [15, Theorem 1] considered a linear system of integro-time delay differential equations (IDDEs) with constant time retardation. In [15], firstly, a generalized double integral inequality was obtained and then less conservative asymptotically stability criteria were proposed by using that double integral inequality and choosing a new Lyapunov-Krasovskii functional (LKF). To the best of the information, we would like to note that the asymptotically stability criteria of Tian et al. [15, Theorem 1] consist of very interesting but strong conditions. However, in this paper, we define a more suitable LKF, then we obtain the result of Tian et al. [15, Theorem 1] for uniformly asymptotically stability under very weaker conditions using the LKF and also investigate the integrability of the norm and boundedness of solutions. To show the effectiveness of our results, two numerical examples are proposed for the uniformly asymptotically stability as well as integrability and boundedness of solutions. By this work, we do contributions to the work of Tian et al. [15, Theorem 1] under very weaker conditions and those in the previous relevant literature, and we obtain two more new results on the integrability of the norm of solutions and boundedness of solutions. The results of this paper are new and they may be useful for researchers working on the topics of this paper.
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## 1. Introduction

The subject of qualitative theory of integral equations and integro-differential equations (IDEs) both with and without time retardation(s) is one of the most useful mathematical tools in both pure and applied mathematics. Essentially, this subject has enormous applications in many physical problems, engineering, mechanics, and medicine and so on. In particular, some problems of standard closed electric RLC circuits, heat transfer, fluid mechanics, radiation, population dynamics, etc. are described by IDEs and IDDEs ([2], [4], [5], [6], [9],[10],[12]). In particular, we would like to mention that the subject of qualitative theory of IDDEs and IDEs both with and without time retardation(s) is a hot and attractive topic, and it deserves investigation. It is worth mentioning that when we look for the relevant literature, various qualitative properties of IDEs and IDDEs, for example,

$$\dot{y}(t) = Ay(t) + \int\limits_{0}^{t} C(t,s)y(s)ds,$$

$$\dot{y}(t) = Ay(t) + \int\limits_{t-h}^{t} C(t,s)y(s)ds$$

and their modified linear and non-linear forms have been investigated since 1970s. Many interesting results have been obtained on the subject (for some recent works, see [1], [3], [7], [8], [11], [13], [16-25] and the biography in these papers). It is also well known that several approaches can be encountered in the literature on this subject. The second Lyapunov method, fixed point methods in various function spaces, the LKF method, semigroup theory, the Lyapunov-Razumikhin method ([14]), construction of the resolvent kernel and its applications, transform theory methods, etc., are used to obtain suitable qualitative conditions for time-delay systems of IDEs (see, [26-29] and the biography of this paper for some of them). However, to the best of information, the LKM method is the most effective method in the relevant literature to investigate the subject for scalar and system of IDDEs. The effectiveness of the LKF method depends upon on the construction or the definition of suitable LKF(s), which lead(s) meaningful and optimal result(s) on the subject. The aim in this paper is to provide this fact for the problems of this paper regrading certain systems of IDDEs.

From this point of view, in 2020, Tian et al. [15] studied the asymptotically stability of the linear system of IDDEs:

$$\dot{y}(t) = Ay(t) + By(t-h) + C\int\limits_{t-h}^{t} y(s)ds, \tag{1.1}$$

$$y(t) = \phi(t), t \in [-h, 0],$$

where $y(t) \in \mathbb{R}^n$ is the sate vector, $t \in [0, \infty)$, $A,\ B,\ C \in \mathbb{R}^{n\times n}$ are constant matrices, $h$ is the constant time retardation and $\phi(t)$ is a continuous initial function.

In Tian et al. [15, Theorem 1], it is presented a generalized double integral inequality. Later, new stability criteria, [15, Theorem 1], are proposed by choosing a new LKF and using the generalized double integral inequality. Both the generalized integral inequality and the new LKF includes multiple integrals, which could yield less conservative results. In [15], two examples are also provided to illustrate the effectiveness of the proposed criteria. We should mention that the asymptotically stability result and the related Lemma 1, Lemma 2 and Lemma 3 (Tian et al. [15]) are new and very interesting, and they have good scientific novelty. However, to the best of knowledge, [15, Theorem 1] has very strong conditions.

Motivated from the results of Tian et al. [15], we consider the following nonlinear system of IDDEs with the constant time retardation:

$$\dot{y}(t) = A(t)y(t) + BF(y(t-h)) + C\int_{t-h}^{t} K(t,s)G(y(s))ds + Q(t,y(t),y(t-h)), \quad (1.2)$$

$$y(t) = \phi(t), t \in [-h,0],$$

where $y(t) \in \mathbb{R}^n$ is the sate vector, $t \in [0,\infty) = \mathbb{R}^+$, $h$ is a positive constant, i.e., constant time retardation, $B,\ C \in \mathbb{R}^{n\times n}$ are $n \times n$ constant matrices, $A(t) = (a_{ij}(t))$, $B = (b_{ij})$, $C = (c_{ij})$,$i,\ j = 1,2,...,n$, $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$ is an $n \times n$ matrix, $F,\ G \in C(\mathbb{R}^n,\mathbb{R}^n)$,$Q \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $F(0) = 0$ and $G(0) = 0$.

## 2. Basic results

In this section, we need to give a theorem, which is taken from the book of Burton [4, Theorem 4. 2.9]) and it will be used in the proofs of qualitative results of this paper. In fact, that theorem of Burton [4, Theorem 4. 2.9] has a vital role in the proofs. Thus, we now consider the non-autonomous system of delay differential equations (DDEs):

$$\frac{dy}{dt} = Z(t, y_t), \quad (2.1)$$

where $Z : \mathbb{R} \times C \to \mathbb{R}^n$, $\mathbb{R} = (-\infty,\infty)$, is a continuous mapping with $Z(t,0) = 0$, and $Z$ takes bounded sets into bounded sets. For some $\tau > 0$, $C = C([-\tau,0],\ \mathbb{R}^n)$ denotes the space of continuous functions $\phi : [-\tau,0] \to \mathbb{R}^n,\ \tau > 0$. For any $a \geq 0$, some $t_0 \geq 0$, and $y \in C([t_0-\tau,t_0+a],\ \mathbb{R}^n)$, it is assumed that $y_t = y(t+\theta)$ for $\theta \in [-\tau,0]$ and $t \geq t_0$.

Let $y \in \mathbb{R}^n$ and norm $\|.\|$ is defined by

$$\|y\| = \sum_{i=1}^{n} |y_i|.$$

Next, let $M \in \mathbb{R}^{n\times n}$. Then, the norm of this matrix is defined by

$$\|M\| = \max_{1\leq j\leq n} \left( \sum_{i=1}^{n} |m_{ij}| \right).$$

We should note that, here in some places, $y$ will be written instead of $y(t)$, without mentioning.

