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WELL-POSEDNESS OF A MIXED HEMIVARIATIONAL-VARIATIONAL PROBLEM

MIRCEA SOFONEA* AND ANDALUZIA MATEI**

*Laboratoire de Mathématiques et Physique,
University of Perpignan Via Domitia,
52 Avenue Paul Alduy, 66860 Perpignan, France
E-mail: sofonea@univ-perp.fr

**Department of Mathematics, University of Craiova,
13 A.I. Cuza, 200585, Craiova, Romania
E-mail: andaluzia.matei@edu.ucv.ro

Abstract. We consider a mixed hemivariational-variational problem, i.e., a system which gathers a hemivariational inequality with a constrained variational inequality. We list the assumptions on the data and prove the existence of a unique solution to the problem. Subsequently, we prove the continuous dependence of the solution with respect to the data. Then, we deduce a criterion of convergence to the solution of the mixed hemivariational-variational inequality, i.e., we formulate necessary and sufficient conditions which guarantee the convergence of a sequence to the unique solution of the system. The proof of our results is based on the particular structure of the problem which allows us to employ a fixed point argument. Finally, we provide two examples which illustrate our abstract results.
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1. Introduction

Mixed variational problems represent a class of problems with a convex structure which arise in the analysis of a large number of nonlinear boundary value problems with constraints. The major ingredient in their structure consists to introduce a new variable, the Lagrange multiplier, associated to the set of constraints. Existence and uniqueness results can be found in [5, 8, 9, 11, 13, 26], for instance. References on their numerical treatment include [1, 12, 14, 15]. As it follows from these references, the numerical treatment of nonlinear problems based on mixed variational formulations is efficient and accurate. This explains why such formulations are widely used in Solid and Contact Mechanics as well as in various Engineering Applications.

Currently, there is an increasing interest in the study of mixed hemivariational-variational problems, i.e., problems which couple a hemivariational inequality with a variational inequality. Such kind of problems are formulated by using the Clarke generalized derivative of a locally Lipschitz function and, consequently, they have both a convex and nonconvex structure. Abstract results and examples arising in Contact Mechanics can be found in [2, 17, 19]. For recent results in the related field we also refer the reader to [16, 20, 21, 32, 31].

In the current paper we study the well-posedness of a mixed hemivariational-variational problem with a perturbed term, the hemivariational inequality being governed by a strongly monotone Lipschitz continuous operator. The functional framework is the following. First, X, Y and Z are real Hilbert spaces endowed with the inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ and $(\cdot,\cdot)_Z$, and the associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We denote by $K_1 \times K_2$ the product of the sets K_1 and K_2 and $X \times Y$ will represent the product space endowed with the canonical inner product. A typical element of $X \times Y$ will be denoted by (u, λ) and, moreover, 0_X, 0_Y represent the zero elements of X and Y, respectively. In addition, we use the notation $\mathcal{L}(X, Y)$ for the space of linear continuous operators defined on X with values in Z, equipped with the canonical norm $\|\cdot\|_{\mathcal{L}(X,Z)}$.

The problem data are the operators $A : X \to X$, $G : X \to Z$, the bilinear forms $b : X \times Y \to \mathbb{R}$ and $c : Y \times Y \to \mathbb{R}$, the function $J : Z \to \mathbb{R}$, the set $\Lambda \subseteq Y$ and the element $f \in X$. We use the notation $J^0(u; v)$ for the generalized derivative of J in the point $u \in Z$ in the direction $v \in Z$, when J is locally Lipschitz. Then, the mixed hemivariational-variational problem we consider consists of the following system.

Problem $\mathcal{P}$. *Find* $(u, \lambda) \in X \times \Lambda$ *such that*

$$(Au, v - u)_X + b(v - u, \lambda) + J^0(Gu; Gv - Gu) \geq (f, v - u)_X, \ \forall\, v \in X, \quad (1.1)$$

$$b(u, \mu - \lambda) - c(\mu, \mu) + c(\lambda, \lambda) \leq 0 \qquad \forall\, \mu \in \Lambda. \quad (1.2)$$

Note that Problem $\mathcal{P}$ can be seen as a generalization of the mixed variational problem

$$a(u, v) + b(v, \lambda) = (f, v)_X \quad \forall\, v \in X, \quad (1.3)$$

$$b(u, \mu) - t^2 c(\lambda, \mu) = 0 \qquad \forall\, \mu \in \Lambda. \quad (1.4)$$

in which $t > 0$ is a given parameter. The system (1.3)–(1.4) represents a saddle point problem with penalty term which can be encountered in the study of elastic plates. Details on this topic can be found in [4, p.131–132] and [3, p.137-138].

Our aim in this paper is threefold. The first one is to provide sufficient conditions which guarantee the unique solvability of Problem $\mathcal{P}$. The second one is to prove the continuous dependence of the solution with respect to the data. The third one is to introduce a general criterion of convergence to the solution. Besides the novelty of the results we present in this paper, we underline that the analysis we provide here is carried out by using as crucial ingredient the fixed-point structure of Problem $\mathcal{P}$. The results we present in this paper find applications in the study of boundary value problems which, in a variational formulation, lead to such kind of mixed problems.

The rest of the paper is structured as follows. In Section 2 we introduce some preliminary material that we need in the next sections. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. The proof is based on a recent result in [18] combined with a fixed point argument. In Section 4 we use the fixed point structure of the Problem $\mathcal{P}$ in order to prove a convergence result, Theorem 4.1. In Section 5 we provide a criterion of convergence to the solution of Problem $\mathcal{P}$, Theorem 5.1. Finally, in Section 6 we provide two examples which illustrate the abstract results provided by Theorems 3.1–5.1.

2. Preliminaries

We now recall some notation and preliminary results which will be used in the rest of the manuscript. For simplicity we restrict ourselves to the Hilbertian case. Therefore, below in this section we assume that $(H,(\cdot,\cdot)_H,\|\cdot\|_H)$ is a real Hilbert space.

Definition 2.1. *An operator $A : H \to H$ is said to be strongly monotone if there exists $m_A > 0$ such that*

$$(Au - Av, u - v)_H \geq m_A\|u - v\|_H^2 \quad \forall\, u, v \in H. \tag{2.1}$$

The operator A is Lipschitz continuous if there exists $L_A > 0$ such that

$$\|Au - Av\|_H \leq L_A\|u - v\|_H \quad \forall\, u, v \in H. \tag{2.2}$$

The following result will be used in Section 3 of this paper.

Lemma 2.2. *Let $A : H \to H$ be a strongly monotone Lipschitz continuous operator with constants m_A and L_A and let $\rho > 0$. Then:*

a) *the operator A is invertible and its inverse $A^{-1} : H \to H$ is strongly monotone Lipschitz continuous with constant $m_{A^{-1}} = \frac{m_A}{L_A^2}$ and $L_{A^{-1}} = \frac{1}{m_A}$;*

b) *the operator $B_\rho : H \to H$ defined by $B_\rho u = u - \rho Au$ for all $u \in H$ satisfies the inequality*

$$\|B_\rho u_1 - B_\rho u_2\|_H \leq k(\rho)\|u_1 - u_2\|_H \quad \forall\ u_1,\, u_2 \in H$$

with $k(\rho) = (1 - 2\rho m_A + \rho^2 L_A^2)^{\frac{1}{2}}$. Moreover, if $\rho \in \left(0, \frac{2m_A}{L_A^2}\right)$ then $0 < k(\rho) < 1$ and, therefore, B_ρ is a contraction on H.

The proof of Lemma 2.2 a) can be found in [28, p. 23]. The proof of Lemma 2.2 b) follows from arguments identical to those used on [28, p. 22] and, therefore, we skip it.

