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ON SOLVING SPLIT EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF α-NONEXPANSIVE MULTI-VALUED MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce α-nonexpansive multi-valued mappings in Hilbert spaces, and prove some properties and the existence of fixed points of these mappings. Further, we apply the Fan-KKM theorem to prove the existence of solutions to split equilibrium problems. Then we study iterative schemes for solving split equilibrium problems and common fixed point problems of a finite family of α-nonexpansive multi-valued mappings in Hilbert spaces. we prove that the proposed iterative algorithms converge weakly and strongly to a common solution of the considered problems. Our results improve and extend the previous results given in the literature.

Key Words and Phrases: α-nonexpaxnsive multi-valued mapping, split equilibrium problem, fixed point.
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1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Notations $\rightharpoonup$ and $\to$ denote strong convergence and weak convergence, respectively. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The equilibrium problem (EP) is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \forall x \in C,$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. The solution set of EP is denoted by $EP(F)$. Since its inception by Blum and Oettli [4] in 1994 , the EP has attracted wide attention due to its applications in a large variety of problems arising in numerous problems in physics, optimizations, and economics. Some methods have been rapidly established for solving this problem (see [9, 17]).

Later, the so-called split equilibrium problem (SEP) was introduced : let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator. The SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \forall x \in C, \tag{1.1}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \forall y \in Q. \tag{1.2}$$

The solution set of SEP is denoted by $SEP(F_1, F_2)$.

In 2012, He [12] constructed some iterative algorithms to solve such problem and obtained some weak and strong convergence theorems. In 2013, Kazmi and Rizvi [14] introduced an iterative scheme of finding the common approximate solution of a split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping under the assumption that the intersection of the solution sets is nonempty. Later on, many iterative algorithms were considered to find a common solution of SEP and other nonlinear problems (see [5, 6, 10, 19, 22, 24]).

We denote by $CB(C)$ and $K(C)$ the collection of all nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The Hausdorff metric H on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}, \forall A, B \in CB(C),$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$ is the distance from a point x to a subset A. Let $T : C \to CB(C)$ be a multi-valued mapping. An element $x \in C$ is called a fixed point of T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x \in Tx\}$.

The study of fixed points for multi-valued mappings using Hausdorff metric was introduced by Markin [16]. Then many fixed point problems of single value mappings were extended to multi-valued cases due to the wide applications in control theory, convex optimization, differential inclusions, game theory and economics. Recently, the existence of fixed points and the convergence theorems of multi-valued mappings have been studied by many authors (see [19, 22, 16, 8, 7, 13, 1, 25]).

Recall that a multi-valued mapping $T : C \to CB(C)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \forall x, y \in C;$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \forall x \in C, p \in F(T);$$

(iii) k-nonspreading [22] if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq kd(x, Ty)^2 + kd(Tx, y)^2, \forall x, y \in C;$$

(iv) hybrid [8] if

$$3H(Tx, Ty)^2 \leq \|x - y\|^2 + d(x, Ty)^2 + d(Tx, y)^2, \forall x, y \in C;$$

(v) λ-hybrid [23] if

$$(1 + \lambda)H(Tx, Ty)^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda d(x, Ty)^2 + \lambda d(Tx, y)^2, \forall x, y \in C.$$

The mapping $T : C \to CB(C)$ is said to be

(i) demiclosed at 0 if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ imply $x \in Tx$.

(ii) sequentially completely continuous if for every bounded sequence $\{x_n\} \subset C$, there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ is convergent;

(iii) hemicompact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$;

In this paper, we introduce the class of α-nonexpansive multi-valued mappings. We say that a mapping $T : C \to CB(C)$ is an α-nonexpansive multi-valued mapping if there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that

$$H(Tx, Ty)^2 \le (1 - 2\alpha)\|x - y\|^2 + \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2, \forall x, y \in C. \tag{1.3}$$

We note that T is α-nonexpansive in the case of single valued mappings proposed by Aoyama and Kohsaka [2] in 2011. The notion of α-nonexpansive mapping was further partially extended to a generalized α-nonexpansive mapping by Pant and Shukla [20] in 2017. A mapping $T : C \to C$ is said to be generalized α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that for each $x, y \in C$,

$$\begin{aligned}&\frac{1}{2}\|x - Tx\| \le \|x - y\|\\ \Rightarrow&\|Tx - Ty\| \le (1 - 2\alpha)\|x - y\| + \alpha\|Tx - y\| + \alpha\|x - Ty\|.\end{aligned}$$

Obviously, unlike our definition, this class of nonlinear mapping does not properly contain α-nonexpansive mappings. Very recently, Iqbal et al. [13] considered the multi-valued version. $T : C \to CB(C)$ is called a multi-valued generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ such that for each $x, y \in C$,

$$\begin{aligned}&\frac{1}{2}d(x, Tx) \le \|x - y\|\\ \Rightarrow&H(Tx, Ty) \le (1 - 2\alpha)\|x - y\| + \alpha d(Tx, y) + \alpha d(x, Ty).\end{aligned}$$

For more discussion on α-nonexpansive mappings, we refer to [1, 3] and the references therein.

Remark 1.1. Taking $\alpha = 0, \frac{1}{2}$ and $\frac{1}{3}$, the mapping T is reduced to the so-called nonexpansive multi-valued mapping, $\frac{1}{2}$-nonspreading multi-valued mapping and hybrid multi-valued mapping, respectively. The class of α-nonexpansive mappings and the class of λ-hybrid mappings are equivalent in a Hilbert space when $\lambda > -1$. Moreover, if T is α-nonexpansive with $\alpha \in [0, 1)$ and $F(T) \ne \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned}H(Tx, Tp)^2 &\le (1 - 2\alpha)\|x - p\|^2 + \alpha d(Tx, p)^2 + \alpha d(x, Tp)^2\\ &\le (1 - 2\alpha)\|x - p\|^2 + \alpha H(Tx, Tp)^2 + \alpha\|x - p\|^2.\end{aligned}$$

It follows that

$$H(Tx, Tp) \le \|x - p\|. \tag{1.4}$$

We now give an example of an α-nonexpansive multi-valued mapping which is neither $\frac{1}{2}$-nonspeading nor nonexpansive.