For any $\phi \in C$, let

$$\|\phi\|_C = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-\tau,0]}$$

and

$$C_H = \{\phi : \phi \in C \text{ and } \|\phi\|_C \leq H < \infty\}.$$

In this paper, it is supposed that the function $Z$ also satisfies the condition of the uniqueness of solutions of (2.1). We should also state that the systems of linear IDDEs (1.1) and nonlinear IDDEs (1.2) are particular cases of the system of DDEs (2.1).

Let $y(t) = y(t, t_0, \phi)$ be a solution of (2.1) on $[t_0-\tau, t_0]$, $t_0 \geq 0$, such that $y(t) = \phi(t)$ on $[t_0 - \tau, t_0]$, where $\phi : [t_0 - \tau, t_0] \to \mathbb{R}^n$ is a continuous initial function. Let $V_1 : \mathbb{R}^+ \times C_H \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, be a continuous functional in $t$ and $\phi$ with $V_1(t, 0) = 0$. Further, let $\frac{d}{dt}V_1(t, y)$ denote the derivative of $V_1(t, y)$ on the right through any solution of the system of DDEs (2.1).

For the proofs of our theorems we need the following theorem.

**Theorem 1.** (Burton [4, Theorem 4. 2.9]). *Assume that*

- **(A1)** *The functional $V_1(t, y)$ is locally Lipschitz in $y$, i.e., for every compact $S \subset \mathbb{R}^n$ and $\gamma > t_0$, there exists a $K_{\gamma s} \in \mathbb{R}$ with $K_{\gamma s} > 0$ such that*

$$|V_1(t, y_t) - V_1(t, x_t)| \leq K_{\gamma s}\|y - x\|_{[t_0-\tau,t]}$$

*for all $t \in [t_0, \gamma]$ and $x, y \in C([t_0 - \tau, t_0], S)$.*

- **(A2)** *Let $Z : \mathbb{R}^+ \times C_H \to \mathbb{R}^+$ be a continuous functional that is one-sided locally Lipschitz in $t$, i.e.,*

$$Z(t_2, \phi) - Z(t_1, \phi) \leq K(t_2 - t_1), 0 < t_1 < t_2 < \infty, K > 0, K \in \mathbb{R}, \phi \in C_H.$$

- **(A3)** *There are four strictly increasing functions $\omega$, $\omega_1$, $\omega_2$, $\omega_3 : \mathbb{R}^+ \to \mathbb{R}^+$ with value $0$ at $0$ such that*

$$\omega(\|\phi(0)\|) + Z(t, \phi) \leq V_1(t, \phi) \leq \omega_1(\|\phi(0)\|) + Z(t, \phi),$$

$$Z(t, \phi) \leq \omega_2(\|\phi\|_C),$$

*and*

$$\frac{d}{dt}V_1(t, y) \leq -\omega_3(\|y(t)\|)$$

*whenever $t \in \mathbb{R}^+$ and $y \in C_H$. Then, the trivial solution $y(t) = 0$ of (2.1) is uniformly asymptotically stable.*

We now state the main result of Tian et al. [15, Theorem 1].

**Theorem 2** (Tian et al. [15, Theorem 1].) *For given scalar $h > 0$, system (1.1) is asymptotically stable if there exist matrices $P \in S_+^{5n}$, $R_1$, $R_2$,$R_3 \in S_+^n$,and any matrices $M_1$, $M_2$,$M_3$,$M_4 \in R^{6n\times n}$ such that*

$$\begin{aligned}\Psi =& Sym\left[\Pi_1^T P \Pi_2\right] + \delta_1^T R_1 \delta_1 - \delta_2^T R_1 \delta_2 + h^2 \delta_0^T R_2 \delta_0 + \frac{h^2}{2}\delta_0^T R_3 \delta_0 \\ &- \Pi_3^T R_3 \Pi_3 - 3\Pi_4^T R_2 \Pi_4 - 5\Pi_5^T R_2 \Pi_5 - 7\Pi_6^T R_2 \Pi_6 - 9\Pi_7^T R_2 \Pi_7 \\ &+ \frac{h^2}{2}\left(M_1 R_3^{-1} M_1^T + \frac{1}{2}M_2 R_3^{-1} M_2^T + \frac{1}{3}M_3 R_3^{-1} M_3^T + \frac{1}{4}M_4 R_4^{-1} M_4^T\right) \\ &+ hSym\left(M_1\Pi_8 + M_2\Pi_9 + M_3\Pi_{10} + M_4\Pi_{11}\right) < 0,\end{aligned}$$

*where*

$$
\begin{aligned}
\Pi_1 &= \begin{bmatrix} \delta_1{}^T & \delta_3^T & \delta_4^T & \delta_5^T & \delta_6^T \end{bmatrix}^T,\\
\Pi_2 &= \begin{bmatrix} \delta_0{}^T & \delta_1^T - \delta_2^T & h\delta_1^T - \delta_3^T & \frac{h^2}{2}\delta_1^T - \delta_4^T & \frac{h^2}{6}\delta_1^T - \delta_5^T \end{bmatrix},\\
\Pi_3 &= \delta_1 - \delta_2, \Pi_4 = \delta_1 + \delta_2 - \frac{2}{h}\delta_3,\\
\Pi_5 &= \delta_1 - \delta_2 + \frac{6}{h}\delta_3 - \frac{12}{h^2}\delta_4,\\
\Pi_6 &= \delta_1 + \delta_2 - \frac{12}{h}\delta_3 + \frac{60}{h^2}\delta_4 - \frac{120}{h_3}\delta_5,\\
\Pi_7 &= \delta_1 - \delta_2 + \frac{20}{h}\delta_3 - \frac{180}{h^2}\delta_4 + \frac{840}{h^3}\delta_5 - \frac{1680}{h^4}\delta_6,\\
\Pi_8 &= \delta_1 - \frac{1}{h}\delta_3,\\
\Pi_9 &= \delta_1 + \frac{2}{h}\delta_3 - \frac{6}{h^2}\delta_4,\\
\Pi_{10} &= \delta_1 - \frac{3}{h}\delta_3 + \frac{24}{h^2}\delta_4 - \frac{60}{h^3}\delta_5,\\
\Pi_{11} &= \delta_1 + \frac{4}{h}\delta_3 - \frac{60}{h^2}\delta_4 + \frac{360}{h^3}\delta_5 - \frac{840}{h^4}\delta_6,\\
\delta_0 &= A\delta_1 + B\delta_2 + C\delta_3,\\
\delta_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_n \; 0_{n\times(7-i)n} \end{bmatrix}, \; i = 1, 2, ..., 6.
\end{aligned}
$$

