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ASYMPTOTICALLY PERIODIC SOLUTIONS FOR FRACTIONAL DIFFERENTIAL VARIATIONAL INEQUALITIES

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Abstract. In this work, we consider a general model of fractional differential variational inequalities formulated by a fractional differential inclusion and a general variational inequality. The existence of mild solutions on a half-line is proved. Furthermore, we study establish the sufficient conditions ensuring the existence and uniqueness of asymptotically periodic solutions based on theory of semigroup and some fixed point arguments. Finally, examples are given to illustrate our theoretical results.
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1. Introduction

For various motivations, differential variational inequalities (DVIs) in Euclidean spaces and general Banach spaces have been studied extensively recently; e.g. Pang and Stewart [28, 31], Avgerinos and Papageorgiou [7], Gwinner [15, 16], Liu Zhenhai et al [24, 23, 22, 21], Anh [2, 3], Ke et al [5, 6, 20] and references cited therein. Built on dynamical systems associated with variational inequalities, differential variational inequalities open up a broad paradigm for the enhanced modeling of complex real-world engineering systems, such as mechanical impact problems, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics and hybrid engineering systems with variable structures.

In the literature, it is seen that classical differential variational inequalities which contain first-order differential systems were systematically studied by Pang and Stewart [28]. Thenceforth, DVIs have been studied extensively and motivated further research on many interesting theoretical problems. Besides, fractional DVIs (DVIs containing fractional derivatives) also play an important role which give us observation numerous phenomena, especially in fractional advection dispersion equation, fluid flow, rheology and the viscoelasticity problem. In [19], authors investigated the existence of solutions for a new class of differential variational inequalities in finite spaces. More recently, in [29], Liu Zhenhai and coauthors studied generalized differential hemivariational inequalities involving the time fractional order derivative operator in Banach spaces with application to a frictional quasistatic contact problem for viscoelastic materials.

Continuing these above works, in this paper, we consider a class of fractional differential variational inequalities in Banach spaces. Our purpose is to study the existence of asymptotically periodic solutions. Let $(X, \|\cdot\|_X)$ be a Banach space and $(U, \|\cdot\|_U)$ be a reflexive Banach space with the dual U^*. We are concerned with the following differential variational inequality

$$^{C}D_t^{\alpha}x(t) - Ax(t) \in F(t, x(t), u(t)), t \in J := [0, \infty), \tag{1.1}$$

$$\langle B(x(t), u(t)) - g(t, x(t), u(t)), v - u(t)\rangle + h(x(t), u(t), v) \geq 0, \forall v \in K, \tag{1.2}$$

$$x(0) = \xi, \tag{1.3}$$

where $^{C}D_t^{\alpha}$ denotes the Caputo's fractional time derivative of order α with the lower limit zero; $(x(\cdot), u(\cdot))$ takes values in $X \times K$, with K being a nonempty convex closed subset of U. The linear part $A : D(A) \subset X \to X$ is a generator of a C_0- semigroup $\{S(t)\}_t \geq 0$ on X; the nonlinear part $F : \mathbb{R}^+ \times X \times U \to \mathcal{P}(X)$ is multivalued. The given single-valued functions $B : X \times U \to U^*, g : \mathbb{R}^+ \times X \times K \to U^*$ and $h : X \times K \times K \to (-\infty; \infty]$ are addressed in the next sections.

In order to highlight the generality of (1.1)-(1.3) and the connections to previous problem classes, we have to mention that in some cases:

(i) *In case that* (1.1) *is classical differential equations/inclusions (involving first-order derivatives),* $h(x, u, v) := \langle \bar{h}(x, u), v - u\rangle$ *for some function* $\bar{h}$*, then* (1.1)-(1.3) *becomes the problem studied in* [9, 28] *with Euclidean phase spaces and in* [6, 2, 15] *with Banach phase spaces.*
For instance, if $X = \mathbb{R}^m, U = \mathbb{R}^n$ *and* $h(x, u, v) := \langle \bar{h}(x, u), v - u\rangle$*, then* (1.1)-(1.3) *becomes the problem studied in* [19]*. In case* $\alpha \in \mathbb{N}^*$*,* (1.1)-(1.3) *return to the DVIs with meaning by Pang and Stewart* [28] *which have been subjected to investigation by many authors* [5, 9, 24, 31].