Definition 2.3. *A function $j : H \to \mathbb{R}$ is said to be locally Lipschitz if for any $x \in H$ there exist a neighborhood of x, U_x, and a constant L_x such that*

$$|j(u) - j(v)| \leq L_x\|u - v\|_H \quad \forall\, u, v \in U_x.$$

The Clarke directional derivative of the locally Lipschitz function $j : H \to \mathbb{R}$ at the point $u \in H$ in the direction $v \in H$ is defined by

$$j^0(u; v) = \limsup_{w\to u,\lambda\downarrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

Moreover, the generalized gradient (Clarke subdifferential) of j at u is the subset of H given by

$$\partial j(u) = \{\, \xi \in H \;:\; j^0(u;v) \geq (\xi, v)_H \quad \forall\, v \in H\,\}. \tag{2.3}$$

Recall also that if j is Lipschitz continuous of rank $L_j > 0$ then

$$|j^0(u;v)| \leq L_j \|v\|_H \qquad \forall\, u,\, v \in H. \tag{2.4}$$

For details on this topic see, e.g., [6, 22, 23, 24, 25, 29].

Given $\rho > 0$ we now consider the following particular case of the mixed hemivariational-variational problem (1.1)–(1.2).

Problem $\mathcal{P}^0$. *Find $(u, \lambda) \in X \times \Lambda$ such that*

$$(u, v-u)_X + \rho b(v-u, \lambda) + \rho J^0(Gu; Gv - Gu) \geq (g, v-u)_X,\ \forall\, v \in X, \tag{2.5}$$

$$b(u, \mu - \lambda) - c(\mu, \mu) + c(\lambda, \lambda) \leq 0 \qquad \forall\, \mu \in \Lambda. \tag{2.6}$$

In the study of this problem we introduce the following assumptions.

$\underline{H(b)}$ $\quad b : X \times Y \to \mathbb{R}$ is a bilinear form such that there exist $M_b > 0$ and $\alpha_b > 0$:

$$\left\{ \begin{array}{l} \text{(a) } |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y \qquad \forall v \in X,\, \mu \in Y; \\[2ex] \text{(b) } \displaystyle\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha_b. \end{array} \right. \tag{2.7}$$

$\underline{H(c)}$ $\quad c : Y \times Y \to \mathbb{R}$ is a symmetric, continuous and Y-elliptic bilinear form.

$\underline{H(J)}$ $\quad J\colon Z \to \mathbb{R}$ is a Lipschitz continuous function of rank $L_J > 0$ and there exists $m_J \geq 0$ sucht that

$$J^0(u_1; v_2 - v_1) + J^0(u_2; v_1 - v_2) \leq m_J\, \|u_1 - u_2\|_X \|v_1 - v_2\|_X, \tag{2.8}$$

$$\forall\, u_1,\, u_2,\, v_1,\, v_2 \in X.$$

$\underline{H(G)}$ $\quad G : X \to Z$ is a linear compact operator, i.e., $u_n \rightharpoonup u$ in X as $n \to \infty \implies Gu_n \to Gu$ in Z as $n \to \infty$.

$\underline{H(sr)}$ $\quad \rho\, m_J \|G\|^2_{\mathcal{L}(X,Z)} < 1.$

$\underline{H(\Lambda)}$ $\quad \Lambda$ is a closed convex subset of Y such that $0_Y \in \Lambda$.

$\underline{H(g)}$ $\quad g \in X$.

The following result represents a version of Theorem 7 recently proved in [18].

Proposition 2.4. *Assume $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(sr)$, $H(\Lambda)$ and $H(g)$. Then Problem $\mathcal{P}^0$ has a unique solution $(u, \lambda) \in X \times \Lambda$.*

The proof of this proposition is based on a number of preliminary results of convex analysis combined with the properties of the subdifferential in the sense of Clarke, a crucial tool being a fixed point theorem for set-valued mappings, see [30].

Proposition 2.4 allows us to introduce the operators $Q : X \to X$ and $R : X \to \Lambda$ defined as follows:

$$u = Qg, \quad \lambda = Rg \quad \Longleftrightarrow \quad (u, \lambda) \text{ is the solution to Problem } \mathcal{P}^0 \tag{2.9}$$

for any $g \in X$. Note that Problem $\mathcal{P}^0$ as well as its solution depend on the parameter $\rho > 0$. Therefore, the operators Q and R depend on ρ, too. Nevertheless, for simplicity, since no confusion arises, we do not mention this dependence. The properties of these operators will play a crucial role in the analysis of Problem $\mathcal{P}$ we shall carry out in the next two sections.

3. An existence and uniqueness result

In this section we provide the unique solvability of the hemivariational-variational problem $\mathcal{P}$. To this end, besides the assumptions already introduced in Section 2, we consider the following assumptions on the operator A, the function J and the element f.

$\underline{H(A)}$ A is a strongly monotone Lipschitz continuous operator with constants $m_A > 0$ and $L_A > 0$, i.e., it satisfies inequalities (2.1) and (2.2) with $H = X$.

$\underline{H(J)_1}$ There exists $\beta_J \geq 0$ such that

$$J^0(Gu_1; Gw) + J^0(Gu_2; -Gw) \leq \beta_J \, \|u_1 - u_2\|_X \|w\|_X \qquad \forall \, u_1, \, u_2, \, w \in X. \tag{3.1}$$

$\underline{H(s)}$ $m_J \|G\|^2_{\mathcal{L}(X,Z)} < m_A$.

$\underline{H(f)}$ $f \in X$.

Note that below in this paper we need the inequality

$$J^0(Gv_1; Gv_2 - Gv_1) + J^0(Gv_2; Gv_1 - Gv_2) \leq m_J \|G\|^2_{\mathcal{L}(X,Z)} \, \|v_1 - v_2\|^2_X \qquad \forall \, v_1, v_2 \in X, \tag{3.2}$$

with some $m_J \geq 0$ which satisfies the smallness assumption $H(s)$. Assume that condition (3.1) is satisfied. Then it follows that (3.2) holds with $m_J = \frac{\beta_J}{\|G\|^2_{\mathcal{L}(X,Z)}}$ and, in this case, $H(s)$ implies that $\beta_J < m_A$. Nevertheless, some elementary examples can be considered in which inequalities (2.8) and (3.1) hold with $m_J \|G\|^2_{\mathcal{L}(X,Z)} < \beta_J$. For this reason, to recover the case when $m_J \|G\|^2_{\mathcal{L}(X,Z)} < m_A < \beta_J$, below in this paper we consider conditions (2.8) and (3.1) separately.

Our main result in this section is the following.

Theorem 3.1. *Assume $H(A)$, $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(J)_1$, $H(\Lambda)$, $H(s)$ and $H(f)$. Then, Problem $\mathcal{P}$ has a unique solution $(u,\lambda) \in X \times \Lambda$.*

Proof. We fix $\rho > 0$ such that $\rho < \frac{1}{m_J \|G\|^2_{\mathcal{L}(X,Z)}}$ to be determined later and, besides the operators Q and R introduced in Section 2, we consider the operators S, T and U defined as follows:

$$S : X \to X, \quad Su = \rho f - \rho A u + u \qquad \forall\, u \in X, \tag{3.3}$$

$$T : X \to \Lambda, \quad Tu = Q(Su) \qquad \forall\, u \in X, \tag{3.4}$$

$$W : X \to \Lambda, \quad Wu = R(Su) \qquad \forall\, u \in X. \tag{3.5}$$

Note that these operators depend on ρ. Nevertheless, for simplicity, in this section we do not mention this dependence. We now proceed in three steps, as follows.

Step i) We prove that the following equivalence holds:

$$(u,\lambda) \in X \times \Lambda \ \text{ is a solution to Problem } \mathcal{P} \quad \Longleftrightarrow \quad u = Tu \text{ and } \quad \lambda = Wu. \tag{3.6}$$

Indeed, using (1.1)–(1.2), (2.5)–(2.6) and the definitions (2.9), (3.3)–(3.5) it is easy to see that the following equivalences hold.

$$(u,\lambda) \in X \times \Lambda \ \text{ is a solution to Problem } \mathcal{P}$$

$$\Longleftrightarrow \quad \left\{ \begin{array}{l} (u, v-u)_X + \rho b(v-u,\lambda) + \rho J^0(Gu; Gv - Gu) \\ \qquad \geq (\rho f - \rho A u + u, v-u)_X \quad \forall\, v \in X, \\[2mm] b(u, \mu - \lambda) - c(\mu,\mu) + c(\lambda,\lambda) \leq 0 \qquad \forall\, \mu \in \Lambda. \end{array} \right.$$

$$\Longleftrightarrow \quad u = Q(\rho f - \rho A u + u), \quad \lambda = R(\rho f - \rho A u + u)$$

$$\Longleftrightarrow \quad u = Q(Su), \quad \lambda = R(Su)$$

$$\Longleftrightarrow \quad u = Tu, \quad \lambda = Wu.$$

It follows from here that the statement (3.6) holds.