The following LKF is used as a basic tool to prove [15, Theorem 1]:

$$
\begin{aligned}
V(y_t) =& \zeta^T(t)P\zeta(t) + \int_{t-h}^{t} y^T(s)R_1y(s)ds + h\int_{t-h}^{t}\int_{u}^{t} \dot{y}^T(s)R_2\dot{y}(s)dsdu\\
&+ \int_{t-h}^{t}\int_{u}^{t}\int_{v}^{t} \dot{y}^T(s)R_3\dot{y}(s)dsdvdu,
\end{aligned}
$$

where

$$
\zeta(t) = \begin{bmatrix} y^T(t) & \int_{t-h}^{t} y^T(s)ds & v_1^T(t) & v_2^T(t) \; v_3^T(t) \end{bmatrix}^T,
$$

$$
v_1^T(t) = \int_{t-h}^{t}\int_{u_1}^{t} y^T(s)dsdu_1,
$$

$$
v_2^T(t) = \int_{t-h}^{t}\int_{u_1}^{t}\int_{u_2}^{t} y^T(s)dsdu_2du_1,
$$

$$v_3^T(t)=\int_{t-h}^{t}\int_{u_1}^{t}\int_{u_2}^{t}\int_{u_3}^{t} y^T(s)dsdu_3du_2du_1.$$

Then, the time derivative of $V(y_t)$ along the trajectories of system (1.1) is as follows:

$$\begin{aligned}\dot{V}(y_t)=&2\zeta^T(t)P\zeta(t)+y^T(t)R_1y(t)-y^T(t-h)R_1y(t-h)\\&+h^2\dot{y}^T(t)R_2\dot{y}(t)+\frac{h^2}{2}\dot{y}^T(t)R_3\dot{y}(t)\\&-h\int_{t-h}^{t}\dot{y}^T(s)R_2\dot{y}(s)ds-\int_{t-h}^{t}\int_{u}^{t}\dot{y}^T(s)R_3\dot{y}(s)dsdu\\=&\eta^T(t)\Big\{Sym(\Pi_1^TP\Pi_2)+\delta_0^TQ\delta_0-\delta_7^TQ\delta_7+\delta_1^TR_1\delta_1\\&-\delta_2^TR_1\delta_2+h^2\delta_0^TR_2\delta_0+\frac{h^2}{2}\delta_0^TR_3\delta_0\}\Big\}\eta(t)\\&-h\int_{t-h}^{t}\dot{y}^T(s)R_2\dot{y}(s)ds-\int_{t-h}^{t}\int_{u}^{t}\dot{y}^T(s)R_3\dot{y}^T(s)dsdu,\end{aligned}$$

where

$$\eta(t)=\left[y^T(t)\ y^T(t-h)\int_{t-h}^{t}y^T(s)ds\ v_1^T(t)\ v_2^T(t)\ v_3^T(t)\right]^T.$$

## 3. Analyses of behaviors of solutions

Assume $Q(t,y(t),y(t-h))\equiv 0$ in the nonlinear system of IDDEs (1.2) with constant time retardation, i.e., we consider

$$\dot{y}(t)=A(t)y(t)+BF(y(t-h))+C\int_{t-h}^{t}K(t,s)G(y(s))ds. \tag{3.1}$$

In this section, we generalize and improve the main result of Tian et al. [15, Theorem 1] under less restrictive conditions and also give an additional qualitative result for non-linear system of IDDEs (3.1) with constant time retardation. The results here are proved by the Lyapunov- Krasovski functional approach.

**A. Assumptions**

The following assumptions are needed in the proofs of our new results.

**(H1)** There are positive constants $g_0$, $f_0$ and $K_0$ such that

$$\begin{aligned}&G(0)=0,\|G(u)-G(v)\|\le g_0\|u-v\| \text{ for all } u,\ v\in\mathbb{R}^n,\\&F(0)=0,\|F(y)-F(x)\|\le f_0\|y-x\| \text{ for all } x,\ y\in\mathbb{R}^n,\\&\|K(t,s)\|\le K_0 \text{ for all } s\le t.\end{aligned}$$

**(H2)** There are constants $f_0,\ g_0,\ K_0$ from $(H1)$, $a_0>0$ and $h>0$ such that

$$a_{ii}(t)+\sum_{j=1,j\neq i}^{n}|a_{ji}(t)|\leq -a_0 \text{ for all } t\in\mathbb{R}^+$$

and

$$a_0-f_0\left\|B\right\|>0, h<\frac{a_0-f_0\left\|B\right\|}{g_0K_0\left\|C\right\|}, \left\|C\right\|\neq 0.$$

The new stability result is the following theorem.

**Theorem 3.** *Assume that the conditions (H1) and (H2) hold. Then, the zero solution of the nonlinear system of IDDEs (3.1) is uniformly asymptotically stable.*

*Proof.* For the proof of this theorem, we define a LKF $V=V(t,y_t)$ by

$$V=\|y\|+\beta\int_{t-h}^{t}\|F(y(s))\|\,ds+\gamma\int_{-h}^{0}\int_{t+\eta}^{t}\|K(t,s)G(y(s))\|\,dsd\eta, \tag{3.2}$$

where $-h\leq\eta\leq 0$ and $\beta,\ \gamma>0,\ \beta,\ \gamma\in\mathbb{R}$ and we will choose $\beta,\gamma$ later in the proof, and

$$\|y\|=\sum_{i=1}^{n}|y_i|.$$

From this step, we see that the functional $V$ satisfies the following equality and inequality:

$$V(t,0)=0, \|y\|\leq V(t,y_t).$$

For the coming step, using the condition $(H1)$, we arrive

$$\begin{aligned}|V(t,y_t)-V(t,x_t)|\leq{}&|\ \ \|y\|-\|x\|\ \ |+\beta\left|\int_{t-h}^{t}[\|F(y(s))\|-\|F(x(s))\|]ds\right|\\&+\gamma\left|\int_{-h}^{0}\int_{t+\eta}^{t}[\|K(t,s)G(y(s))\|-\|K(t,s)G(x(s))\|]dsd\eta\right|\\\leq{}&\sum_{i=1}^{n}|y_i-x_i|+\beta\int_{t-h}^{t}\|F(y(s))-F(x(s))\|\,ds\\&+\gamma\int_{-h}^{0}\int_{t+\eta}^{t}\|K(t,s)\|\ \ \|G(y(s))-G(x(s))\|\,dsd\eta\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{n} |y_i - x_i| + \beta f_0 \int_{t-h}^{t} \|y(s) - x(s)\| \, ds \\
&\quad + \gamma g_0 K_0 \int_{-h}^{0} \sup_{t+\eta \le s \le t} \|y(s) - x(s)\| d\eta \\
&\le \|y(t) - x(t)\| + \beta h f_0 \sup_{t-h \le s \le t} \|y(s) - x(s)\| \\
&\quad + \gamma h g_0 K_0 \sup_{t+\eta \le s \le t} \|y(s) - x(s)\| \\
&\le (1 + \beta h f_0 + \gamma h g_0 K_0) \sup_{t+\eta \le s \le t} \|y(s) - x(s)\| \\
&= M_0 \sup_{t+\eta \le s \le t} \|y(s) - x(s)\| ,
\end{aligned}$$

where $M_0 = 1 + \beta h f_0 + \gamma h g_0 K_0$.