(ii) *If* $B(x, u) := \bar{B}(u)$ *for some* $\bar{B}$ *and* $h(x, u, v) := \phi(v) - \phi(u)$ *where* $\phi : U \to (-\infty, \infty]$ *is a proper, convex and lower semicontinuous function and* $\alpha = 1$*, then* (1.1)-(1.3) *becomes the problem investigated in* [2, 6].

(iii) *In* [30]*, authors investigated a class of differential variational inequalities, when investigating the variational constraints, the expression of* B *was given as* (1.2)*, while the function* h *was given by a subdifferential of a convex function and involving the memory term.*

The concept of asymptotically periodic solutions is introduced for fractional differential equations by C. Cuevas et al in [10, 11, 12, 13] when author studied some class of fractional differential equations and fractional order functional integro-differential equations. It should be mentioned that, the study of asymptotically periodic solutions to fractional differential systems have attracted much attention from many authors (see [32, 14, 19, 26, 27, 25, 34]) due to the fact that the mathematical modeling of a variety of physical processes gives rise to asymptotically periodic solutions. Our motivation for the present work is that, no attempt has been made to establish the results on asymptotically periodic problems for the general model of fractional differential variational inequality (1.1)-(1.3). Then, together with generalization the previous model of fractional DVIs, the main contribution presented in this paper includes that we establish sufficient conditions ensuring the existence of asymptotically periodic solutions to fractional DVIs.

The structure of the paper is as follows. In Section 2 we provide the necessary preliminary facts. Section 3 gives the existence results on a half-line. Section 4 accounts for the treatment of the existence of asymptotically periodic solutions to considering differential variational inequalities. In the last Section 5, we give two examples to show the applicability of our theoretical results.

2. Preliminaries

2.1. **Measure of noncompactness.** Let E be a Banach space. Denote

$$\mathcal{P}(E) = \{B \subset E : B \neq \emptyset\},$$
$$\mathcal{B}(E) = \{B \in \mathcal{P}(E) : B \text{ is bounded}\}.$$

The Hausdorff measure of noncompactness (MNC) $\chi(\cdot)$ is defined as follows, for $\Omega \in \mathcal{B}(E)$,

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon - \text{net}\}.$$

Denote by $L^1(0,T;E)$ the space of functions defined on $[0,T]$, taking values in E and being integrable in the sense of Bochner. Let $D \subset L^1(0,T;E)$ be such that, for all $f \in D, \|f(t)\| \leq \nu(t)$ for a.e. $t \in [0,T]$, where $\nu \in L^1(0,T;\mathbb{R})$ is a nonnegative function, then we say that D is integrable bounded. We will frequently appeal to the following fundamental result, which the proof is similar to [17, Theorem 4.2.1 and Corollary 4.2.5].

Proposition 2.1. *If $\{w_n\} \subset L^1(0,T;E)$ is integrable bounded, then we have*

$$\chi\left(\left\{\int_0^t w_n(s)ds\right\}\right) \leq 2\int_0^t \chi(\{w_n(s)\})ds,$$

for $t \in [0,T]$.

The following proposition can be used (see e.g. [6, 19]).

Proposition 2.2. *Let $D \subset L^1(0,T;E)$ be such that*

(1) *D is integrable bounded,*
(2) *$\chi(D(t)) \leq q(t)$ for a.e. $t \in [0,T]$,*

where $q \in L^1(0,T;\mathbb{R})$. Then

$$\chi\left(\int_0^t D(s)ds\right) \leq 4\int_0^t q(s)ds;$$

here $\int_0^t D(s)ds = \left\{\int_0^t \xi(s)ds : \xi \in D\right\}$.

In the sequel, the concept of χ-norm of a bounded linear operator $\mathcal{T}$ ($\mathcal{T} \in \mathcal{L}(E)$) will be used:

$$\|\mathcal{T}\|_\chi = \inf\{\beta > 0 : \chi(\mathcal{T}(B)) \le \beta\chi(B) \text{ for all bounded set } B \subset E\}. \tag{2.1}$$

It is noted that the χ- norm of $\mathcal{T}$ can be formulated by

$$\|\mathcal{T}\|_\chi = \chi(\mathcal{T}(\mathbf{B_1})),$$

where $\mathbf{B_1}$ is a unit ball in E. It is know that

$$\|\mathcal{T}\|_\chi \le \|\mathcal{T}\|_{\mathcal{L}(E)},$$

where the last norm is understood as the operator norm in $\mathcal{L}(E)$. Obviously, $\mathcal{T}$ is a compact operator iff $\|\mathcal{T}\|_\chi = 0$.