Step ii) We now prove the following inequalities concerning the operators Q and R:

$$\|Qg_1 - Qg_2\|_X \leq \frac{1}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|g_1 - g_2\|_X \qquad \forall\ g_1,\, g_2 \in X. \tag{3.7}$$

$$\|Rg_1 - Rg_2\|_Y \leq \frac{2 + \rho(\beta_J - m_J \|G\|^2_{\mathcal{L}(X,Z)})}{\rho \alpha_b (1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)})} \|g_1 - g_2\|_X \ \forall\ g_1,\, g_2 \in X. \tag{3.8}$$

These inequalities are necessary to study the existence of a unique fixed point of the operator T, suggested by equality $u = Tu$ in (3.6). Their proof is as follows. Let g_1, $g_2 \in X$ and, for simplicity, denote

$$u_1 = Qg_1, \quad u_2 = Qg_2, \quad \lambda_1 = Rg_1, \quad \lambda_2 = Rg_2. \tag{3.9}$$

Then, using the definition (2.9) we deduce that

$$(u_1, v-u_1)_X + \rho b(v-u_1, \lambda_1) + \rho J^0(Gu_1; Gv-Gu_1) \geq (g_1, v-u_1)_X \ \forall\, v \in X, \quad (3.10)$$

$$b(u_1, \mu - \lambda_1) - c(\mu, \mu) + c(\lambda_1, \lambda_1) \leq 0 \qquad \forall\, \mu \in \Lambda \quad (3.11)$$

and, moreover,

$$(u_2, v-u_2)_X + \rho b(v-u_2, \lambda_2) + \rho J^0(Gu_2; Gv-Gu_2) \geq (g_2, v-u_2)_X \quad \forall\, v \in X, \quad (3.12)$$

$$b(u_2, \mu - \lambda_2) - c(\mu, \mu) + c(\lambda_2, \lambda_2) \leq 0 \qquad \forall\, \mu \in \Lambda. \quad (3.13)$$

We now take $v = u_2$ in (3.10), then $v = u_1$ in (3.12) and add the resulting inequalities to find that

$$\|u_1 - u_2\|_X^2 \leq \rho b(u_2 - u_1, \lambda_1 - \lambda_2)$$
$$+\rho J^0(Gu_1; Gu_2 - Gu_1) + \rho J^0(Gu_2; Gu_1 - Gu_2) + (g_1 - g_2, u_1 - u_2)_X.$$

Next, we use inequality (3.2) to deduce that

$$(1-\rho m_J \|G\|^2_{\mathcal{L}(X,Z)})\|u_1-u_2\|_X^2 \leq \rho b(u_2-u_1, \lambda_1-\lambda_2) + \|g_1-g_2\|_X \|u_1-u_2\|_X. \quad (3.14)$$

On the other hand, we take $\mu = \lambda_2$ in (3.11), then $\mu = \lambda_1$ in (3.13) and add the resulting inequalities to find that

$$b(u_2 - u_1, \lambda_1 - \lambda_2) \leq 0. \quad (3.15)$$

We now combine inequalities (3.14) and (3.15) to deduce that

$$(1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)})\|u_1 - u_2\|_X \leq \|g_1 - g_2\|_X.$$

As $1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)} > 0$, we immediately get (3.7).

Consider now an arbitrary element $w \in X$, $w \neq 0_X$. We take $v = u_1 + w$ in (3.10), then $v = u_2 - w$ in (3.12) and add the resulting inequalities to find that

$$\rho b(w, \lambda_2 - \lambda_1) \leq (u_1 - u_2, w)_X + (g_1 - g_2, w)_X + \rho J^0(Gu_1; Gw) + \rho J^0(Gu_2; -Gw).$$

Then, using assumption $H(J)_1$ yields

$$\rho b(w, \lambda_2 - \lambda_1) \leq \Big[(1 + \rho\beta_J)\|u_1 - u_2\|_X + \|g_1 - g_2\|_X\Big]\|w\|_X. \quad (3.16)$$

Assume now that $\lambda_1 \neq \lambda_2$. Then inequality (3.16) implies that

$$\rho\, \frac{b(w, \lambda_2 - \lambda_1)}{\|w\|_X \|\lambda_1 - \lambda_2\|_Y} \leq \Big[(1 + \rho\beta_J)\|u_1 - u_2\|_X + \|g_1 - g_2\|_X\Big] \frac{1}{\|\lambda_1 - \lambda_2\|_Y}$$

and, using assumption (2.7)(b) we find that

$$\|\lambda_1 - \lambda_2\|_Y \leq \frac{1}{\rho\alpha_b}\Big[(1 + \rho\beta_J)\|u_1 - u_2\|_X + \|g_1 - g_2\|_X\Big]. \quad (3.17)$$

Note that, obviously, this inequality holds even in the case when $\lambda_1 = \lambda_2$. We now use (3.17), equalities (3.9) and inequality (3.7) to deduce that (3.8) holds, which concludes the proof of this step.

Step iii) We prove that, with a convenient choice of ρ, the operator $T : X \to X$ is a contraction. Indeed, assume that u_1, $u_2 \in X$ and recall the definition (3.4) which shows that $Tu_1 = Q(Su_1)$ and $Tu_2 = Q(Su_2)$. Then, using inequality (3.7) with $g_1 = Su_1$ and $g_2 = Su_2$ we find that

$$\|Tu_1 - Tu_2\|_X = \|Q(Su_1) - Q(Su_2)\|_X \le \frac{1}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|Su_1 - Su_2\|_X$$

and, therefore, (3.3) yields

$$\|Tu_1 - Tu_2\|_X \le \frac{1}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|(u_1 - \rho A u_1) - (u_2 - \rho A u_2)\|_X.$$

We now use assumption $H(A)$ and Lemma 2.2 to deduce that

$$\|Tu_1 - Tu_2\|_X \le \frac{k(\rho)}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|u_1 - u_2\|_X \tag{3.18}$$

where, recall, $k(\rho) = (1 - 2\rho m_A + \rho^2 L_A^2)^{\frac{1}{2}}$.

Consider now the real valued function F given by

$$F(\rho) = k(\rho) + \rho m_J \|G\|^2_{\mathcal{L}(X,Z)} = (1 - 2\rho m_A + \rho^2 L_A^2)^{\frac{1}{2}} + \rho m_J \|G\|^2_{\mathcal{L}(X,Z)},$$

for all $\rho \in \Big(0, \frac{1}{m_J \|G\|^2_{\mathcal{L}(X,Z)}}\Big)$. Then, using the smallness assumption $H(s)$ we deduce that $F'(0) = m_J \|G\|^2_{\mathcal{L}(X,Z)} - m_A < 0$. This implies that F is strictly decreasing in a neighborhood of the origin and, since $F(0) = 1$, we deduce that for $\rho > 0$ small enough we can assume that $F(\rho) < 1$. We now use inequality (3.18) to see that for such ρ the operator T is a contraction, as claimed.

Step iv) Existence and uniqueness. We use Step iii) to chose ρ such that the operator T is a contraction on X. Then, the Banach contraction principle implies that there exists a unique element $u \in X$ such that $u = Tu$. We now define $\lambda = Wu$ and use (3.6) to see that the pair (u, λ) is a solution to Problem $\mathcal{P}$. This concludes the existence part of the theorem. The uniqueness part is a direct consequence of the equivalence (3.6) and the uniqueness of the fixed point of the operator T. □

4. A continuous dependence result

In this section we study the continuous dependence of the solution of Problem $\mathcal{P}$ with respect to the data A, b, c and f. To this end, we assume in what follows that $H(A)$, $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(J)_1$, $H(\Lambda)$, $H(s)$ and $H(f)$ hold and we consider the sequences $\{A_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{f_n\}$ such that, for each $n \in \mathbb{N}$, the following conditions hold.