Example 1.2. Consider $C = \{0, \frac{9}{10}, 1\}$ with the usual norm. Define $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, \ x = 0 \\ \{0, \frac{9}{10}, 1\}, \ x = \frac{9}{10} \\ \{0, 1\}, \ x = 1 \end{cases}$$

Now, we show that T is $(-\frac{1}{4})$-nonexpansive. In fact, we have the following cases:
Case 1: If $x = y = 0, \frac{9}{10}, 1$, then $H(Tx, Ty) = d(x, y) = d(Tx, y) = d(x, Ty) = 0$.
Case 2: If $x = 0, y = \frac{9}{10}$, then $H(Tx, Ty) = 1$, $d(x, y) = \frac{9}{10}$, $d(Tx, y) = \frac{9}{10}$ and $d(x, Ty) = 0$. This implies that

$$\frac{3}{2} \times (\frac{9}{10})^2 - \frac{1}{4} \times (\frac{9}{10})^2 - \frac{1}{4} \times 0^2 = \frac{81}{80} > 1^2.$$

Case 3: If $x = 0, y = 1$, then $H(Tx, Ty) = 1$, $d(x, y) = 1$, $d(Tx, y) = 1$ and $d(x, Ty) = 0$. This implies that

$$\frac{3}{2} \times 1^2 - \frac{1}{4} \times 1^2 - \frac{1}{4} \times 0^2 = \frac{5}{4} > 1^2.$$

Case 4: If $x = \frac{9}{10}, y = 1$, then $H(Tx, Ty) = \frac{1}{10}$, $d(x, y) = \frac{1}{10}$, $d(Tx, y) = 0$ and $d(x, Ty) = \frac{1}{10}$. This implies that

$$\frac{3}{2} \times (\frac{1}{10})^2 - \frac{1}{4} \times 0^2 - \frac{1}{4} \times (\frac{1}{10})^2 = \frac{1}{80} > (\frac{1}{10})^2.$$

On the other hand, T is not $\frac{1}{2}$-nonspreading since for $x = 0$ and $y = 1$, we have $Tx = \{0\}$ and $Ty = \{0, 1\}$. This shows that

$$2H(Tx, Ty)^2 = 2 > 1^2 + 0^2 = d(Tx, y)^2 + d(x, Ty)^2.$$

Since $H(T(0), T(\frac{9}{10})) = 1 > d(x, y)$, this also implies that T is not nonexpansive.

In 2016, Suantai et al. [22] proposed the iterative algorithm to solve the problems for finding a common solution of the split equilibrium problem and the fixed point problem of an $\frac{1}{2}$-nonspreading multi-valued mapping in Hilbert space, given sequence $\{x_n\}$ by

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \end{cases}$$

where $T_{r_n}^{F_i}, (i = 1, 2)$ is resolvent operator. The authors proved that $\{x_n\}$ converges weakly to an element of $F(S) \bigcap SEP(F_1, F_2)$ under some conditions.

In 2019, Li [20] introduce the concept of split Nash equilibrium problems and prove an existence theorem. Motivated and inspired by the above results and related literature, we further study the existence of solutions to split equilibrium problems and the existence of fixed points of α-nonexpansive multi-valued mappings. Then we propose some iterative algorithms for finding a common element of the set of solutions of split equilibrium problems and the set of common fixed points of a finite family of α-nonexpansive multi-valued mappings in real Hilbert spaces. Finally we prove some weak and strong convergence theorems which extend and improve the corresponding results of Suantai et al. [22] and Cholamjiak et al. [8] and many others.

2. Preliminaries

We now recall some concepts and results which are needed in sequel.

Definition 2.1. Let $T : C \to H$ be a nonlinear mapping. Then T is called α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \forall x, y \in C.$$

Particularly, T is called firmly nonexpansive when $\alpha = 1$.

A mapping P_C is said to be metric projection of H onto C if for every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

It is well known that P_C is firmly nonexpansive mapping. Moreover, every nonexpansive mapping $T : H \to H$ satisfies the inequality

$$\langle (x - Tx) - (y - Ty), T(y) - T(x)\rangle \leq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2, \forall x, y \in H.$$

Therefore, we get

$$\langle x - Tx, y - T(x)\rangle \leq \frac{1}{2}\|T(x) - x\|^2, \forall x \in H, y \in F(T). \tag{2.1}$$

A multi-valued mapping $T : C \to CB(C)$ is said to satisfy the endpoint condition if $Tp = \{p\}$ for all $p \in F(T)$. It is well known that the best approximation operator P_T, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, satisfies $F(T) = F(P_T)$ and the endpoint condition.

Lemma 2.2. *In a real Hilbert space H, the following well known results hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \forall x, y \in H$;

(2) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1], x, y \in H$;

(3) If $\{x_n\}$ is a sequence in H which converges weakly to $z \in H$, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \forall y \in H.$$

Lemma 2.3. [21] *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H. Let $u, v \in H$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

For solving equilibrium problems, let us give the following assumptions for the bifunction $F : C \times C \to \mathbb{R}$:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x \in C$;

(A3) for each $x, y, z \in C, \lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.4. [9] *Suppose that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the conditions (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$J_r^F x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then the following conclusions hold:

(i) J_r^F is single-valued;

(ii) J_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|J_r^F x - J_r^F y\|^2 \leq \langle x - y, J_r^F x - J_r^F y\rangle;$$

(iii) $F(J_r^F) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.5. [8] *Let C be a closed and convex subset of a real Hilbert space H and $T : C \to CB(C)$ be a multivalued mapping. Suppose that there exist $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded and, for all $y \in C$, there exists $a \in Ty$ such that*

$$\mu_n\|z_n - a\|^2 \leq \mu_n\|z_n - y\|^2$$

for a Banach limit μ. Then T has a fixed point in C.

Lemma 2.6. [7] *Let H be a real Hilbert space. Let $m \in \mathbb{N}$ be fixed and let $\{x_i\}_{i=1}^m \subset H$. For $\alpha_i \in (0, 1)$, $i = 1, 2, \cdots, m$, such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i\|x_i\|^2 - \sum_{1\leq i\leq j\leq m} \alpha_i\alpha_j\|x_i - x_j\|^2.$$

Definition 2.7. Let K be a nonempty subset of a linear space B. A set-valued mapping $T : K \to 2^B\backslash\{\emptyset\}$ is said to be a KKM mapping if for any finite subset$\{y_1, y_2, \cdots, y_n\}$ of K, we have co$\{y_1, y_2, \cdots, y_n\} \subset \bigcup_{1\leq i\leq n} T(y_i)$, where co$\{y_1, y_2, \cdots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \cdots, y_n\}$.