Thus, condition (A1) of Theorem 1 holds. Hence, the LKF $V(t, y_t)$ is locally Lipschitz in $y$.

Let

$$Z(t, y) = \gamma \int_{-h}^{0} \int_{t+\eta}^{t} \|K(t, s)G(y(s))\| \, ds d\eta.$$

Then, using $Z(t, y)$ and the functional $V$, it therefore follows that

$$\beta_1 \|y\| + Z(t, y) \le V(t, y_t) \le \beta_2 \|y\| + \beta f_0 h \|y(s)\|_{[t-h,t]} + Z(t, y),$$
$$0 < \beta_1 \le 1, \beta_2 \ge 1,$$

and

$$\begin{aligned}
Z(t, y) &= \gamma \int_{-h}^{0} \int_{t+\eta}^{t} \|K(t, s)G(y(s))\| \, ds d\eta \\
&\le \gamma h g_0 K_0 \sup_{t+\eta \le s \le t} \|y(s)\| = \gamma h g_0 K_0 \|y(s)\|_{[t+\eta,t]}.
\end{aligned}$$

For the verification of the condition (A2) of Theorem 1, we consider

$$\begin{aligned}
Z(t_2, y) - Z(t_1, y) =& \gamma \int_{-h}^{0} \int_{t_2+\eta}^{t_2} \|K(t, s)G(y(s))\| \, ds d\eta \\
& - \gamma \int_{-h}^{0} \int_{t_1+\eta}^{t_1} \|K(t, s)G(y(s))\| \, ds d\eta.
\end{aligned}$$

For the next step, add and subtract the following term to this equality:

$$\gamma \int_{-h}^{0} \int_{t_1+\eta}^{t_2+\eta} \|K(t, s)G(y(s))\| \, ds.$$

Using the condition (H1), we then obtain

$$\begin{aligned}
Z(t_2,y)-Z(t_1,y) =&\gamma \int_{-h}^{0}\int_{t_2+\eta}^{t_2} \|K(t,s)G(y(s))\|\; ds d\eta \\
&-\gamma \int_{-h}^{0}\int_{t_1+\eta}^{t_1} \|K(t,s)G(y(s))\|\; ds d\eta \\
&+\gamma \int_{-h}^{0}\int_{t_1+\eta}^{t_2+\eta} \|K(t,s)G(y(s))\|\; ds d\eta \\
&-\gamma \int_{-h}^{0}\int_{t_1+\eta}^{t_2+\eta} \|K(t,s)G(y(s))\|\; ds d\eta \\
=&\gamma \int_{-h}^{0}\int_{t_1}^{t_2} \|K(t,s)G(y(s))\|\; ds d\eta \\
&-\gamma \int_{-h}^{0}\int_{t_1+\eta}^{t_2+\eta} \|K(t,s)G(y(s))\|\; ds d\eta \\
\leq&\gamma \int_{-h}^{0}\int_{t_1}^{t_2} \|K(t,s)G(y(s))\|\; ds d\eta \\
\leq&\gamma \int_{-h}^{0}\int_{t_1}^{t_2} \|K(t,s)G(y(s))\|\; ds d\eta \\
\leq&\gamma h g_0 K_0 K_1 (t_2-t_1),
\end{aligned}$$

where

$$K_1 = \sup_{t_1\leq s\leq t_2} \|y(s)\|\,, 0<t_1<t_2<\infty.$$

Thus, the condition (A2) of Theorem 1 holds.

Calculating the derivative of the LKF $V$ in (3.2) along the solutions of system (3.1), it follows

$$\begin{aligned}
\frac{d}{dt}V(t,y_t) =&\sum_{i=1}^{n} {y'}_i(t) sgn y_i(t+0) + \beta\,\|F(y(t))\| - \beta\,\|F(y(t-h))\| \\
&+ h\gamma\;\|K(t,t)G(y(t))\| - \gamma \int_{-h}^{0} \|K(t,t+\eta)G(y(t+\eta))\|\, d\eta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} y'_i(t) sgny_i(t+0) + \beta \left\| F(y(t)) \right\| - \beta \left\| F(y(t-h)) \right\| \\
&\quad + h\gamma \ \left\| K(t,t)G(y(t)) \right\| - \gamma \int\limits_{t-h}^{t} \left\| K(t,s)G(y(s)) \right\| ds \\
&\le \sum_{i=1}^{n} y'_i(t) sgny_i(t+0) + \beta f_0 \left\| y(t) \right\| - \beta \left\| F(y(t-h)) \right\| \\
&\quad + h\gamma g_0 K_0 \left\| y(t) \right\| - \gamma \int\limits_{t-h}^{t} \left\| K(t,s) \right\| \left\| F(y(s)) \right\| ds
\end{aligned}
$$

Using the condition (H2) , we obtain

$$
\begin{aligned}
\sum_{i=1}^{n} sgny_i(t+0) y'_i(t) &\le \sum_{i=1}^{n} a_{ii}(t) \left| y_i(t) \right| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left| a_{ji}(t) \right| \ \left| y_i(t) \right| \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left| b_{ij} \right| \ \left| F_j(y(t-h)) \right| \\
&\quad + \left\| C \right\| \int\limits_{t-h}^{t} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| K_{ij}(t,s) \right| \ \left| G_j(y(s)) \right| ds \\
&= \sum_{i=1}^{n} \left( a_{ii}(t) + \sum_{j=1, j\neq i}^{n} \left| a_{ji}(t) \right| \right) \left| y_i(t) \right| \\
&\quad + \left\| B \right\| \ \left\| F(y(t-h)) \right\| + \left\| C \right\| \int\limits_{t-h}^{t} \left\| K(t,s) \right\| \ \ \left\| F(y((s)) \right\| ds \\
&\le - a_0 \left\| y(t) \right\| + \left\| B \right\| \ \ \left\| F(y(t-h)) \right\| \\
&\quad + \left\| C \right\| \int\limits_{t-h}^{t} \left\| K(t,s) \right\| \left\| F(y(s) \right\|) ds.
\end{aligned}
$$