$\underline{H(A_n)}$ $A_n : X \to X$ is a strongly monotone Lipschitz continuous operator with positive constants m_n and L_n.

$\underline{H(b_n)}$ $b_n : X \times Y \to \mathbb{R}$ is a bilinear form which satisfies condition (2.7) with positive constants α_n and M_n.

$\underline{H(c_n)}$ $c_n : Y \times Y \to \mathbb{R}$ is a symmetric, continuous and Y-elliptic bilinear form.

$\underline{H(sn)}$ $\quad m_J\|G\|^2_{\mathcal{L}(X,Z)} < m_n$.

$\underline{H(f_n)}$ $\quad f_n \in X$.

Then, using Theorem 3.1 it follows that for each $n \in \mathbb{N}$ there exists a unique solution to the following problem.

Problem $\mathcal{P}_n$. *Find* $(u_n, \lambda_n) \in X \times \Lambda$ *such that*

$$(A_n u_n, v - u_n)_X + b_n(v - u_n, \lambda_n) + J^0(Gu_n; Gv - Gu_n) \geq (f_n, v - u_n)_X \quad \forall\, v \in X,$$

$$b_n(u_n, \mu - \lambda_n) - c_n(\mu, \mu) + c_n(\lambda_n, \lambda_n) \leq 0 \qquad \forall\, \mu \in \Lambda.$$

Consider now the following additional assumptions.

$$A_n v \to Av \quad \text{for all } v \in X, \quad \text{as} \quad n \to \infty. \tag{4.1}$$

$$\text{There exist } m_0,\ L_0 > 0 \text{ such that } m_0 \leq m_n \leq L_n \leq L_0 \quad \forall\, n \in \mathbb{N}. \tag{4.2}$$

$$\text{There exists } \alpha_0 > 0 \text{ such that } \ \alpha_0 \leq \alpha_n \quad \forall\, n \in \mathbb{N}. \tag{4.3}$$

$$\begin{cases} \text{For each } n \in \mathbb{N} \text{ there exists } d_n \geq 0 \ \text{ such that} \\ \text{(a) } |b_n(u,\lambda) - b(u,\lambda)| \leq d_n \|u\|_X \|\lambda\|_Y \quad \text{for all } u \in X,\ \lambda \in Y. \\ \text{(b) } d_n \to 0 \ \text{ as } \ n \to \infty. \end{cases} \tag{4.4}$$

$$\begin{cases} \text{For each } n \in \mathbb{N} \text{ there exists } \theta_n \geq 0 \ \text{ such that} \\ \text{(a) } |c_n(\mu,\lambda) - c(\mu,\lambda)| \leq \theta_n \|\mu\|_Y \|\lambda\|_Y \quad \text{for all } \mu,\ \lambda \in Y. \\ \text{(b) } \theta_n \to 0 \ \text{ as } \ n \to \infty. \end{cases} \tag{4.5}$$

$$f_n \to f \quad \text{in} \quad X. \tag{4.6}$$

Our main result in this section is the following.

Theorem 4.1. *Assume* $H(A)$, $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(J)_1$, $H(\Lambda)$, $H(s)$ *and* $H(f)$ *and, for each* $n \in \mathbb{N}$ *assume* $H(A_n)$, $H(b_n)$, $H(c_n)$, $H(sn)$ *and* $H(f_n)$. *Moreover, assume that* (4.1)–(4.6) *hold. Then, the solution* (u_n, λ_n) *of Problem* $\mathcal{P}_n$ *converges to the solution* (u, λ) *of Problem* $\mathcal{P}$, *i.e.,*

$$u_n \to u \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty. \tag{4.7}$$

$$\lambda_n \to \lambda \quad \text{in } Y, \quad \text{as} \quad n \to \infty. \tag{4.8}$$

Proof. The proof is split into four steps, as follows.

Step i) Preliminaries. Let $n \in \mathbb{N}$, $\rho \in \left(0, \frac{1}{m_J\|G\|^2_{\mathcal{L}(X,Z)}}\right)$ and denote by Problem $\mathcal{P}^0_n$ the problem obtained replacing in Problem $\mathcal{P}^0$ the bilinear forms b and c by the forms

b_n and c_n, respectively. It follows from Proposition 2.4 that we are in a position to introduce the operators $Q_n : X \to X$ and $R_n : X \to \Lambda$ defined by the equivalence below.

$$u = Q_n g, \quad \lambda = R_n g \quad \Longleftrightarrow \quad (u, \lambda) \text{ is the solution to Problem } \mathcal{P}_n^0 \tag{4.9}$$

for any $g \in X$. Consider also the operators S_n, T_n and W_n defined as follows

$$S_n : X \to X, \quad S_n u = \rho f_n - \rho A_n u + u \qquad \forall\, u \in X, \tag{4.10}$$

$$T_n : X \to X, \quad T_n u = Q_n(S_n u) \qquad \forall\, u \in X, \tag{4.11}$$

$$W_n : X \to \Lambda, \quad W_n u = R_n(S_n u) \qquad \forall\, u \in X.$$

Then, it follows from the proof of Step i) in Theorem 3.1 that

$$u_n = T_n u_n = Q_n(S_n u_n) \quad \text{and} \quad \lambda_n = W_n u_n = R_n(S_n u_n).$$

Moreover, inequalities (3.7) and (3.8) combined with assumption (4.3) imply that

$$\|Q_n g_1 - Q_n g_2\|_X \le \frac{1}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|g_1 - g_2\|_X \qquad \forall\, g_1,\, g_2 \in X, \tag{4.12}$$

$$\|R_n g_1 - R_n g_2\|_Y \le \frac{2 + \rho(\beta_J - m_J \|G\|^2_{\mathcal{L}(X,Z)})}{\rho\alpha_0 (1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)})} \|g_1 - g_2\|_X \;\forall\, g_1,\, g_2 \in X. \tag{4.13}$$

Let $m = \min\{m_0, m_A\}$, $L = \max\{L_0, L_A\}$ and let $k(\rho) = (1 - 2\rho m + \rho^2 L^2)^{\frac{1}{2}}$. Then, it follows from $H(A)$, $H(A_n)$ and (4.2) that the operators A_n and A are strongly monotone and Lipschitz continuous with constants m and L. In other words, we may assume that the constants of strong monotonicity and Lipschitz continuity of the operators A_n and A are the same, and are denoted by m and L, respectively. This property combined with (3.4), (3.7) and Lemma 2.2 shows that

$$\|Tu_1 - Tu_2\|_X \le \frac{k(\rho)}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|u_1 - u_2\|_X \quad \forall\, u_1,\, u_2 \in X.$$

A similar argument based on (4.11) and (4.12) yields

$$\|T_n u_1 - T_n u_2\|_X \le \frac{k(\rho)}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}} \|u_1 - u_2\|_X \quad \forall u_1,\, u_2 \in X.$$

As a consequence, a careful analysis of the proof of Step iii) in Theorem 3.1 shows that we can choose a positive real $\rho_0 \in \left(0, \frac{1}{m_J \|G\|^2_{\mathcal{L}(X,Z)}}\right)$ such that the operators T and T_n are contractions with the same constant $k_0 \in [0, 1)$, which does not depend on n. Therefore,

$$\|T_n u_1 - T_n u_2\|_X \le k_0 \|u_1 - u_2\|_X, \quad \|Tu_1 - Tu_2\|_X \le k_0 \|u_1 - u_2\|_X \tag{4.14}$$

for all $u_1,\, u_2 \in X$.