We also employ a relation between the so-called k-condensing and k-Lipschitz properties of a nonlinear map. Let $\widetilde{E}$ be another Banach space and $\widetilde{\chi}$ the Hausdorff MNC on $\widetilde{E}$. A mapping $\Phi : E \to \widetilde{E}$ is said condensing with respect to a constant k (k-condensing) if

$$\widetilde{\chi}(\Phi(\Omega)) \le k\chi(\Omega), \forall\Omega \in \mathcal{P}(E).$$

It is well known in [1] that if Φ is a Lipschitz map with a constant k (k-Lipschitz), that is,

$$\|\Phi(x) - \Phi(\widetilde{x})\|_{\widetilde{E}} \le k\|x - \widetilde{x}\|_E, \forall x, \widetilde{x} \in E,$$

then Φ is k-condensing.

Here is the definition of a condensing multimap with respect to the MNC χ.

Definition 2.1. Let D be a subset in E. An upper continuous multimap $\mathcal{F} : D \to P(X)$ is called to be condensing with respect to MNC χ (χ-condensing) iff for any bounded set $\Omega \subset D$, the relation $\chi(\Omega) \le \chi(\mathcal{F}(\Omega))$ implies that Ω is relatively compact.

Theorem 2.3. [1, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of X and let $F : \mathcal{M} \to K_v(\mathcal{M})$ be a closed and χ-condensing multimap. Then, $Fix(F) := \{x \in \mathcal{M} : x \in F(x)\}$ is nonempty.*

Let $BC(J; E)$ be the space of bounded continuous functions on J taking values in E. By the idea of this fixed point theorem, we consider the MNC in $BC(J; E)$ as follows: Let B be a bounded set in $BC(J; E)$ and $\pi_T : BC(J; E) \to C([0, T]; E)$ is the restriction of $u \in BC(J; E)$ to the interval $[0, T]$, define

$$d_\infty(B) = \lim_{T\to\infty} \sup_{u\in B} \sup_{t\ge T} \|u(t)\|,$$

$$\chi_\infty(B) = \sup_{T>0} \chi_T(B),$$

where χ_T is Hausdorff MNC in $C([0, T]; E)$. Put

$$\chi^*(B) = d_\infty(B) + \chi_\infty(B).$$

It is shown that χ^* satisfies all properties stated in Definition 2.1. In addition, if $\chi^*(B) = 0$ then B is relatively compact in $BC(J; E)$ (see [4, 5]). Specially, if $u \in C(\mathbb{R}^+; E)$ then $d_\infty(\{u\}) = 0$ iff $u \in BC(J; E)$.

2.2. **Fractional calculus.** Let $L^1(0,T;E)$ be the space of integrable functions on interval $[0,T]$, in the Bochner sense.

Definition 2.2. The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;E)$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function, provided the integral converges.

Definition 2.3. Let $N \geq 1$ be an integer. For a function $f \in C^N([0,T];E)$, the Caputo fractional derivative of order $\alpha \in (N-1,N)$ is defined by

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)ds.$$

For $u \in C^N([0,T];E)$, we have the following formulas

$$\begin{aligned} {}^C D_0^\alpha I_0^\alpha u(t) &= u(t), \\ I_0^\alpha \, {}^C D_0^\alpha u(t) &= u(t) - \sum_{k=0}^{N-1} \frac{u^k(0)}{k!} t^k. \end{aligned}$$

We consider the problem

$$\begin{aligned} {}^C D_0^\alpha y(t) &= Ay(t) + f(t), t \in J, && (2.2) \\ y(0) &= \xi. && (2.3) \end{aligned}$$

where A is the generator of a C_0-semigroup $S(\cdot)$ such that

$$\|S(t)\| \leq M, \forall t \geq 0.$$

Then, based on [34, Lemma 3.1], we have

$$y(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s)ds, t \geq 0,$$

where

$$\begin{aligned} \mathcal{S}_\alpha(t)x &= \int_0^\infty \phi_\alpha(\theta) S(t^\alpha \theta) x d\theta, \\ \mathcal{P}_\alpha(t)x &= \alpha \int_0^\infty \theta \phi_\alpha(\theta) S(t^\alpha \theta) x d\theta, \forall x \in E, \end{aligned}$$

here ϕ_α being a probability density function defined on $(0,\infty)$ that has the following expression

$$\phi_\alpha(\theta) = \frac{1}{\alpha\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin n\pi\alpha, \theta \in (0,\infty).$$

Due to [4] and [26], we have the below lemma, which reveals several important properties of $\{\mathcal{S}_\alpha(t), t \geq 0\}$ and $\{\mathcal{P}_\alpha(t), t \geq 0\}$.