From the discussion made, it follows that

$$
\begin{aligned}
\frac{d}{dt} V(t, y_t) &\le - a_0 \left\| y(t) \right\| + \left\| B \right\| \ \ \left\| F(y(t-h)) \right\| + \left\| C \right\| \int\limits_{t-h}^{t} \left\| K(t,s) \right\| \ \ \left\| F(y(s) \right\|) ds \\
&\quad + \beta f_0 \left\| y(t) \right\| - \beta \left\| F(y(t-h)) \right\| \\
&\quad + h\gamma g_0 K_0 \left\| y(t) \right\| - \gamma \int\limits_{t-h}^{t} \left\| K(t,s) \right\| \ \ \left\| F(y(s)) \right\| ds
\end{aligned}
$$

$$= - \ (a_0 - \beta f_0 - h\gamma g_0 K_0) \|y(t)\| - (\beta - \|B\|) \|F(y(t-h))\| + \|C\| \int\limits_{t-h}^{t} \|K(t,s)\| \ \|F(y(s))\| \, ds - \gamma \int\limits_{t-h}^{t} \|K(t,s)\| \ \|F(y(s))\| \, ds.$$

Let $\beta = \|B\|$ and $\gamma = \|C\|$. Then, for a sufficiently small positive constant $\delta$, we have

$$\frac{d}{dt} V(t, y_t) \le - \ (a_0 - f_0 \|B\| - h g_0 K_0 \|C\|) \|y(t)\| \le -\delta \|y(t)\| < 0, (\|y(t)\| \ne 0),$$

by the condition (H2). Thus, the condition (A3) of Theorem 1 holds.

From the whole discussion, we see that the conditions of (A1)-(A3) of Theorem 1 hold (see Burton [4, Theorem 4. 2.9]). Therefore, the zero solution of the nonlinear system of IDDEs (3.1) is uniformly asymptotically stable.

As the second, the new integrability result is the following Theorem 4.

**Theorem 4.** *The norm of solutions of the system of IDDEs (3.1) is integrable in the sense of Lebesgue on $\mathbb{R}^+$ if conditions (H1) and (H2) hold.*

*Proof.* We prove Theorem 4 using the LKF $V = V(t, y_t)$, which is defined by (3.2). Then, it is obvious that

$$\frac{d}{dt} V(t, y_t) \le - \ \delta \|y(t)\| \tag{3.3}$$

by the conditions (H1) and (H2).

From this point of view, it is clear that $V$ is a decreasing functional. Hence, integrating the inequality (3.3), we get

$$\delta \int\limits_{t_0}^{t} \|y(s)\| ds \le V(t_0, \phi(t_0)) - V(t, y_t) \le V(t_0, \phi(t_0))$$

By this inequality, it follows that

$$\int\limits_{t_0}^{\infty} \|y(s)\| ds < \infty.$$

Hence, the proof of Theorem 4 is completed.

## 4. Numerical application

In this section, as an application for a particular case of the system of IDDEs (3.1), a numerical example is given. Hence, we prove that the conditions of Theorems 3-4 can be satisfied, and Theorems 3 and 4 can be applied in the particular case.

**Example 1.** In a particular case of the system of IDDEs (3.1), we consider the system of nonlinear IDDEs with constant time retardation:

$$\begin{pmatrix} {y'}_1 \\ {y'}_2 \end{pmatrix} = -\begin{pmatrix} 25+\frac{1}{1+t^2} & \frac{1}{1+t^2} \\ \frac{1}{1+t^2} & 25+\frac{1}{1+t^2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
- \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \sin y_1(t-1) \\ \sin y_2(t-1) \end{pmatrix}
+ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\int\limits_{t-1}^{t}\begin{pmatrix} \frac{1}{1+s^2+t^2} & 1 \\ 1 & \frac{1}{1+s^2+t^2} \end{pmatrix}\begin{pmatrix} \sin y_1(s) \\ \sin y_2(s) \end{pmatrix} ds, \qquad (4.1)$$

where $t \geq 1 = h$, which is the constant time retardation.

When comparing the systems of IDDEs (4.1) and IDDEs (3.1), we have the following:

$$A(t) = -\begin{pmatrix} 25+\frac{1}{1+t^2} & \frac{1}{1+t^2} \\ \frac{1}{1+t^2} & 25+\frac{1}{1+t^2} \end{pmatrix},$$

$$B = -\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

$$F(y(t-1)) = \begin{bmatrix} f_1(y_1(t-1)) \\ f_2(y_2(t-1)) \end{bmatrix} = \begin{bmatrix} \sin(y_1(t-1)) \\ \sin(y_2(t-1)) \end{bmatrix},$$

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$K(t,s) = \begin{bmatrix} \frac{1}{1+s^2+t^2} & 1 \\ 1 & \frac{1}{1+s^2+t^2} \end{bmatrix},$$

$$G(y) = \begin{bmatrix} \sin y_1 \\ \sin y_2 \end{bmatrix}.$$

From this point of view, clearly, $F(0) = 0$. Next, using $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$, by some simple computations, we have

$$\begin{aligned}
\|F(y) - F(x)\| &= \left\| \begin{bmatrix} \sin y_1 - \sin x_1 \\ \sin y_2 - \sin x_2 \end{bmatrix} \right\| \\
&= |\sin y_1 - \sin x_1| + |\sin y_2 - \sin x_2| \\
&=2\left|\cos\left(\frac{y_1+x_1}{2}\right)\sin\left(\frac{y_1-x_1}{2}\right)\right| \\
&\quad + 2\left|\cos\left(\frac{y_2+x_2}{2}\right)\sin\left(\frac{y_2-x_2}{2}\right)\right| \\
&\leq 2\left(\frac{|y_1-x_1|}{2} + \frac{|y_2-x_2|}{2}\right) \\
&= \|y-x\|,\ f_0 = 1,\ \text{for all } y,\ x \in \mathbb{R}^2.
\end{aligned}$$

Next, clearly, $G(0) = 0$. By the same way, using $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$, some computations give

$$\|G(y) - G(x)\| = \left\| \begin{pmatrix} \sin y_1 - \sin x_1 \\ \sin y_2 - \sin x_2 \end{pmatrix} \right\| \leq \|y - x\|, g_0 = 1, \text{ for all } x,\ y \in \mathbb{R}^2.$$

In view of the matrix $A(t)$, we have

$$a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| = -25 < -24 = -a_0$$

since

$$a_{11}(t) + |a_{21}(t)| = -25 - \frac{1}{1+t^2} + \frac{1}{1+t^2} = -25 < -24 = -a_0.$$
$$a_{22}(t) + |a_{12}(t)| = -25 - \frac{1}{1+t^2} + \frac{1}{1+t^2} = -25 < -24 = -a_0.$$

Thus, it follows that

$$a_{ii}(t) + \sum_{j=1, j\neq i}^{2} |a_{ji}(t)| < -a_0 = -24 \text{ for all } t \in \mathbb{R}^+.$$