Step ii) We prove that for each $g \in X$ the following convergences hold.

$$Q_n g \to Qg \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty, \tag{4.15}$$

$$R_n g \to Rg \quad \text{in } Y, \quad \text{as} \quad n \to \infty. \tag{4.16}$$

Let $g \in X$ and, for simplicity, denote

$$w_n = Q_n g, \quad w = Qg, \quad \delta_n = R_n g, \quad \delta = Rg. \tag{4.17}$$

Then, using the definition (2.9) and (4.9) we deduce that

$$(w, v - w)_X + \rho b(v - w, \delta) + \rho J^0(Gw; Gv - Gw) \geq (g, v - w)_X \quad \forall\, v \in X, \tag{4.18}$$

$$b(w, \mu - \delta) - c(\mu, \mu) + c(\delta, \delta) \leq 0 \qquad \forall\, \mu \in \Lambda \tag{4.19}$$

and

$$(w_n, v - w_n)_X + \rho b_n(v - w_n, \delta_n) + \rho J^0(Gw_n; Gv - Gw_n) \geq (g, v - w_n)_X \quad \forall\, v \in X, \tag{4.20}$$

$$b_n(w_n, \mu - \delta_n) - c_n(\mu, \mu) + c_n(\delta_n, \delta_n) \leq 0 \qquad \forall\, \mu \in \Lambda. \tag{4.21}$$

We now take $v = w$ in (4.20), then $v = w_n$ in (4.18) and add the resulting inequalities to find that

$$\|w_n - w\|_X^2 \leq \rho b(w_n - w, \delta) + \rho b_n(w - w_n, \delta_n)$$
$$+\rho J^0(Gw_n; Gw - Gw_n) + \rho J^0(Gw; Gw_n - Gw).$$

Next, we use (3.2) to deduce that

$$(1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)})\|w_n - w\|_X^2 \leq \rho b(w_n - w, \delta) + \rho b_n(w - w_n, \delta_n). \tag{4.22}$$

On the other hand, since

$$b(w_n - w, \delta) + b_n(w - w_n, \delta_n) = b(w_n - w, \delta - \delta_n) + b(w_n - w, \delta_n) + b_n(w - w_n, \delta_n),$$

using (4.4), we get

$$b(w_n - w, \delta) + b_n(w - w_n, \delta_n) \leq b(w_n - w, \delta - \delta_n) + d_n \|w_n - w\|_X \|\delta_n\|_Y. \tag{4.23}$$

Consequently, (4.22) and (4.23) yield

$$(1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)})\|w_n - w\|_X^2 \leq \rho b(w_n - w, \delta - \delta_n) + \rho d_n \|w_n - w\|_X \|\delta_n\|_Y. \tag{4.24}$$

Next, we take $\mu = \delta$ in (4.21), then $\mu = \delta_n$ in (4.19) and add the resulting inequalities to find that

$$b(w, \delta_n - \delta) + b_n(w_n, \delta - \delta_n) \leq c(\delta_n, \delta_n) - c(\delta, \delta) + c_n(\delta, \delta) - c_n(\delta_n, \delta_n).$$

Therefore,

$$b(w - w_n, \delta_n - \delta) \leq b_n(w_n, \delta_n - \delta) - b(w_n, \delta_n - \delta)$$
$$+c(\delta_n, \delta_n) - c(\delta, \delta) + c_n(\delta, \delta) - c_n(\delta_n, \delta_n).$$

Using again (4.4) we can write

$$b(w - w_n, \delta_n - \delta) \le d_n \|w_n\|_X \|\delta_n - \delta\|_Y$$
$$+ c(\delta_n, \delta_n) - c(\delta, \delta) + c_n(\delta, \delta) - c_n(\delta_n, \delta_n).$$

As

$$c(\delta_n, \delta_n) - c(\delta, \delta) + c_n(\delta, \delta) - c_n(\delta_n, \delta_n)$$
$$= c(\delta_n, \delta_n - \delta) - c_n(\delta_n, \delta_n - \delta) + c(\delta_n - \delta, \delta) - c_n(\delta_n - \delta, \delta),$$

using assumption (4.5) we deduce that

$$c(\delta_n, \delta_n) - c(\delta, \delta) + c_n(\delta, \delta) - c_n(\delta_n, \delta_n) \le \theta_n \big(\|\delta_n\|_Y + \|\delta\|_Y\big) \|\delta_n - \delta\|_Y. \tag{4.25}$$

We now combine inequalities (4.24)–(4.25) to deduce that

$$(1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}) \|w_n - w\|_X^2 \le \rho d_n \|w_n\|_X \|\delta_n - \delta\|_Y$$
$$+ \rho \theta_n \big(\|\delta_n\|_Y + \|\delta\|_Y\big) \|\delta_n - \delta\|_Y + \rho d_n \|w_n - w\|_X \|\delta_n\|_Y$$

and, therefore,

$$(1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}) \|w_n - w\|_X^2 \le \rho d_n \|w_n\|_X (\|\delta_n\|_Y + \|\delta\|_Y) \tag{4.26}$$
$$+ \rho \theta_n \big(\|\delta_n\|_Y + \|\delta\|_Y\big)^2 + \rho d_n (\|w_n\|_X + \|w\|_X) \|\delta_n\|_Y.$$

Note that the sequences $\{w_n\} \subset X$ and $\{\delta_n\} \subset \Lambda$ are bounded. Indeed, to prove this statement we fix $n \in \mathbb{N}$. We set $\mu = 0_Y$ in (4.21) to find that

$$b_n(w_n, -\delta_n) \le 0 \tag{4.27}$$

Next, we set $v = 0_X$ in (4.20) and use (4.27), (2.4) to deduce that

$$\|w_n\|_X^2 \le \rho J^0(Gw_n; -Gw_n) + (g, w_n)_X \le \rho\, L_J \, \|G\|_{\mathcal{L}(X,Z)} \|w_n\|_X + \|g\|_X \|w_n\|_X.$$

As a result, we have

$$\|w_n\|_X \le \rho\, L_J \, \|G\|_{\mathcal{L}(X,Z)} + \|g\|_X. \tag{4.28}$$

Moreover, using (4.20) with $v = w_n - \frac{\widetilde{w}}{\|\widetilde{w}\|_X}$ where $\widetilde{w}$ is an arbitrary element of X such that $\widetilde{w} \ne 0_X$, we find that

$$\rho \frac{b_n(\widetilde{w}, \delta_n)}{\|\widetilde{w}\|_X} \le -(w_n, \frac{\widetilde{w}}{\|\widetilde{w}\|_X}) + \rho J^0(Gw_n; -G\frac{\widetilde{w}}{\|\widetilde{w}\|_X}) + (g, \frac{\widetilde{w}}{\|\widetilde{w}\|_X})_X.$$

Consequently, (4.3) and (2.4) imply that

$$\rho\, \alpha_n \, \|\delta_n\|_Y \le \|w_n\|_X + \rho\, L_J \, \|G\|_{\mathcal{L}(X,Z)} + \|g\|_X.$$

Using now (4.28) and (4.3) we deduce that

$$\|\delta_n\|_Y \le \frac{2}{\rho\, \alpha_0} (\rho\, L_J \, \|G\|_{\mathcal{L}(X,Z)} + \|g\|_X). \tag{4.29}$$

Inequalities (4.28) and (4.29) show that the sequences $\{w_n\}$ and $\{\delta_n\}$ are bounded in X and Y, respectively, as claimed. Therefore, using (4.26) and assumptions $H(sr)$, (4.4) (b) and (4.5) (b) we find that

$$w_n \to w \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty. \tag{4.30}$$

Consider now an arbitrary element $\widetilde{w} \in X$, $\widetilde{w} \neq 0_X$. We take $v = w + \widetilde{w}$ in (4.18), then $v = w_n - \widetilde{w}$ in (4.20), add the resulting inequalities and use $H(J)_1$ to find that

$$\rho\, b_n(\widetilde{w}, \delta_n) - \rho b(\widetilde{w}, \delta) \leq (w - w_n, \widetilde{w})_X + \rho\, \beta_J\, \|w - w_n\|_X \|\widetilde{w}\|_X.$$

Hence,

$$\begin{aligned}\rho\, b(\widetilde{w}, \delta_n - \delta) &\leq \rho\, b(\widetilde{w}, \delta_n) - \rho\, b_n(\widetilde{w}, \delta_n)\\ &\quad + \|w - w_n\|_X \|\widetilde{w}\|_X + \rho\, \beta_J\, \|w - w_n\|_X \|\widetilde{w}\|_X,\end{aligned}$$

and, keeping in mind $H(b)$ (b) and (4.4), we get

$$\rho\, \alpha_b\, \|\delta_n - \delta\|_Y \leq \rho\, d_n\, \|\delta_n\|_Y + (\rho\, \beta_J + 1)\|w - w_n\|_X.$$

We now use (4.29), (4.4) (b), and (4.30) to obtain that

$$\delta_n \to \delta \quad \text{in} \quad Y, \quad \text{as} \quad n \to \infty. \tag{4.31}$$

Finally, we use notation (4.17) and the convergences (4.30), (4.31) to deduce (4.15) and (4.16), respectively.