Lemma 2.8. [11] *Let K be a nonempty closed convex subset of a Hausdorff topological vector space B and let $T : K \to 2^B\backslash\{\emptyset\}$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $T(y_0)$ is a compact subset, then $\bigcap_{y\in K} T(y) \neq \emptyset$.*

3. Existence results

We need the following concept, convexity direction preserved , for operators from H_1 to H_2. It is an important condition for the operator A for the existence of solutions to SEP. Furthermore, we assume that $AC = Q$ is satisfied below.

Definition 3.1. Let C, Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. A bounded linear operator $A : H_1 \to H_2$ is said to be convexity direction preserved with respect to F_1 and F_2 on C^2 and Q^2 if, for any given points $(u, Au), (v, Av) \in C\times Q$ and for their arbitrary convex combination $w = \lambda u+(1-\lambda)v$, where $0 \leq \lambda \leq 1$, we have either that $F_1(w,u) \geq 0$ and $F_2(Aw,Au) \geq 0$ both hold or that $F_1(w,v) \geq 0$ and $F_2(Aw,Av) \geq 0$ both hold.

Remark 3.2. For any given points $u, v \in C$ and their convex combination of

$$w = \lambda u + (1-\lambda)v,$$

we must have

$$\text{either} \quad F_1(w,u) \geq 0, \text{or} \quad F_1(w,v) \geq 0, \tag{3.1}$$

and

$$\text{either} \quad F_2(Aw,Au) \geq 0, \text{or} \quad F_2(Aw,Av) \geq 0. \tag{3.2}$$

To see (3.1), we assume by contradiction that $F_1(w,u) < 0$ and $F_1(w,v) < 0$ both hold. Then from A1 and A4, we have

$$0 = F_1(w,w) = F_1(w, \lambda u + (1-\lambda)v) \leq \lambda F_1(w,u) + (1-\lambda)F_1(w,v) < 0.$$

This is a contradiction. Hence (3.1) must hold. Since $A : H_1 \to H_2$ is a bounded linear operator and $AC = Q$, it follows that

$$Aw = \lambda Au + (1-\lambda)Av.$$

Then we can similarly show that (3.2) holds. The convexity direction preserved property of A insures that one of the two inequalities in (3.1) and the corresponding inequality in (3.2) must simultaneously hold for the point u or the point v.

Theorem 3.3. *Let H_1, H_2 be two real Hilbert space and $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying (A1)-(A4), and that A satisfies the following conditions:*
(C1) $AC = Q$;
(C2) A is convexity direction preserved with respect to F_1 and F_2.
If there is a point $(t, At) \in C \times Q$ such that

$$\{(z, Az) \in C \times Q : F_1(z,t) \geq 0, F_2(Az, At) \geq 0\}$$

is compact, then $SEP(F_1, F_2) \neq \emptyset$.

Proof. It follows from (C1) that $A|_C \in \mathcal{L}(C,Q)$. we denote G its graph, i.e.,

$$G = \{(x, Ax) : x \in C\} \subseteq C \times Q.$$

Since H_1 and H_2 are Hilbert spaces , by the closed graph theorem in Banach spaces, the graph of the bounded linear operator $A|_C$ from C to Q is a closed subset of $C \times Q$ with respect to the product topology. Then the convexity of C and Q and the linearity of A imply that G is a nonempty closed and convex subset of $C \times Q$.

On this underlying space G, we define a mapping $T : G \to 2^G \backslash \{\emptyset\}$ by

$$T(x, Ax) = \{(z, Az) \in G : F_1(z,x) \geq 0, F_2(Az, Ax) \geq 0\}.$$

From the continuity of the mappings A and (A3), we have $(x, Ax) \in T(x, Ax)$ which yields that, for every $(x, Ax) \in G$, $T(x, Ax)$ is a nonempty closed subset of G.

Next we show that T is a KKM mapping. For any points $(u, Au), (v, Av) \in G$, we arbitrarily take a convex combination of $w = \lambda u + (1-\lambda)v$, where $0 < \lambda < 1$. Since $A \in \mathcal{L}(H_1, H_2)$, it follows that

$$Aw = \lambda Au + (1-\lambda)Av.$$

Then

$$(w, Aw) = \lambda(u, Au) + (1-\lambda)(v, Av).$$

From condition (C2), we have either

$$F_1(w, u) \geq 0 \text{ and } F_2(Aw, Au) \geq 0,$$

or

$$F_1(w, v) \geq 0 \text{ and } F_2(Aw, Av) \geq 0.$$

That is, either $(w, Aw) \in T(u, Au)$, or $(w, Aw) \in T(v, Av)$. It implies that

$$(w, Aw) \in T(u, Au) \cup T(v, Av).$$

Similarly, we can extend this to finite convex combinations. Hence, T is a KKM mapping. For the given point $(t, At) \in G$ from the assumptions of this theorem, we know

$$\{(z, Az) \in C \times Q : F(z, t) \geq 0 \text{ and } G(Az, At) \geq 0\}$$

is a nonempty compact subset of G.

By applying the Lemma 2.8, we obtain that $\bigcap_{(x,Ax)\in K} T(x, Ax) \neq \emptyset$. Then, by taking any $(x^*, Ax^*) \in \bigcap_{(x,Ax)\in K} T(x, Ax)$, we have

$$F_1(x^*, x) \geq 0, \forall x \in C, \tag{3.3}$$

and

$$F_2(Ax^*, Ax) \geq 0, \forall x \in C. \tag{3.4}$$

From (C1) of this theorem, (3.4) is equivalent to

$$F_2(Ax^*, y) \geq 0, \forall y \in Q, \tag{3.5}$$

Combining (3.3), (3.5), we have $x^* \in SEP(F_1, F_2)$.

Corollary 3.4. *Let H_1,H_2 be two real Hilbert space and $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty compact convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to R$ are bifunctions satisfying (A1)-(A4), and that A satisfies the following conditions:*

(C1) $AC = Q$;

(C2) A is convexity direction preserved with respect to F_1 and F_2 .

Then $SEP(F_1, F_2) \neq \emptyset$.

Now, we are prepared to prove the existence theorem of the fixed point of α-nonexpansive multi-valued mappings. To this end, we first prove some crucial lemmas, which are also used in the next section.