We also obtain

$$\|B\| = \left\| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right\| = 4,$$
$$\|C\| = \left\| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\| = 3,$$
$$\|C\| \neq 0,$$
$$\|K(t,s)\| = \max_{1\leq j\leq 2} \sum_{i=1}^{2} |K_{ij}|$$
$$= \max\left\{ \frac{1}{1+s^2+t^2} + 1,\ \frac{1}{1+s^2+t^2} + 1 \right\}$$
$$= \frac{1}{1+s^2+t^2} + 1 \leq 2 = K_0.$$

In view of the discussion done, we find

$$a_0 - f_0 \|B\| = 24 - 4 = 20 > 0,$$
$$g_0 K_0 \|C\| = 1 \times 2 \times 3 = 6,$$
$$\frac{a_0 - f_0 \|B\|}{g_0 K_0 \|C\|} = \frac{10}{3},$$
$$1 = h < \frac{a_0 - f_0 \|B\|}{g_0 K_0 \|C\|} = \frac{10}{3}.$$

From this point of view, the conditions (H1)-(H2), of Theorems 3-4 hold. For this reason, we conclude that the zero solution of the system of IDDEs (4.1) is uniformly asymptotically stable as well as the norm of the solutions of the same system is integrable in the sense of Lebesgue on $\mathbb{R}^+$.

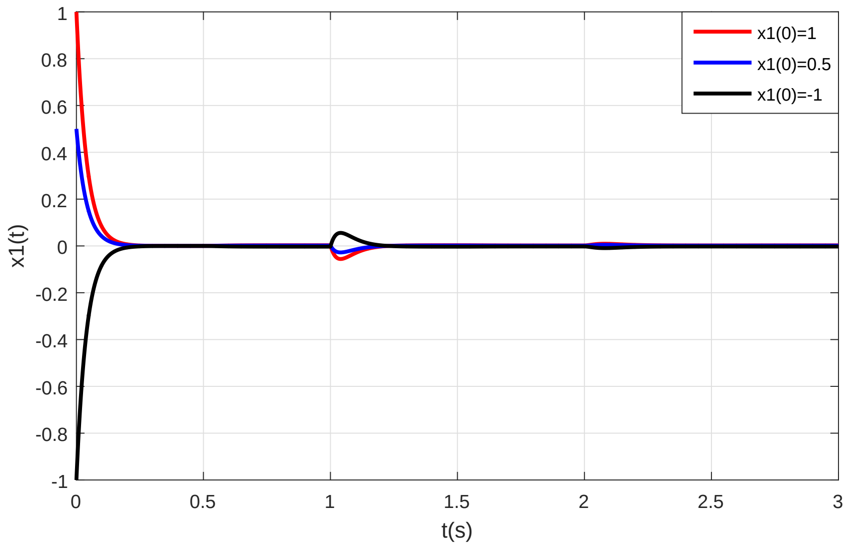


FIGURE 1. The orbits of solution $x_1 = y_1$ of the system of IDDEs (4.1) for different initial values.

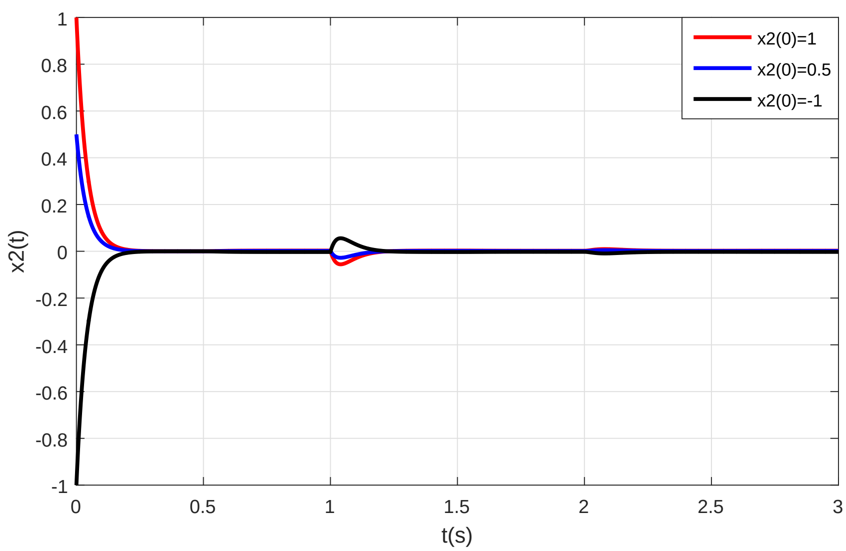


FIGURE 2. The orbits of solution $x_2 = y_2$ of the system of IDDEs (4.1) for different initial values.

## 5. Analysis of boundedness

In this section, we prove a new theorem such that the solutions of the system of nonlinear IDDEs (1.2) are bounded at the infinity. Hence, it is needed to impose the following condition in addition to that given above by (H1):

**B. Assumption**

**(H3)** There exist positive constants $f_0$, $g_0$, $K_0$ and $a_0$, $h$ from (H1) and (H2), respectively, and a continuous function $Q_0 \in C(\mathbb{R}, \mathbb{R})$ such that

$$a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \leq -a_0 \text{ for all } t \in \mathbb{R}^+,$$

$$\|Q(t, y(t), y(t-h))\| \leq |Q_0(t)| \ \ \|y(t)\| \text{ for all } t \in \mathbb{R}^+ \text{ and for all } y,\ y(t-h) \in \mathbb{R}^n$$

and

$$a_0 - f_0 \|B\| - hg_0K_0 \|C\| - |Q_0(t)| \geq 0 \text{ for all } t \in \mathbb{R}^+.$$

**Theorem 5.** *The solutions of the system of IDDEs (1.2) are bounded at infinity if conditions (H1) and (H3) hold.*

*Proof.* In the proof of this theorem, we use the LKF $V$ in (3.2). Using conditions (H1) and (H3), we can conclude

$$\frac{d}{dt}V(t, y_t) \leq 0.$$

Integrating this inequality, we obtain

$$V(t, y_t) \leq V(t_0, \phi(t_0)) \equiv \text{ a positive constant.}$$

Let

$$V(t_0, \phi(t_0)) \equiv B > 0.$$

As the next step, it follows that

$$\|y(t)\| \leq V(t, y_t) \leq B, i.e., \|y(t)\| \leq B.$$

When $t \to +\infty$, it is derived that

$$\lim_{t\to+\infty} \||y(t)|\| \leq \lim_{t\to+\infty} B = B.$$

Thus, clearly, the solutions of the nonlinear system of IDDEs (1.2) are bounded as $t \to +\infty$. The proof of Theorem 5 is completed.

## 6. Numerical application

In this section, as an application for a particular case of the system of nonlinear IDDEs (1.2), we give a numerical example . Hence, it is shown that the conditions of Theorem 5 can be hold.