Lemma 2.4. *The operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ have the following properties:*

(1) *If* $\sup_{t\geq 0}\|S(t)\|_{\mathcal{L}(E)} \leq M$, *then for any fixed* $t \geq 0$, $\mathcal{S}_\alpha(t)$ *and* $\mathcal{P}_\alpha(t)$ *are linear and bounded operators, i.e. for any* $x \in E$,

$$\|\mathcal{S}_\alpha(t)x\| \leq M\|x\|, \quad \|\mathcal{P}_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

(2) $\{\mathcal{S}_\alpha(t), t \geq 0\}$ *and* $\{\mathcal{P}_\alpha(t), t \geq 0\}$ *are strongly continuous.*
(3) $\{\mathcal{S}_\alpha(t), t > 0\}$ *and* $\{\mathcal{P}_\alpha(t), t > 0\}$ *are compact if* $\{S(t), t > 0\}$ *is compact.*
(4) *If* $S(t)$ *is exponentially stable, i.e. there exists a positive number* δ *such that*

$$\|S(t)\|_{\mathcal{L}(E)} \leq Me^{-\delta t},$$

then

$$\|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(E)} \leq \frac{m}{(1+t)^\alpha}, \quad \|\mathcal{P}_\alpha(t)\|_{\mathcal{L}(E)} \leq \frac{m}{(1+t)^{2\alpha}},\ t > 0 \tag{2.4}$$

for a constant $m > 0$ *given by*

$$m = M\max\left\{\sup_{t>0} E_{\alpha,1}(-at^\alpha)(1+t)^\alpha;\ \sup_{t>0} E_{\alpha,\alpha}(-at^\alpha)(1+t)^{2\alpha}\right\}, \tag{2.5}$$

where $E_{\alpha,\beta}$ *is the Mittag-Leffler function defined by*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}.$$

Now, let $p > \dfrac{1}{\alpha}$, we define the *Cauchy operator*

$$\mathcal{W}_\alpha : L^p(0,T;E) \to C([0,T];E),$$
$$\mathcal{W}_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds. \tag{2.6}$$

By [18, Proposition 2.5], we have the estimates and properties imposed on $\mathcal{W}_\alpha$ given below.

Proposition 2.5. *Suppose that* A *generates a norm-continuous* C_0*-semigroup* $\{S(t)\}_{t\geq 0}$ *in* E, *i.e., the map* $(0,\infty) \mapsto S(t) \in \mathcal{L}(E)$ *is continuous. Then*

(1) *For each bounded set* $\Omega \subset L^p(0,T;E)$, $\mathcal{W}_\alpha(\Omega)$ *is an equicontinuous set in* $C([0,T];E)$. *Moreover, we have the following estimate*

$$\chi_{\mathcal{C}}(\mathcal{W}_\alpha(\Omega)) \leq 4\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|_\chi \chi(\Omega(s))ds,$$

where $\|\cdot\|_\chi$ *is the* χ*-norm given by* (2.1), χ_T *is Hausdorff MNC on* $C([0,T];E)$.
(2) *If* $\{f_n\} \subset L^p(0,T;E), p > 1$, *is a semicompact sequence, then* $\mathcal{W}_\alpha(f_n)$ *is relatively compact in* $C([0,T];E)$. *Moreover, if* $f_n \rightharpoonup f$ *in* $L^p(0,T;E)$, *then* $\mathcal{W}_\alpha(f_n) \to \mathcal{W}_\alpha(f^*)$ *in* $C([0,T];E)$.