Lemma 3.5. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be an α-nonexpansive multivalued mapping such that $\alpha \in [0, 1)$. If $x, y \in C$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \le H(Tx, Ty)^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\langle x - a, y - b\rangle.$$

Proof. Let $x, y \in C$ and $a \in Tx$. By Nadler's theorem (see [18]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \le H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned} H(Tx, Ty)^2 &\le (1 - 2\alpha)\|x - y\|^2 + \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2 \\ &\le (1 - 2\alpha)\|x - y\|^2 + \alpha\|a - y\|^2 + \alpha\|x - b\|^2 \\ &= (1 - 2\alpha)\|x - y\|^2 + \alpha[\|a - x\|^2 + 2\langle a - x, x - y\rangle + \|x - y\|^2 \\ &\quad + \|x - a\|^2 + 2\langle x - a, a - b\rangle + \|a - b\|^2] \\ &= (1 - \alpha)\|x - y\|^2 + \alpha[2\|a - x\|^2 + \|a - b\|^2 + 2\langle a - x, x - a - (y - b)\rangle] \\ &= (1 - \alpha)\|x - y\|^2 + \alpha[\|a - b\|^2 + 2\langle x - a, y - b\rangle] \\ &\le (1 - \alpha)\|x - y\|^2 + \alpha H(Tx, Ty)^2 + 2\alpha\langle x - a, y - b\rangle. \end{aligned}$$

This implies that

$$H(Tx, Ty)^2 \le \|x - y\|^2 + \frac{2\alpha}{1-\alpha}\langle x - a, y - b\rangle.$$

This completes the proof.

Lemma 3.6. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be an α-nonexpansive multivalued mapping such that $\alpha \in [0, 1)$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Proof. Let $\{x_n\}$ be a sequence in C which converges weakly to p and let $y_n \in Tx_n$ be such that $\|x_n - y_n\| \to 0$.

Now, we show that $p \in F(T)$. By Lemma 3.5, there exists $z_n \in Tp$ such that

$$\|y_n - z_n\|^2 \le \|x_n - p\|^2 + \frac{2\alpha}{1-\alpha}\langle x_n - y_n, p - z_n\rangle.$$

Since Tp is compact and $z_n \in Tp$, there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \to z \in Tp$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H$, define a function $f : H \to [0, \infty)$ by

$$f(x) := \limsup_{i\to\infty} \|x_{n_i} - x\|^2.$$

Then, by Lemma 2.2, we obtain

$$f(x) = \limsup_{i\to\infty}(\|x_{n_i} - p\|^2 + \|p - x\|^2)$$

for all $x \in H$. Thus $f(x) = f(p) + \|p - x\|^2$ for all $x \in H$. It follows that

$$f(z) = f(p) + \|p - z\|^2. \tag{3.6}$$

We observe that

$$\begin{aligned} f(z) &= \limsup_{i\to\infty} \|x_{n_i} - z\|^2 = \limsup_{i\to\infty} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \\ &\leq \limsup_{i\to\infty} \|y_{n_i} - z\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} f(z) &\leq \limsup_{i\to\infty} \|y_{n_i} - z\|^2. \\ &= \limsup_{i\to\infty} (\|y_{n_i} - z_{n_i} + z_{n_i} - z\|^2). \\ &\leq \limsup_{i\to\infty} (\|x_{n_i} - p\|^2 + \frac{2\alpha}{1-\alpha}\langle x_{n_i} - y_{n_i}, p - z_{n_i}\rangle). \\ &= \limsup_{i\to\infty} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \tag{3.7}$$

Hence it follows from (3.6) and (3.7) that$\|p - z\| = 0$. This completes the proof.

Lemma 3.7. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be an α-nonexpansive multivalued mapping such that $\alpha \in [0, 1)$. Then $F(T)$ is closed. Moreover, if T satisfies the endpoint condition, then $F(T)$ is convex.*

Proof. If $F(T) = \emptyset$, then it is closed. Assume that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. Using (1.4), we have

$$\begin{aligned} d(x, Tx) &\leq \|x - x_n\| + d(x_n, Tx) \\ &\leq \|x - x_n\| + H(Tx_n, Tx) \\ &\leq 2\|x - x_n\|. \end{aligned}$$

It follows that $d(x, Tx) = 0$. Hence $x \in F(T)$. We conclude that $F(T)$ is closed.

To show that $F(T)$ is convex. Let $p = tp_1 + (1 - t)p_2$, where $p_1, p_2 \in F(T)$ and $t \in (0, 1)$. Let $z \in Tp$. It follows from Lemma 2.2 that

$$\begin{aligned} \|z - p\|^2 &= \|t(z - p_1) + (1 - t)(z - p_2)\|^2 \\ &= t\|z - p_1\|^2 + (1 - t)\|z - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= td(z, Tp_1)^2 + (1 - t)d(z, p_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &\leq tH(Tp, Tp_1)^2 + (1 - t)H(Tp, Tp_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &\leq t\|p - p_1\|^2 + (1 - t)\|p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\ &= \|t(p - p_1) + (1 - t)(p - p_2)\|^2 \\ &= 0. \end{aligned}$$

and hence $p = z$. Therefore, $p \in F(T)$. This completes the proof.

Theorem 3.8. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be an α-nonexpansive multivalued mapping with $\alpha \in [0, 1)$. Then, $F(T) \neq \emptyset$ if and only if there exist $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded.*

Proof. The proof of necessity is obvious. To prove sufficiency, assume that there exist $z_0 \in C$ and $z_n \in Tz_{n-1}$ for all $n \geq 1$ such that $\{z_n\}$ is bounded. Let $y \in C$. From Lemma 3.5, there exists $b \in Ty$ such that

$$\begin{aligned}\|z_{n+1} - b\|^2 &\leq \|z_n - y\|^2 + \frac{2\alpha}{1-\alpha}\langle z_n - z_{n+1}, y - b\rangle \\ &= \|z_n - y\|^2 + \frac{\alpha}{1-\alpha}(\|z_n - b\|^2 + \|z_{n+1} - y\|^2 \\ &\quad - \|z_n - y\|^2 - \|z_{n+1} - b\|^2),\end{aligned}$$

therefore

$$\left(1 + \frac{\alpha}{1-\alpha}\right)\|z_{n+1} - b\|^2 - \frac{\alpha}{1-\alpha}\|z_n - b\|^2 \leq \left(1 - \frac{\alpha}{1-\alpha}\right)\|z_n - y\|^2 + \frac{\alpha}{1-\alpha}\|z_{n+1} - y\|^2.$$

Let μ be a Banach limit on l^∞. For any $n \in \mathbb{N}$, we have

$$\frac{1}{1-\alpha}\mu_n\|z_{n+1} - b\|^2 - \frac{\alpha}{1-\alpha}\mu_n\|z_n - b\|^2 \leq \frac{1-2\alpha}{1-\alpha}\mu_n\|z_n - y\|^2 + \frac{\alpha}{1-\alpha}\mu_n\|z_{n+1} - y\|^2.$$

This implies that

$$\mu_n\|z_n - b\|^2 \leq \mu_n\|z_n - y\|^2.$$

By Lemma 2.5, T has a fixed point in C. This completes the proof.