**Example 2.** We now deal with the system of nonlinear IDDEs with constant time retardation:

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = -\begin{pmatrix} 25+\frac{1}{1+t^2} & \frac{1}{1+t^2} \\ \frac{1}{1+t^2} & 25+\frac{1}{1+t^2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$-\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \sin y_1(t-1) \\ \sin y_2(t-1) \end{pmatrix}$$
$$+\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\int_{t-1}^{t}\begin{pmatrix} \frac{1}{1+s^2+t^2} & 1 \\ 1 & \frac{1}{1+s^2+t^2} \end{pmatrix}\begin{pmatrix} \sin y_1(s) \\ \sin y_2(s) \end{pmatrix} ds,$$
$$+\begin{pmatrix} \frac{y_1}{1+t^2+y_1^2(t-1)} \\ \frac{y_2}{1+t^2+y_2^2(t-1)} \end{pmatrix} \tag{6.1}$$

where $t \geq 1 = h$, which is the constant time retardation. When we compare the system of IDDEs (6.1) and the system of IDDEs (1.2), the satisfaction of the condition (H1) can be seen form Example 1. As regarding the satisfaction of the condition (H3), from Example 1, we have

$$a_{ii}(t) + \sum_{j=1, j\neq i}^{2} |a_{ji}(t)| < -a_0 = -24 \text{ for all } t \in \mathbb{R}^+.$$

Next, we have

$$Q(t, y, y(t-1)) = \begin{pmatrix} \frac{y_1}{1+t^2+y_1^2(t-1)} \\ \frac{y_2}{1+t^2+y_2^2(t-1)} \end{pmatrix}.$$

For the next step, we derive

$$\begin{aligned} \|Q(t, y, y(t-1))\| &= \left\| \begin{pmatrix} \frac{y_1}{1+t^2+y_1^2(t-1)} \\ \frac{y_2}{1+t^2+y_2^2(t-1)} \end{pmatrix} \right\| \\ &= \frac{|y_1|}{1+t^2+y_1^2(t-1)} + \frac{|y_2|}{1+t^2+y_2^2(t-1)} \\ &\leq \frac{|y_1|}{1+t^2} + \frac{|y_2|}{1+t^2} \\ &= \frac{1}{1+t^2}\left[|y_1| + |y_2|\right] = |Q_0(t)| \, \|y\|, \end{aligned}$$

where

$$|Q_0(t)| = \frac{1}{1+t^2}, |y_1| + |y_2| = \|y\|.$$

Finally, it follows that

$$\begin{aligned} a_0 - & f_0 \|B\| - hg_0K_0 \|C\| - |Q_0(t)| \\ &= 24 - 4 - 2\times 3 - \frac{1}{1+t^2} \geq 13 > 0. \end{aligned}$$

Hence, the condition (H3) holds. Therefore , the solutions of the system of nonlinear IDDEs (6.1) with constant time retardation are bounded as $t \to \infty$.

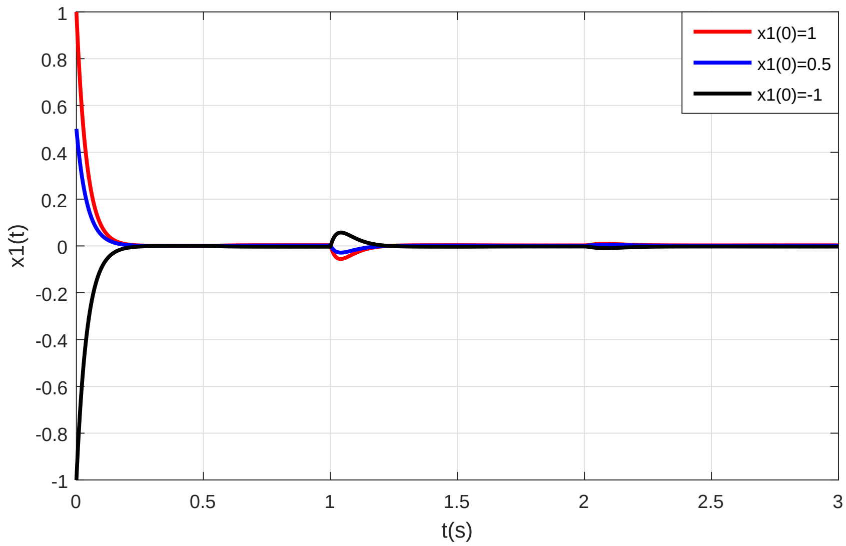


FIGURE 3. The orbits of bounded solution $x_1 = y_1$ of the system of IDDEs (6.1) for different initial values.

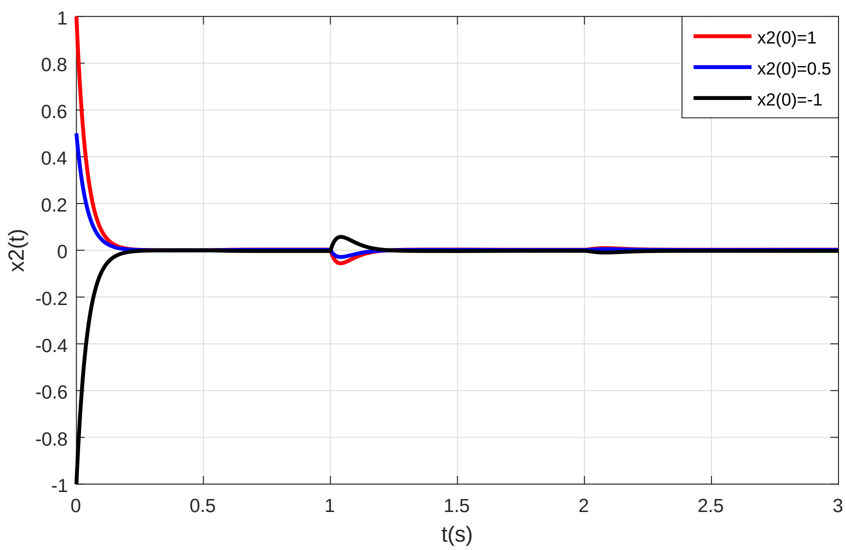


FIGURE 4. The orbits of bounded solution $x_2 = y_2$ of the system of IDDEs (6.1) for different initial values.

## 7. Contributions

We now comment the contributions of Theorems 3-5 to the qualitative properties of solutions of IDDEs and the past literature on the subject.

**1)** It is clear that the system of IDDEs (1.2) extend and improve the system of IDDEs (1.1) in Tian et al. [15, Theorem 1] from linear case to the a more general non-linear case.