Step iii) We prove the convergence (4.7). Let $n \in \mathbb{N}$. We use equalities $u_n = T_n u_n$ and $u = Tu$ to deduce that

$$\|u_n - u\|_X = \|T_n u_n - Tu\|_X \leq \|T_n u_n - T_n u\|_X + \|T_n u - Tu\|_X,$$

and, using (4.14) we find that

$$\|u_n - u\|_X \leq k_0\, \|u_n - u\|_X + \|T_n u - Tu\|_X.$$

Equivalently,

$$\|u_n - u\|_X \leq \frac{1}{1 - k_0}\, \|T_n u - Tu\|_X. \tag{4.32}$$

Next, using (4.11) and (3.4) yields

$$\|T_n u - Tu\|_X \leq \|Q_n(S_n u) - Q_n(Su)\|_X + \|Q_n(Su) - Q(Su)\|_X. \tag{4.33}$$

We shall prove that each term in the right hand side of this inequality converges to zero. To this end we use the bound (4.12) to see that

$$\|Q_n(S_n u) - Q_n(Su)\|_X \leq \frac{1}{1 - \rho m_J \|G\|^2_{\mathcal{L}(X,Z)}}\, \|S_n u - Su\|_X. \tag{4.34}$$

Next, using assumptions (4.1) and (4.6) and definitions (4.10) and (3.3) of the operators S_n and S it follows that

$$S_n u \to Su \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty. \tag{4.35}$$

We now combine inequality (4.34) and convergence (4.35) to find that

$$Q_n(S_n u) - Q_n(Su) \to 0_X \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty. \tag{4.36}$$

On the other hand we use (4.15) to deduce that

$$Q_n(Su) \to Q(Su) \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty. \tag{4.37}$$

The convergence (4.7) is now a direct consequence of the inequalities (4.32), (4.33) and the convergences (4.36), (4.37).

Step iv) We prove the convergence (4.8). Let $n \in \mathbb{N}$. We use equalities $\lambda_n = W_n u_n$ and $\lambda = Wu$ to deduce that

$$\|\lambda_n - \lambda\|_Y = \|W_n u_n - Wu\|_Y \leq \|W_n u_n - W_n u\|_Y + \|W_n u - Wu\|_Y,$$

and, using (4.11), (3.5), we find that

$$\|\lambda_n - \lambda\|_Y \leq \|R_n(S_n u) - R_n(Su)\|_Y + \|R_n(Su) - R(Su)\|_Y. \tag{4.38}$$

We shall prove that each term in the right hand side of this inequality converges to zero. To this end we use the bound (4.13) to see that

$$\|R_n(S_n u) - R_n(Su)\|_Y \leq \frac{2 + \rho(\beta_J - m_J\|G\|^2_{\mathcal{L}(X,Z)})}{\rho\alpha_0(1 - \rho m_J\|G\|^2_{\mathcal{L}(X,Z)})}\,\|S_n u - Su\|_X.$$

Then, using (4.35) we find that

$$R_n(S_n u) - R_n(Su) \to 0_Y \quad \text{in} \quad Y, \quad \text{as} \quad n \to \infty. \tag{4.39}$$

On the other hand we use (4.16) to deduce that

$$R_n(Su) \to R(Su) \quad \text{in} \quad Y, \quad \text{as} \quad n \to \infty. \tag{4.40}$$

The convergence (4.8) is now a direct consequence of the inequality (4.38) and the convergences (4.39), (4.40). □

5. A convergence criterion

In this section we formulate a criterion of convergence to the solution of the mixed hemivariational-variational Problem $\mathcal{P}$. Our main result in this section is the following.

Theorem 5.1. *Assume that $H(A)$, $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(J)_1$, $H(\Lambda)$, $H(s)$ and $H(f)$ hold, let $\{(u_n,\lambda_n)\} \subset X \times \Lambda$ be an arbitrary sequence and let (u,λ) be the solution of Problem $\mathcal{P}$ obtained in Theorem 3.1. Then, the convergences*

$$u_n \to u \quad \textit{in} \quad X, \quad \textit{as } n \to \infty, \tag{5.1}$$

$$\lambda_n \to \lambda \quad \textit{in} \quad Y, \quad \textit{as } n \to \infty, \tag{5.2}$$

hold if and only if the following convergences hold, too:

$$u_n - Tu_n \to 0_X \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty, \tag{5.3}$$

$$\lambda_n - Wu_n \to 0_Y \quad \text{in} \quad Y, \quad \text{as} \quad n \to \infty. \tag{5.4}$$

Proof. Assume that (5.1) and (5.2) hold. Let $n \in \mathbb{N}$. Then, using equality $u = Tu$, see (3.6), we have

$$\|u_n - Tu_n\|_X \leq \|u_n - u\|_X + \|Tu_n - Tu\|_X,$$

and, using (4.14) we find that

$$\|u_n - Tu_n\|_X \leq (1 + k_0)\,\|u_n - u\|_X.$$

We now use this inequality and the convergence (5.1) to see that (5.3) holds.

On the other hand, using equality $\lambda = Wu$ in (3.6) and the definition (3.5) we have

$$\|\lambda_n - Wu_n\|_Y \le \|\lambda_n - \lambda\|_Y + \|R(Su) - R(Su_n)\|_Y. \tag{5.5}$$

It follows from (3.3) and (5.1) that

$$Su_n \to Su \quad \text{in} \quad X, \quad \text{as} \ \ n \to \infty,$$

and, using (3.8), we deduce that

$$\|R(Su) - R(Su_n)\|_Y \to 0 \ \ \text{as} \ \ n \to \infty. \tag{5.6}$$

We now combine (5.5), (5.2) and (5.6) and find that (5.4) holds, too.

Conversely, assume now that (5.3) and (5.4) hold. Let $n \in \mathbb{N}$. We use equality $u = Tu$, again, to see that

$$\|u_n - u\|_X \le \|u_n - Tu_n\|_X + \|Tu_n - Tu\|_X.$$

Next, using (4.14) we find that

$$\|u_n - u\|_X \le \|u_n - Tu_n\|_X + k_0 \, \|u_n - u\|_X.$$

This implies that

$$\|u_n - u\|_X \le \frac{1}{1 - k_0} \|u_n - Tu_n\|_X$$

and, using (5.3) we deduce that (5.1) holds.

Finally, since $\lambda = Wu$, using (3.5), again, we find that

$$\|\lambda_n - \lambda\|_Y \le \|\lambda_n - Wu_n\|_Y + \|R(Su_n) - R(Su)\|_Y. \tag{5.7}$$

The convergences (5.4) and (5.1), combined with definition (3.3) and inequality (3.8), show that each term in inequality (5.7) converges to zero. This implies that (5.2) holds and concludes the proof. □

We complete the statement of Theorem 5.1 with the following remark.

Remark 5.2. *Theorem* 5.1 *provides necessary and sufficient conditions which guarantee the convergences* (5.1) *and* (5.2)*, i.e., it represents a criterion of convergence. This criterion is intrinsic, since no reference to the solution* (u, λ) *of Problem* $\mathcal{P}$ *is made in the statement of conditions* (5.3) *and* (5.4)*.*

6. Two examples

In this section we present two examples of mixed hemivariational-variational problems. The first one illustrates our abstract results in Theorems 3.1–5.1. The second one provides an existence, uniqueness and continuous dependence result in the study of a nonlinear elastic constitutive law with internal state variable.