4. Convergence results

Theorem 4.1. *Let H_1, H_2 be two real Hilbert space and $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying (A1)-(A4), and F_2 is upper semicontinuous in the first argument. For $i = 1, 2, \cdots, m$, let $T_i : C \to K(C)$ be a family of α-nonexpansive multi-valued mappings with $\alpha \in [0, 1)$ such that*

$$\Theta = \bigcap_{i=1}^{m} F(Ti) \bigcap SEP(F_1, F_2) \neq \emptyset.$$

Define a sequence $\{x_n\}$ by $x_1 \in C$ arbitrary and

$$\begin{cases} u_n = J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_{0,n}x_n + \sum_{i=1}^{m}\alpha_{i,n}T_i u_n, \end{cases} \tag{4.1}$$

where $\alpha_{i,n} \in (0, 1)$ for all $i = 0, 1, 2, \cdots, m$,

$$\sum_{i=0}^{m}\alpha_{i,n} = 1 \text{ for all } n \geq 1,\ r_n > 0$$

and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) T_i satisfies satisfies the endpoint condition for all $i = 1, 2, \cdots, m$;

(C2) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for all* $i = 1, 2, \cdots, m$;

(C3) $\liminf_{n\to\infty} r_n > 0$.

Then

(i) The sequence $\{x_n\}$ *generated by (4.1) converges weakly to* $p \in \Theta$.

(ii) If one of the T_i *is sequentially completely continuous, then* $\{x_n\}$ *converges strongly to* $p \in \Theta$.

(iii) If one of the T_i *is hemicompact, then* $\{x_n\}$ *converges strongly to* $p \in \Theta$.

Proof. Now we prove conclusion (i).

We first show that $A^*(I - J_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $J_{r_n}^{F_2}$ is firmly nonexpansive and $I - J_{r_n}^{F_2}$ is firmly nonexpansive, we see that

$$\begin{aligned}
&\|A^*(I - J_{r_n}^{F_2})Ax - A^*(I - J_{r_n}^{F_2})Ay\|^2 \\
&= \langle A^*(I - J_{r_n}^{F_2})(Ax - Ay), A^*(I - J_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= \langle (I - J_{r_n}^{F_2})(Ax - Ay), AA^*(I - J_{r_n}^{F_2})(Ax - Ay)\rangle \\
&\le L\langle (I - J_{r_n}^{F_2})(Ax - Ay), (I - J_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= L\|(I - J_{r_n}^{F_2})(Ax - Ay)\|^2 \\
&\le L\langle Ax - Ay, (I - J_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= L\langle x - y, A^*(I - J_{r_n}^{F_2})Ax - A^*(I - J_{r_n}^{F_2})Ay)\rangle
\end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - J_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - J_{r_n}^{F_2})A$ is nonexpansive.

Now, we divide the proof into five steps as follows:

Step 1. Show that $\{x_n\}$ is bounded.

Let $p \in \Theta$. Then $p = J_{r_n}^{F_1}p$ and $(I - \gamma A^*(I - J_{r_n}^{F_2})A)p = p$. Thus we have

$$\begin{aligned}
\|u_n - p\| &= \|J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)p\| \\
&\le \|(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - J_{r_n}^{F_2})A)p\| \\
&\le \|x_n - p\|. \qquad (4.2)
\end{aligned}$$

Let

$$x_{n+1} = \alpha_{0,n}x_n + \sum_{i=1}^{m} \alpha_{i,n}y_n^i,\ y_n^i \in T_iu_n.$$

From (C1), we have

$$\|y_n^i - p\| = d(y_n^i, T_ip) \le H(T_iu_n, T_ip) \le \|u_n - p\| \le \|x_n - p\|. \qquad (4.3)$$

for all $i = 1, 2, \cdots, m$. It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_{0,n}(x_n - p) + \sum_{i=1}^{m} \alpha_{i,n}(y_n^i - p)\| \\
&\le \alpha_{0,n}\|x_n - p\| + \sum_{i=1}^{m} \alpha_{i,n}\|y_n^i - p\| \\
&\le \|x_n - p\|.
\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded.
Step 2. Show that

$$\lim_{n\to\infty}\|y_n^i - x_n\| = 0$$

for all $i = 1, 2, \ldots, m$. From the lemma 2.6 and (4.3), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_{0,n}(x_n - p) + \sum_{i=1}^{n}\alpha_{i,n}(y_n^i - p)\|^2 \\ &= \alpha_{0,n}\|x_n - p\|^2 + \sum_{i=1}^{n}\alpha_{i,n}\|y_n^i - p\|^2 - \sum_{i=1}^{n}\alpha_{0,n}\alpha_{i,n}\|x_n - y_n^i\|^2 \\ &\le \|x_n - p\|^2 - \sum_{i=1}^{n}\alpha_{0,n}\alpha_{i,n}\|x_n - y_n^i\|^2.\end{aligned}$$

It follows that

$$\sum_{i=1}^{n}\alpha_{0,n}\alpha_{i,n}\|x_n - y_n^i\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From (C2) and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we obtain

$$\lim_{n\to\infty}\|x_n - y_n^i\| = 0 \tag{4.4}$$

for all $i = 1, 2, \ldots, m$.
Step 3. Show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $\lim_{n\to\infty}\|y_n^i - u_n\| = 0$ for all $i = 1, 2, \ldots, m$.

For $p \in \Theta$ and by using (2.1), we estimate

$$\begin{aligned}\|u_n - p\|2 &= \|J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - J_{r_n}^{F_1}p\|^2 \\ &\le \|x_n - \gamma A^*(I - J_{r_n}^{F_2})Ax_n - p\|^2 \\ &\le \|x_n - p\|^2 + \gamma^2\|A^*(I - J_{r_n}^{F_2})Ax_n\|^2 + 2\gamma\langle p - x_n, A^*(I - J_{r_n}^{F_2})Ax_n\rangle \\ &= \|x_n - p\|^2 + \gamma^2\langle Ax_n - J_{r_n}^{F_2}Ax_n, AA^*(I - J_{r_n}^{F_2})Ax_n\rangle \\ &\quad + 2\gamma\langle A(p - x_n), Ax_n - J_{r_n}^{F_2}Ax_n\rangle \\ &\le \|x_n - p\|^2 + L\gamma^2\langle Ax_n - J_{r_n}^{F_2}Ax_n, (I - J_{r_n}^{F_2})Ax_n\rangle \\ &\quad + 2\gamma\langle A(p - x_n), Ax_n - J_{r_n}^{F_2}Ax_n\rangle \\ &\le \|x_n - p\|^2 + L\gamma^2\|Ax_n - J_{r_n}^{F_2}Ax_n\|^2 \\ &\quad + 2\gamma(\langle Ap - J_{r_n}^{F_2}Ax_n, Ax_n - J_{r_n}^{F_2}Ax_n\rangle - \|Ax_n - J_{r_n}^{F_2}Ax_n\|^2) \\ &\le \|x_n - p\|^2 + L\gamma^2\|Ax_n - J_{r_n}^{F_2}Ax_n\|^2 \\ &\quad + 2\gamma(\frac{1}{2}\|Ax_n - J_{r_n}^{F_2}Ax_\|^2 - \|Ax_n - J_{r_n}^{F_2}Ax_n\|^2) \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - J_{r_n}^{F_2}Ax_n\|^2\end{aligned}$$