**2)** In 2020, Tian et al. [15, Theorem 1] constructed the following LKF to prove the related stability result:

$$\begin{aligned} V(y_t) =& \zeta^T(t)P\zeta(t) + \int_{t-h}^{t} y^T(s)R_1y(s)ds + h\int_{t-h}^{t}\int_{u}^{t} \dot{y}^T(s)R_2\dot{y}(s)dsdu \\ &+ \int_{t-h}^{t}\int_{u}^{t}\int_{v}^{t} \dot{y}^T(s)R_3\dot{y}(s)dsdvdu, \end{aligned} \tag{7.1}$$

where

$$\zeta(t) = \begin{bmatrix} y^T(t) & \int_{t-h}^{t} y^T(s)ds & v_1^T(t) & v_2^T(t) & v_3^T(t) \end{bmatrix}^T,$$

$$v_1^T(t) = \int_{t-h}^{t}\int_{u_1}^{t} y^T(s)dsdu_1,$$

$$v_2^T(t) = \int_{t-h}^{t}\int_{u_1}^{t}\int_{u_2}^{t} y^T(s)dsdu_2du_1,$$

$$v_3^T(t) = \int_{t-h}^{t}\int_{u_1}^{t}\int_{u_2}^{t}\int_{u_3}^{t} y^T(s)dsdu_3du_2du_1.$$

Since $P,\ R_i,\ i$=1,2,3,in the LKF (7.1) are positive definite symmetric matrices, then it is clear that the LKF (7.1) is positive definite. In [15], the authors used the LKF (7.1) as a basic tool to prove Theorem 2, which was stated in Section 2.

For the next step, Tian et al. [15, Theorem 1] calculated the time derivative of the LKF (7.1) along the linear system of IDDEs (1.1) and derived the following equality:

$$\begin{aligned} \dot{V}(y_t) =& \eta^T(t)\Big\{ Sym(\Pi_1^T P\Pi_2) + \delta_0^T Q\delta_0 - \delta_7^T Q\delta_7 + \delta_1^T R_1\delta_1 - \delta_2^T R_1\delta_2 + h^2\delta_0^T R_2 \\ &+ \frac{h^2}{2}\delta_0^T R_3\delta_0\}\Big\}\eta(t) \\ &- h\int_{t-h}^{t} \dot{y}^T(s)R_2\dot{y}(s)ds - \int_{t-h}^{t}\int_{u}^{t} \dot{y}^T(s)R_3\dot{y}^T(s)R_3\dot{y}(s)dsdu, \end{aligned} \tag{7.2}$$

where

$$\eta(t) = \left[ y^T(t)\ y^T(t-h) \int\limits_{t-h}^{t} y^T(s)ds\ v_1^T(t)\ v_2^T(t)\ v_3^T(t) \right]^T$$

and the arguments of $\dot{V}(y_t)$ are given clearly in Theorem 2 at Section 2.

Hence, based Lemma 1, Lemma 2 and Lemma 3, the LKF (7.1) and its time derivative in (7.2) as the same in Tian et al. [15, Theorem 1], which is negative definite, the authors proved that the linear system of IDDEs (1.1) is asymptotic stable. In fact, the LKF (7.1) and its time derivative in (7.2) satisfy the conditions of Lyapunov-Krasovskii's asymptotically stability theorem (see [4]). By this result, i.e., Theorem 2 given in Section 2, a new and interesting delay-dependent stability condition is obtained. In Tian et al. [15, Theorem 1], a less conservative stability criterion is obtained by using the double integral inequality and choosing the new LKF.

**3)** In this paper, we define a more optimal the LKF than that given by (7.1) such that

$$V = \|y\| + \beta \int\limits_{t-h}^{t} \|F(y(s))\|\, ds + \gamma \int\limits_{-h}^{0} \int\limits_{t+\eta}^{t} \|K(t,s)G(y(s))\|\, ds d\eta.$$

Then, we prove that the LKF satisfies the conditions of the well-known Theorem 1 of Burton, [4, Theorem 4. 2.9]) on the uniformly asymptotically stability of the zero solution. In spite of Tian et al. [15, Theorem 1] proved the asymptotic stability of the linear system of IDDEs (1.1), we discuss the uniformly asymptotically stability such that the uniformly asymptotically stability implies asymptotically stability, but the converse is not true. Hence, we improve and obtain the result of Tian et al. [15, Theorem 1] under weaker conditions, i.e., less restrictive conditions.

Next, it should be noted that the work of Tian et al. [15, Theorem 1] is very interesting and has a good novelty. However, the weaker and less restrictive conditions of Theorem 3 of this paper can be clearly observed and checked if we compare the conditions of Tian et al. [15, Theorem 1] with that of Theorem 3 such that taking into account Lemma 1, Lemma 2 and Lemma 3 of Tian et al. [15], the LKF (7.1) and its time derivative (7.2) and our LKF (3.2) and its time derivative, respectively. Here, in fact, when we compare the conditions of Theorem 3 with that of Tian et al. [15, Theorem 1], we see that the conditions of Theorem 3 are very appropriate and much optimal, easier to verify and apply as seen in Example 1. Therefore, we would not like to give more details about proper discussions. These are the novelty and originality of this paper. Next, the mentioned observations are desirable facts for proper works to be done in the literature.

**4)** Tian et al. [15, Theorem 1] investigated the asymptotically stability of the system of linear IDDEs (1.1). Here, we investigate uniformly asymptotically stability of the zero solution, the integrability of the norm of solutions of the system of nonlinear IDDEs (3.1) with constant time retardation as well as the boundedness of solutions of the system of IDDEs (1.2). In fact, we give two more new results in addition to that of Tian et al. [15, Theorem 1].

**5)** In this paper, two numerical examples are proposed. These examples satisfy the conditions of Theorems 3-5 and show the applications of the results of this paper.

**6)** An advantage of the new LKF (3.2) used here is also that it eliminates using Gronwall's inequality for the boundedness of solutions at infinity. Compared to related results in the literature, the conditions here are more general, simple, and convenient to apply.

## 8. Conclusion

In this paper, we consider a mathematical model, a system of nonlinear IDDEs with constant time retardation. In [15, Theorem 1], for particular case of that system of IDDEs, asymptotically stability of linear system of IDDEs has been discussed by a generalized double integral inequality and suitable LKF. In this paper, we obtain the result of Tian et al. [15, Theorem 1] under very less restrictive conditions, extend and improve that result. In fact, we give three new theorems, Theorems 3-5, such that they are related to uniformly asymptotically stability as well as integrability of norm of solutions and boundedness of solutions at infinity. The technique of the proofs depends upon the construction of a new and more suitable LKF and its usage in the proofs. Two new examples are provided to illustrate the applications of the results. Compared with qualitative results in the literature related to the IDDEs, our results improve and extend the classical result of Tian et al. [15, Theorem 1] which allows new contributions to theory of integro-time delay differential equations and the related results that can be found in the relevant literature.

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