Example 1. We consider Problem $\mathcal{P}$ in the particular case when $X = Y = Z$,

$$Av = v \qquad \forall\, v \in X,$$

$$b(v,\mu) = (v,\mu)_X \qquad \forall\, v,\ \mu \in X,$$

$$c(\mu,\lambda) = \frac{1}{2}(\mu,\lambda)_X \qquad \forall\, \mu,\lambda \in X,$$

$$J \equiv 0,$$

and we assume that $H(\Lambda)$, $H(f)$ hold. Note that in this particular case Problem $\mathcal{P}$ is stated as follows.

Problem $\widetilde{\mathcal{P}}$. *Find $(u,\lambda) \in X \times \Lambda$ such that*

$$(u, v-u)_X + (v-u,\lambda)_X \geq (f, v-u)_X \quad \forall\, v \in X, \tag{6.1}$$

$$(u, \mu-\lambda)_X - \frac{1}{2}\,\|\mu\|_X^2 + \frac{1}{2}\,\|\lambda\|_X^2 \leq 0 \qquad \forall\, \mu \in \Lambda. \tag{6.2}$$

It is easy to see that in this case assumptions $H(A)$, $H(b)$, $H(c)$, $H(J)$, $H(G)$, $H(J)_1$, and $H(s)$ hold with $m_A = L_A = 1$, $m_J = \beta_J = 0$ and any linear compact operator G. Therefore, Theorem 3.1 guarantees the unique solvability of Problem $\widetilde{\mathcal{P}}$.

We now provide an explicit formula for the solution of Problem $\widetilde{\mathcal{P}}$. First, it is easy to see that (6.1)–(6.2) is equivalent with the system

$$(u + \lambda - f, v-u)_X = 0 \quad \forall\, v \in X,$$

$$\|\lambda - u\|_X^2 \leq \|\mu - u\|_X^2 \quad \forall\, \mu \in \Lambda$$

and, denoting by $P_\Lambda : X \to \Lambda$ the projection operator on Λ, we obtain

$$u + \lambda = f, \qquad \lambda = P_\Lambda u. \tag{6.3}$$

Let $B : X \to X$ be the operator given by

$$Bv = v + P_\Lambda v \qquad \forall\, v \in X. \tag{6.4}$$

Then, it is easy to see that B is a strongly monotone Lipschitz continuous operator with constants $m_B = 1$ and $L_B = 2$. Therefore, Lemma 2.2 a) guarantees that B is invertible and its inverse B^{-1} is a strongly monotone Lipschitz continuous operator with constants $m_{B^{-1}} = \frac{1}{4}$ and $L_{B^{-1}} = 1$. In addition, (6.3) shows that the solution of Problem $\widetilde{P}$ above exists, is unique, and is given by

$$u = B^{-1}f, \qquad \lambda = P_\Lambda u = P_\Lambda(B^{-1}f). \tag{6.5}$$

The unique solvability of Problem $\widetilde{P}$ obtained above represents a validation of the statement in Theorem 3.1.

We now move to Theorem 4.1 and, to this end, we assume that $\{a_n\} \subset \mathbb{R}_+$, $\{\beta_n\} \subset \mathbb{R}_+$, $\{\gamma_n\} \subset \mathbb{R}_+$ and $\{f_n\} \subset X$ are sequences such that (4.6) holds and, moreover,

$$a_n \to 1, \tag{6.6}$$

$$\beta_n \to 1, \quad \gamma_n \to 1. \tag{6.7}$$

Then, for each $n \in \mathbb{N}$ we define the operators A_n and the forms b_n, c_n by equalities

$$A_n v = a_n v \qquad \forall\, v \in X,$$

$$b_n(v,\mu) = \beta_n (v,\mu)_X \qquad \forall\, v,\ \mu \in X,$$

$$c_n(\mu,\lambda) = \frac{\gamma_n}{2}(\mu,\lambda)_X \qquad \forall\, \mu,\lambda \in X.$$

Moreover, as previously, we assume that $J \equiv 0$ and $H(\Lambda)$, $H(f)$ hold. Note that in this particular case Problem $\mathcal{P}_n$ is stated as follows.

Problem $\widetilde{\mathcal{P}}_n$. *Find $(u_n, \lambda_n) \in X \times \Lambda$ such that*

$$(a_n u_n, v - u_n)_X + \beta_n (v - u_n, \lambda_n)_X \geq (f_n, v - u_n)_X \quad \forall\, v \in X,$$

$$\beta_n (u_n, \mu - \lambda_n)_X - \frac{\gamma_n}{2}\|\mu\|_X^2 + \frac{\gamma_n}{2}\|\lambda_n\|_X^2 \leq 0 \qquad \forall\, \mu \in \Lambda.$$

It is easy to see that all the assumptions of Theorems 3.1 and 4.1 hold. This guarantees the unique solvability of Problem $\widetilde{\mathcal{P}}_n$ as well as its convergence of its solution to the solution of Problem $\widetilde{\mathcal{P}}$, see (4.7) and (4.8).

This convergence can be proved directly. Indeed, for each $n \in \mathbb{N}$ denote by $B_n : X \to X$ the operator given by

$$B_n v = a_n v + \beta_n P_\Lambda\Big(\frac{\beta_n}{\gamma_n}\, v\Big) \qquad \forall\, v \in X. \tag{6.8}$$

Then, it is easy to see that B_n is a strongly monotone Lipschitz continuous operator with constants $m_n = a_n$ and $L_n = a_n + \frac{\beta_n^2}{\gamma_n}$. Therefore, Lemma 2.2 a) guarantees that B_n is invertible and its inverse B_n^{-1} is a strongly monotone Lipschitz continuous operator with constants

$$m_{B_n^{-1}} = \frac{a_n}{\Big(a_n + \frac{\beta_n^2}{\gamma_n}\Big)^2} \qquad \text{and} \qquad L_{B_n^{-1}} = \frac{1}{a_n}\,. \tag{6.9}$$

In addition, similar arguments as those used in the proof of (6.5) show that

$$u_n = B_n^{-1} f_n, \qquad \lambda_n = P_\Lambda\Big(\frac{\beta_n}{\gamma_n} u_n\Big) = P_\Lambda\Big(\frac{\beta_n}{\gamma_n} B_n^{-1} f_n\Big). \tag{6.10}$$

We now state and prove the following claim.

Claim. *Under the previous assumptions, the following convergence holds:*

$$B_n^{-1} f_n \to B^{-1} f \quad \text{in} \quad X. \tag{6.11}$$

Proof. Let $n \in \mathbb{N}$ and denote

$$B_n^{-1} f = z_n, \qquad B^{-1} f = z \tag{6.12}$$

which implies that

$$B_n z_n = Bz = f. \tag{6.13}$$

We use the strong monotonicity of the operator B_n with constant a_n together with equality (6.13) to see that

$$a_n\|z_n - z\|_X^2 \le (B_n z_n - B_n z, z_n - z)_X = (Bz - B_n z, z_n - z)_X$$

which implies that

$$\|z_n - z\|_X \le \frac{1}{a_n}\,\|B_n z - Bz\|_X. \tag{6.14}$$

We now use the definitions (6.8) and (6.4) combined with the convergences (6.6) and (6.7) to deduce that

$$\|B_n z - Bz\|_X \to 0. \tag{6.15}$$

Therefore, (6.12), (6.14) and (6.15) yield

$$\|B_n^{-1} f - B^{-1} f\|_X \to 0. \tag{6.16}$$

Next, we write

$$\|B_n^{-1} f_n - B^{-1} f\|_X \le \|B_n^{-1} f_n - B_n^{-1} f\|_X + \|B_n^{-1} f - B^{-1} f\|_X$$

and, therefore, the Lipschitz continuity of the operator B_n^{-1} with constant $L_{B_n^{-1}}$ in (6.9) implies that

$$\|B_n^{-1} f_n - B^{-1} f\|_X \le \frac{1}{a_n}\,\|f_n - f\|_X + \|B_n^{-1} f - B^{-1} f\|_X. \tag{6.17}$$

We now combine inequality (6.17) with convergences (6.6), (4.6) and (6.16) to deduce (6.11), which concludes the proof of the claim. □

Next we use equalities (6.10), (6.5) and (6.11) to deduce that (4.7) holds. The convergence (4.8) is now a direct consequence of equalities (6.10), (6.5) and convergences $\beta_n \to 1$, $\gamma_n \to 1$, see (6.7).