It follows from (4.3) that, for all $y_n^i \in T_i u_n$,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_{0,n}(x_n - p) + \sum_{i=1}^{m} \alpha_{i,n}(y_n^i - p)\|^2 \\ &\le \alpha_{0,n}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{i,n}\|y_n^i - p\|^2 \\ &\le \alpha_{0,n}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{i,n}\|u_n - p\|^2 \\ &\le \|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{i,n}\gamma(L\gamma - 1)\|Ax_n - J_{r_n}^{F_2}Ax_n\|^2 \qquad (4.5)\end{aligned}$$

Therefore, we have

$$-\sum_{i=1}^{m} \alpha_{i,n}\gamma(L\gamma - 1)\|Ax_n - J_{r_n}^{F_2}Ax_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $\gamma(L\gamma - 1) < 0$, it follows by (C2) and the existence of $\lim_{n\to\infty}\|x_n - p\|$ that

$$\lim_{n\to\infty}\|Ax_n - J_{r_n}^{F_2}Ax_n\| = 0. \qquad (4.6)$$

Since $J_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(I - J_{r_n}^{F_2})A$ is nonexpansive, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - J_{r_n}^{F_1}p\|^2 \\ &\le \langle J_{r_n}^{F_1}(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - J_{r_n}^{F_1}p, (I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - p\rangle \\ &= \langle u_n - p, (I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|(I - \gamma A^*(I - J_{r_n}^{F_2})A)x_n - p\|^2 \\ &\quad - \|u_n - x_n + \gamma A^*(I - J_{r_n}^{F_2})Ax_n\|^2) \\ &\le \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n + \gamma A^*(I - J_{r_n}^{F_2})Ax_n\|^2) \\ &\le \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \gamma^2\|A^*(I - J_{r_n}^{F_2})Ax_n\|^2 \\ &\quad + 2\gamma\langle u_n - x_n, A^*(I - J_{r_n}^{F_2})Ax_n\rangle)\},\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - x_n\|^2 - 2\gamma\langle u_n - x_n, A^*(I - J_{r_n}^{F_2})Ax_n\rangle \\ &\le \|x_n - p\|^2 - \|u_n - x_n\|^2 - 2\gamma\|u_n - x_n\|\|A^*(I - J_{r_n}^{F_2})Ax_n\|. \qquad (4.7)\end{aligned}$$

This implies by (4.5) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_{0,n}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{i,n}\|u_n - p\|^2 \\ &\le \|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{i,n}(2\gamma\|u_n - x_n\|\|A^*(I - J_{r_n}^{F_2})Ax_n\| - \|u_n - x_n\|^2).\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{m}\alpha_{i,n}\|u_n-x_n\|^2\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\sum_{i=1}^{m}\alpha_{i,n}2\gamma M\|A^*(I-J_{r_n}^{F_2})Ax_n\|,$$

where $M=\sup\{\|u_n-x_n\|:n\in\mathbb{N}\}$. This implies by (C2),(4.6) and the existence of $\lim_{n\to\infty}\|x_n-p\|$ that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{4.8}$$

From (4.4) and (4.8), we have

$$\|u_n-y_n^i\|\leq\|u_n-x_n\|+\|x_n-y_n^i\|\to 0 \tag{4.9}$$

for all $i=1,2,\dots,m$.

Step 4. Show that $\omega_w(x_n)\subseteq\Theta$, where $\omega_w(x_n)=\{x\in H_1: x_{n_i}\rightharpoonup x,\{x_{n_i}\}\subseteq\{x_n\}\}$.

Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega_w(x_n)$is nonempty. Let $q\in\omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}\subseteq\{x_n\}$ converging weakly to q. From (4.8), it implies that $u_{n_i}\rightharpoonup q$ as $i\to\infty$. By (4.9) and Lemma 3.7, we have $q\in\bigcap_{i=1}^{m}F(T_i)$.

Next, we show that $q\in EP(F_1)$. Since $u_n=J_{r_n}^{F_1}(I-\gamma A^*(I-J_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n+\gamma A^*(I-J_{r_n}^{F_2})Ax_n\rangle\geq 0,\forall y\in C,$$

which implies that

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle+\frac{1}{r_n}\langle y-u_n,\gamma A^*(I-J_{r_n}^{F_2})Ax_n\rangle\geq 0,\forall y\in C.$$

From (A2), we have

$$\frac{1}{r_{n_i}}\langle y-u_{n_i},u_{n_i}-x_{n_i}\rangle+\frac{1}{r_{n_i}}\langle y-u_{n_i},\gamma A^*(I-J_{r_{n_i}}^{F_2})Ax_{n_i}\rangle\geq F_1(y,u_{n_i}),\forall y\in C.$$

This implies by $u_{n_i}\rightharpoonup q$, (C3), (4.6), (4.8) and (A4) that

$$F_1(y,q)\leq 0,\forall y\in C.$$

For $t\in(0,1]$ and $y\in C$, let $y_t=ty+(1-t)q$. Since $y\in C$ and $q\in C$, we get $y_t\in C$ and hence $F_1(y_t,q)\leq 0$. So, by (A1) and (A4), we have

$$0=F_1(y_t,y_t)\leq tF_1(y_t,y)+(1-t)F_1(y_t,q)\leq tF_1(y_t,y),\forall y\in C.$$

Letting $t\to 0$, by (A3), we have

$$F_1(q,y)\geq 0,\forall y\in C.$$

This implies that $q\in EP(F_1)$. Since A is a bounded linear operator, we have $Ax_{n_i}\rightharpoonup Aq$. Then it follows from (4.6) that

$$J_{r_{n_i}}^{F_2}Ax_{n_i}\rightharpoonup Aq, \tag{4.10}$$

as $i\to\infty$. By the definition of $J_{r_{n_i}}^{F_2}Ax_{n_i}$, we have

$$F_2(J_{r_{n_i}}^{F_2}Ax_{n_i},y)+\frac{1}{r_{n_i}}\langle y-J_{r_{n_i}}^{F_2}Ax_{n_i},J_{r_{n_i}}^{F_2}Ax_{n_i}-Ax_{n_i}\rangle\geq 0,\forall y\in Q. \tag{4.11}$$

Since F_2 is upper semicontinuous in the first argument, it implies by (4.11) that

$$F_2(Aq, y) \geq 0, \forall y \in Q. \tag{4.12}$$

This shows that $Aq \in EP(F_2)$. Therefore, $q \in SEP(F_1, F_2)$ and hence $q \in \Theta$.