We now illustrate the use of Theorem 5.1 in the proof of the convergences (4.7), (4.8). To this end we start with the remark that, since $m_J = 0$, $m_A = L_A = 1$, inequality (3.18) shows that the operator T is a contraction for $\rho = 1$. This allows use to make the choice $\rho = 1$ in the proof of Theorem 3.1. Therefore, with this choice we deduce that in the particular case we present here, both Problems $\mathcal{P}$ and $\mathcal{P}_0$ reduce to Problem $\widetilde{\mathcal{P}}$. Let $g \in X$ and recall that the solution of Problem $\widetilde{\mathcal{P}}$ for $f = g$ is given by (6.5). It follows now from (2.9) that

$$Qg = B^{-1} g, \qquad Rg = P_\Lambda(Qg) = P_\Lambda(B^{-1} g)$$

and, since (3.3) implies that $Sv = g$ for all $v \in X$, (3.4) and (3.5) show that

$$Tg = B^{-1} g, \qquad Wg = P_\Lambda(B^{-1} g). \tag{6.18}$$

In addition, recall that the solution u is a fixed point for the operator T, i.e., $Tu = u$. Therefore, using (6.5) and (6.18)we deduce that

$$B^{-1}(B^{-1} f) = B^{-1} f. \tag{6.19}$$

We now use (6.10) and (6.18) to see that

$$u_n - Tu_n = B_n^{-1} f_n - B^{-1}(B_n^{-1} f_n).$$

Then, keeping in mind the convergence (6.11), the continuity of the operator B^{-1} and equality (6.19) we deduce that $u_n - Tu_n \to 0_X$ in X. Moreover, (6.10) and (6.18) imply that

$$\lambda_n - Wu_n = P_\Lambda\Big(\frac{\beta_n}{\gamma_n}\, u_n\Big) - P_\Lambda(B^{-1}u_n) = P_\Lambda\Big(\frac{\beta_n}{\gamma_n}\, B_n^{-1} f_n\Big) - P_\Lambda\big(B^{-1}(B_n^{-1} f_n)\big).$$

We now use the convergences (6.7), (6.11) and equality (6.19) to see that $\lambda_n - Wu_n \to 0$ in X. We are now in a position to use Theorem 5.1 in order to deduce the convergences (4.7) and (4.8).

Example 2. We now move to the second example which concerns a nonlinear elastic constitutive law. More details and preliminaries on the rheological arguments we develop below can be find in [7, 10, 27], for instance.

Everywhere below $d \in \{1, 2, 3\}$, $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. We take $X = Y = Z = \mathbb{S}^d$ and we still use the notation by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ for the canonical inner product and the Euclidean norm on the space on $X = \mathbb{S}^d$. Recall that, since now X is a finite dimensional space, the identity map of X is a compact operator.

We now consider a rheological model $\mathcal{R}$ obtained by connecting in parallel three nonlinear elastic springs $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$. Then, it is well known that the stress in the model $\mathcal{R}$ is given by

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 \tag{6.20}$$

where σ_i represents the stress in the spring $\mathcal{S}_i$, $i = 1, 2, 3$. Moreover, the strain in the model $\mathcal{R}$ is given by

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \tag{6.21}$$

where ε_i represents the strain in the spring $\mathcal{S}_i$, $i = 1, 2, 3$. We assume that the constitutive laws of the springs $\mathcal{S}_i$ are given by

$$\sigma_1 = A\varepsilon_1, \qquad \sigma_2 \in \partial J(\varepsilon_2), \qquad \sigma_3 = \beta P_\Lambda(\gamma\varepsilon_3) \tag{6.22}$$

where $A : X \to X$ is a nonlinear elasticity operator which satisfies condition $H(A)$, $J : Z \to \mathbb{R}$ is a potential function which satisfies condition $H(J)$, $G : X \to Z$ $G\tau = \tau$, $\Lambda \subset X$ is a set which satisfies condition $H(\Lambda)$ with $Y = X$, $P_\Lambda : X \to \Lambda$ denotes the projection operator on Λ and β, γ are positive elastic coefficients.

We turn now to a variational formulation of the constitutive law of the rheological element $\mathcal{R}$. To this end we introduce a new unknown, $\lambda \in X$, defined by

$$\lambda = P_\Lambda(\gamma\boldsymbol{\varepsilon}). \tag{6.23}$$

Note that, from mechanical point of view, we can interpret λ as being an internal state variable which, obviously, depends on the deformation field. Then, using (6.20)–(6.22) we deduce that

$$\sigma = A\varepsilon + \sigma_2 + \beta\lambda$$

and, therefore,

$$(A\varepsilon, \tau - \varepsilon)_X + (\sigma_2, \tau - \varepsilon)_X + (\beta\lambda, \tau - \varepsilon)_X = (\sigma, \tau - \varepsilon)_X \quad \forall\, \tau \in X.$$

Then, using (6.22) and the definition (2.3) of the Clarke subdifferential we find that

$$(A\varepsilon, \tau - \varepsilon)_X + \beta(\lambda, \tau - \varepsilon)_X + J^0(\varepsilon; \tau - \varepsilon) \geq (\sigma, \tau - \varepsilon)_X \quad \forall\, \tau \in X. \tag{6.24}$$

On the other hand, (6.23) and the variational characterization of the projection yields

$$\lambda \in \Lambda, \qquad (\lambda - \gamma\varepsilon, \mu - \lambda)_X \geq 0 \qquad \forall\, \mu \in \Lambda,$$

which implies that

$$\lambda \in \Lambda, \qquad \beta(\varepsilon, \mu - \lambda)_X - \frac{\beta}{2\gamma}\, \|\mu\|_X^2 + \frac{\beta}{2\gamma}\, \|\lambda\|_X^2 \leq 0 \qquad \forall\, \mu \in X.$$

Finally, we consider the bilinear forms $b : X \times X \to \mathbb{R}$ and $c : X \times X \to \mathbb{R}$ defined by

$$b(v, \mu) = \beta(v, \mu)_X, \qquad c(\mu, \lambda) = \frac{\beta}{2\gamma}(\mu, \lambda)_X, \qquad \forall\, v,\ \mu,\ \lambda \in X. \tag{6.25}$$

We now combine (6.24)–(6.25) to consider the following problem.

Problem Σ. *Given a stress $\sigma \in X$, find a strain $\varepsilon \in X$ and an internal state variable $\lambda \in X$ such that,*

$$(A\varepsilon, \tau - \varepsilon)_X + (\beta\lambda, \tau - \varepsilon)_X + J^0(\varepsilon; \tau - \varepsilon) \geq (\sigma, \tau - \varepsilon)_X \quad \forall\, \tau \in X, \tag{6.26}$$

$$b(\varepsilon, \mu - \lambda)_X - c(\mu, \mu) + c(\lambda, \lambda) \leq 0 \qquad \forall\, \mu \in X. \tag{6.27}$$

Inequalities (6.26) and (6.27) represent a variational formulation of the constitutive law (6.20)–(6.22). All the results presented in Sections 3 and 4 can be applied in the study of this problem, under appropriate smallness assumptions. In particular, Theorem 3.1 states its unique solvability and Theorem 4.1 provides the continuous dependence of the solution with respect to the elasticity operator A and elasticity coefficients β and γ. Finally, note that if the solution (ε, λ) of Problem Σ is known then, using (6.20)–(6.22) we can define the stresses σ_1, σ_2 and σ_3 in the springs $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, respectively. These variables are determined in a unique way and depend continuously of the data A, β and γ. All these results are important from mechanical point of view since they represent a mathematical validation of the well-posedness of the nonlinear constitutive law described by (6.20)–(6.22).

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