Step 5. Show that $\{x_n\}$ converges weakly to an element of Θ. It is sufficient to show that $\omega_w(x_n)$ is a singleton set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}, \{x_{n_m}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (4.8), we also have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By (4.9) and Lemma 3.7, we see that $p, q \in \bigcap_{i-1}^m F(T_i)$. Applying Lemma 2.3, we obtain $p = q$. The proof of conclusion (i) is completed.

Next we prove conclusion (ii).

From Lemma 3.5, for $y_n^i \in T_i u_n$, there exists $b_n^i \in T_i x_n$ such that

$$\begin{aligned} H(T_i u_n, T_i x_n)^2 &\leq \|u_n - x_n\|^2 + \frac{2\alpha}{1-\alpha}\langle u_n - y_n^i, x_n - b_n^i\rangle \\ &\leq \|u_n - x_n\|^2 + \frac{2\alpha}{1-\alpha}\|u_n - y_n^i\|\|x_n - b_n^i\|. \end{aligned}$$

It follows from (4.8) and (4.9) that

$$\lim_{n\to\infty} H(T_i u_n, T_i x_n) = 0 \tag{4.13}$$

for all $i \in \{1, 2, \cdots, m\}$. This implies that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, T_i x_n) &\leq \lim_{n\to\infty} (d(x_n, T_i u_n) + H(T_i u_n, T_i x_n)) \\ &\leq \lim_{n\to\infty} (\|x_n - y_n^i\| + H(T_i u_n, T_i x_n)) = 0 \end{aligned} \tag{4.14}$$

for all $i \in \{1, 2, \cdots, m\}$. Suppose that T_{i_0} is sequentially completely continuous for some $i_0 \in \{1, 2, \cdots, m\}$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} d(T_{i_0} x_{n_k}, p) = 0,$$

for some $p \in C$. It follows from (4.14) that

$$\|x_{n_k} - p\| \leq d(x_{n_k}, T_{i_0} x_{n_k}) + d(T_{i_0} x_{n_k}, p) \to 0, \tag{4.15}$$

as $k \to \infty$. From Lemma 3.5, for $y_{n_k}^i \in T_i x_{n_k}$, there exists $c_{n_k}^i \in T_i p$ such that

$$\begin{aligned} H(T_i x_{n_k}, T_i p)^2 &\leq \|u_{n_k} - p\|^2 + \frac{2\alpha}{1-\alpha}\langle u_{n_k} - y_{n_k}^i, p - c_{n_k}^i\rangle \\ &\leq \|u_{n_k} - p\|^2 + \frac{2\alpha}{1-\alpha}\|u_{n_k} - y_{n_k}^i\|\|p - c_{n_k}^i\| \\ &\leq (\|u_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|)^2 + \frac{2\alpha}{1-\alpha}\|u_{n_k} - y_{n_k}^i\|\|p - c_{n_k}^i\|. \end{aligned}$$

It follows from (4.8) (4.9) and (4.15) that

$$\lim_{k\to\infty} H(T_i u_{n_k}, T_i p) = 0 \tag{4.16}$$

for all $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_i p) \leq \|p - x_{n_k}\| + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i u_{n_k}) + H(T_i u_{n_k}, T_i p).$$

From (4.13), (4.14) and (4.16), we obtain $d(p, T_ip) = 0$ for all $i \in \{1, 2, \cdots, m\}$. Since T_ip is closed, so $p \in \bigcap_{i=1}^m F(T_i)$ and hence $p \in \Theta$. This implies by the existence of $\lim_{n\to\infty} \|x_n - p\|$ that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof of conclusion (ii).

Finally, we prove conclusion (iii).

Suppose that T_{i_0} is hemicompact for some $i_0 \in \{1, 2, \cdots, m\}$. From (4.14), we have $\lim_{n\to\infty} d(x_n, T_{i_0}x_n) = 0$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$. From Lemma 3.5, for $y^i_{n_k} \in T_iu_{n_k}$, there exists $c^i_{n_k} \in T_ip$ such that

$$\begin{aligned} H(T_iu_{n_k}, T_ip)^2 &\leq \|u_{n_k} - p\|^2 + \frac{2\alpha}{1-\alpha}\langle u_{n_k} - y^i_{n_k}, p - c^i_{n_k}\rangle. \\ &\leq \|x_{n_k} - p\|^2 + \frac{2\alpha}{1-\alpha}\|u_{n_k} - y^i_{n_k}\|\|p - c^i_{n_k}\|. \end{aligned}$$

It follows from (4.9) that

$$\lim_{k\to\infty} H(T_iu_{n_k}, T_ip) = 0. \tag{4.17}$$

for all $i \in \{1, 2, \cdots, m\}$. For each $i \in \{1, 2, \cdots, m\}$, we have

$$d(p, T_ip) \leq \|p - x_{n_k}\| + d(x_{n_k}, T_ix_{n_k}) + H(T_ix_{n_k}, T_iu_{n_k}) + H(T_iu_{n_k}, T_ip).$$

Since $x_{n_k} \to p \in C$, by (4.14), (4.15) and (4.17), we obtain $d(p, T_ip) = 0$ for all $i \in \{1, 2, \cdots, m\}$. Since T_ip is closed, so $p \in \bigcap_{i=1}^m F(T_i)$. This implies by the existence of $\lim_{n\to\infty} \|x_n - p\|$ that $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof of conclusion (iii).

Since P_{T_i} satisfies the endpoint condition for all $i \in \{1, 2, \cdots, m\}$, we then obtain the following result.

Corollary 4.2. *Let H_1, H_2 be two real Hilbert space and $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying (A1)-(A4), and F_2 is upper semicontinuous in the first argument. For $i = 1, 2, \cdots, m$, let $T_i : C \to K(C)$ be a family of nonexpansive multi-valued mappings with $\alpha \in [0, 1)$ such that $\Theta = \bigcap_{i=1}^m F(T_i) \bigcap SEP(F_1, F_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in C$ arbitrary and*

$$\begin{cases} u_n = J^{F_1}_{r_n}(I - \gamma A^*(I - J^{F_2}_{r_n})A)x_n, \\ x_{n+1} \in \alpha_{0,n}x_n + \sum_{i=1}^m \alpha_{i,n}P_{T_i}u_n, \end{cases} \tag{4.18}$$

where $\alpha_{i,n} \in (0, 1)$ for all $i = 0, 1, 2, \cdots, m$, $\sum_{i=0}^m \alpha_{i,n} = 1$ for all $n \geq 1$, $r_n > 0$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ *for all $i = 1, 2, \cdots, m$;*

(C2) $\liminf_{n\to\infty} r_n > 0$.

Then

(i) The sequence $\{x_n\}$ generated by (4.18) converges weakly to $p \in \Theta$.

(ii) If one of the P_{T_i} is sequentially completely continuous, then $\{x_n\}$ converges strongly to $p \in \Theta$.

(iii) If one of the P_{T_i} is hemicompact, then $\{x_n\}$ converges strongly to $p \in \Theta$.

Proof. By the same proof as that of Theorem 4.1, we have $u_n \to y_n^i \in P_{T_i} u_n$. This implies that

$$d(u_n, T_i u_n) \leq d(u_n, P_{T_i} u_n) \leq \|u_n - y_n^i\| \to 0,$$

as $n \to \infty$ for all $i = 0, 1, 2, \ldots, m$. Since $I - T_i$ is demiclosed at 0, we obtain this results.

Remark 4.3. (i) Theorem 4.1 and Corollary 4.2 extend the corresponding one of Suantai et al. [17] to α-nonexpansive multi-valued mapping and to a common fixed point problem of a family of multi-valued mappings. In fact, if $\alpha = \frac{1}{2}$ and $m = 1$, then we get the Theorems 3.3 and 3.5 in [17]. In addition, we have obtained strong convergence results.

(ii) It is well known that the class of α-nonexpansive multi-valued mappings contains the classes of nonexpansive multi-valued mappings, nonspreading multi-valued mappings and hybrid multi-valued mappings. Thus, Theorem 4.1 and Corollary 4.2 can be applied to these classes of mappings.

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References

[1] M. Abbas, H. Iqbal, J.-C. Yao, *A new iterative algorithm for approximation of fixed points of multivalued generalized α- nonexpansive mappings*, J. Nonlinear Convex Anal., **22**(2021), no. 3, 471-486.

[2] K. Aoyama, F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Analysis: Theory, Methods and Applications, **74**(2011), no. 13, 4387-4391.

[3] B.A. Bin Dehaish, R.K. Alharbi, *On convergence theorems for generalized alpha nonexpansive mappings in Banach spaces*, Journal of Function Spaces, **2021**(2021), 6652741.

[4] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud., **63**(1994), no. 1-4, 123-145.

[5] A. Bnouhachem, *Algorithms of common solutions for a variational inequality, a split equilibrium problem and a hierarchical fixed point problem*, Fixed Point Theory Appl., **2013**(2013), 278.

[6] A. Bnouhachem, *Strong convergence algorithm for split equilibrium problems and hierarchical fixed point problems*, The Scientific World Journal, **2014**(2014), 390956.

[7] C. Chidume, J. Ezeora, *Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings*, Fixed Point Theory Appl., **2014**(2014), 111.

[8] P. Cholamjiak, W. Cholamjiak, *Fixed point theorems for hybrid multivalued mappings in Hilbert spaces*, J. Fixed Point Theory Appl., **18**(2016), no. 3, 673-688.

[9] P.L. Combettes, S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6**(2005), no. 1, 117-136.

[10] J. Deepho, et al., *Convergence analysis of hybrid projection with Cesaro mean method for the split equilibrium and general system of finite variational inequalities*, Journal of Computational and Applied Mathematics, **318**(2017), no. 2-4, 658-673.

[11] K. Fan, *A Minimax Inequality and Applications*, In: Inequalities **3** (O. Shisha, Ed.), Academic Press, San Diego, 1972, 103-113.

[12] Z. He, *The split equilibrium problems and its convergence algorithms*, J. Inequal. Appl., **2012**(2012), 162.

[13] H. Iqbal, M. Abbas, S.M. Husnine, *Existence and approximation of fixed points of multivalued generalized α-nonexpansive mappings in Banach spaces*, Numerical Algorithms, **85**(2020), no. 3, 1029-1049.

[14] K.R. Kazmi, S.H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egypt. Math. Soc., **21**(2013), no. 1, 44-51.
[15] J. Li, *Split equilibrium problems for related games and applications to economic theory*, Optimization, **68**(2019), no. 6, 1203-1222.
[16] J.T. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc., **74**(1968), no. 4, 639-640.
[17] A. Moudafi, M. Théra, *Proximal and dynamical approaches to equilibrium problems*, Lecture Notes in Economics and Mathematical Systems, **477**(1999), 187-201.
[18] S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.
[19] N. Onjai-Uea, W. Phuengrattana, *On solving split mixed equilibrium problems and fixed point problems of hybrid-type multivalued mappings in Hilbert spaces*, J. Inequal. Appl., **2017**(2017), 137.
[20] R. Pant, R. Shukla, *Approximating fixed points of generalized nonexpansive mappings in Banach spaces*, Numerical Functional Analysis and Optimization, **38**(2017), no. 2, 248-266.
[21] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **311**(2005), no. 2, 506-517.
[22] S. Suantai, P. Cholamjiak, Y.J. Cho, et al., *On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces*, Fixed Point Theory Appl., **2016**(2016), 35.
[23] S. Suantai, W. Phuengrattana, *Existence and convergence theorems for λ-hybrid mappings in Hilbert spaces*, Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal., **22**(2015), no. 3-4, 177-188.
[24] U. Witthayarat, A.A.N. Abdou, Y.J. Cho, *Shrinking projection methods for solving split equilibrium problems and fixed point problems for asymptotically nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2015**(2015), 200.
[25] C. Zhang, Y. Li, Y. Wang, *On solving split generalized equilibrium problems with trifunctions and fixed point problems of demicontractive multi-valued mappings*, J. Nonlinear Convex Anal., **21**(2020), no. 9, 2027-2042.

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