2.3. **General variational inequalities.** We consider the general variational inequality problem in a reflexive Banach space U with the dual U^*:

Let K be a nonempty convex closed subset of U, $h : K \times K \to [-\infty, \infty]$ and $B : U \to U^*$. Find $u \in K$ such that

$$\langle Bu, v - u\rangle + h(u, v) \geq 0, \forall v \in K. \tag{2.7}$$

Assume that h and B satisfy the following standing assumptions:

(**B**) B is strongly monotone, i.e. there exists a constant $m_B > 0$ such that

$$\langle Bu - Bv, u - v\rangle \geq m_B \|u - v\|^2,$$

for all $u, v \in U$. Moreover, assume that B satisfies

$$\liminf_{\lambda\to 0^+} \langle B(\lambda u + (1 - \lambda)v), u - v\rangle \leq \langle B(v), v - u\rangle,\ \forall u, v \in K.$$

(**H**) h satisfies that:

(1) *$D_1(h) = \{u \in U : h(u, v) \neq -\infty, \forall v \in K\}$ is nonempty.*

(2) *$h(u, u) = 0$ for all $u \in K$.*

(3) *$u \to h(u, v)$ is concave and weakly upper semicontinuous for all $v \in K$, that is,*

$$h(\lambda u_1 + (1 - \lambda)u_2, v) \geq \lambda h(u_1, v) + (1 - \lambda)h(u_2, v),$$

for all $u_1, u_2, v \in K, \lambda \in (0, 1)$. Moreover, if every $u \in K$ and any $\{u_n\} \subset K$ such that $u_n \rightharpoonup u$ in K, we have

$$\limsup_{n\to\infty} h(u_n, v) \leq h(u, v).$$

(4) *$v \to h(u, v)$ is convex for all $v \in K$, that is,*

$$h(u, \lambda v_1 + (1 - \lambda)v_2) \leq \lambda h(u, v_1) + (1 - \lambda)h(u, v_2),$$

for all $u, v_1, v_2 \in K$ and for all $\lambda \in (0, 1)$.

(5) *$h(u, v) + h(v, u) \leq \kappa_h \|u - v\|^2, \forall u, v \in K.$*

Remark 2.1. (i) A special case of the condition (**B**) has been treated in [2, 6]. For instance, in these works, B is a linear continuous operator from U to U^* defined by

$$\langle u, Bv\rangle = b(u, v), \forall u, v \in U,$$

where $b : U \times U \to \mathbb{R}$ is a bilinear continuous function on $U \times U$ such that

$$b(u, u) \geq \eta_B \|u\|_U^2, \forall u \in U.$$

(ii) If $h(u, v) = \phi(v) - \phi(u)$ where $\phi : U \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function such that $K := V \cap dom\phi \neq \emptyset$, then $D_1(h) = K_\phi$ and h satisfies (**H**). In this case, we obtain an elliptic variational inequalities and our abstract system (1.1)-(1.3) becomes the differential variational inequalities of parabolic-elliptic type. Such classes of differential variational inequalities have been studied in the recent papers [2, 6].

(iii) The condition (**H**)(1)-(**H**)(4) were considered in [24] when authors established the sufficient conditions ensuring the convexity and closedness properties of the solution set for variational inequality (2.7).

Lemma 2.6. *Let K be a nonempty, convex and closed subset of U. Assume that* (**B**) *and* (**H**) *hold. Then the general variational inequality* (2.7) *has a unique solution provided that $m_B > \kappa_h$ and there exist $u_0 \in U$ and $r > 0$ such that*

$$\langle Bv, v - u_0\rangle - h(v, u_0) > 0 \text{ for all } v \in U \text{ with } \|v\|_U > r.$$

Proof. The existence of solutions to (2.7) is proved by a similar process to that in [24, Lemma 3.2] via KKM principle argument. It remains to prove the uniqueness. Let u_1, u_2 be two solutions to (2.7), then

$$\langle Bu_1, v - u_1\rangle + h(u_1, v) \geq 0, \forall v \in K, \tag{2.8}$$

$$\langle Bu_2, v - u_2\rangle + h(u_2, v) \geq 0, \forall v \in K. \tag{2.9}$$

Inserting $v = u_2 \in K$ in (2.8) and $v = u_1$ in (2.9), then we sum the resulting inequalities and combine with hypotheses (**B**) and (**H**5) to deduce

$$-(m_B - \kappa_h)\|u_1 - u_2\|_U^2 \geq 0,$$

which yields $u_1 = u_2$. □

3. Initial value problem on a half-line

In this section, we study the existence of mild solution on $\mathbb{R}^+$ to (1.1)-(1.2) with given initial condition (1.3). We introduce the hypotheses as follows:

($\mathcal{H}_A$) *A is a closed linear operator generating a norm-continuous C_0-semigroup $\{S(t)\}_{t\geq 0}$ in X. Moreover, $\{S(t)\}_{t\geq 0}$ is supposed to be exponentially stable, namely,*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{-\delta t}, \quad \forall t \geq 0, \text{ for some } \delta > 0.$$

($\mathcal{H}_B$) *$B : X \times U \to U^*$ satisfies*

(1) *$x \mapsto B(x, u)$ is Lipschitz continuous for all $u \in U$, i.e.*

$$\|B(x_1, u) - B(x_2, u)\|_{U^*} \leq L_B\|x_1 - x_2\|_U,$$

for all $x_1, x_2 \in X$ and for all $u \in U$.

(2) *$u \mapsto B(x, u)$ is strongly monotone for all $x \in X$, i.e. there exists a constant $m_B > 0$ so that*

$$\langle B(x, u_1) - B(x, u_2), u_1 - u_2\rangle \geq m_B\|u_1 - u_2\|_U^2,$$

for all $x \in X$ and for all $u_1, u_2 \in U$. In addition, $B(x, \cdot)$ satisfies

$$\liminf_{\lambda\to 0^+}\langle B(x, \lambda u + (1 - \lambda)v), u - v\rangle \leq \langle B(x, v), v - u\rangle, \ \forall u, v \in K.$$

($\mathcal{H}_F$) *$F : \mathbb{R}^+ \times X \times U \to \mathcal{P}(X)$ has nonempty, convex, weakly compact and*

(1) *For each $(x, y) \in X \times U$, $F(\cdot, x, u) : \mathbb{R}^+ \to \mathcal{P}(X)$ is strongly measurable.*

(2) *For each t, $F(t, \cdot, \cdot) : X \times U \to \mathcal{P}(X)$ is upper semicontinuous.*

(3) *If $S(t)$ is noncompact, there exist $p(\cdot), q(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that*

$$\chi F(t, C, D) \leq p(t)\chi(C) + q(t)\mathcal{U}(D),$$

here χ,$\mathcal{U}$ stand for the Hausdorff MNC in X and U, respectively.

(4) *There exist* $a(\cdot), b(\cdot), c(\cdot) \in L^p_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ *such that*

$$\|F(t,x,u)\| := \sup\{\|\xi\|_X : \xi \in F(t,x,u)\} \le a(t)\|x\|_X + b(t)\|u\|_U + c(t),$$

for all $t \ge 0, x \in X$ *and* $u \in U$.

($\mathcal{H}_h$) $h : X \times K \times K \to (-\infty,\infty]$ *satisfy*

(1) *For all* $x_1, x_2 \in X$ *and* $u, v \in K$, *one has*

$$h(x_1,u,v) + h(x_2,v,u) \le m_h\|x_1 - x_2\|_X\|u - v\|_U + \kappa_h\|u - v\|_U^2,$$

for some $m_h > 0$.

(2) *For each* $x \in X$, $h(x,\cdot,\cdot) : K \times K \to (-\infty,\infty]$ *satisties* $(\mathbf{H})(1)$- $(\mathbf{H})(4)$.

($\mathcal{H}_g$) $g : \mathbb{R}^+ \times X \times U \to U^*$ *is Lipschitz continuous, namely,*

$$\|g(t,x,u) - g(t,y,v)\|_{U^*} \le \eta_{1g}\|x - y\|_X + \eta_{2g}\|u - v\|_U,$$

for all $t \ge 0, x, y \in X$ *and* $u, v \in U$.

Remark 3.1. (1) The condition $(\mathcal{H}_h)(1)$ ensures that $(\mathbf{H})(5)$ holds.

(2) We can find functions h which satisfies $(\mathcal{H}_h)$. For a very simple example, let $X = U = \mathbb{R}, K = [1,2]$ and define $h : X \times K \times K \to (-\infty;\infty]$ by $h(x,u,v) := |x|(v - u) + (v^2 - uv)$.

Let $p > \dfrac{1}{\alpha}$ be fixed. For $x \in BC(J;X)$ and $u \in BC(J;U)$, we denote

$$Sel^p_F(x,u) := \{f \in L^p_{loc}(\mathbb{R}^+;X) : f(t) \in F(t,x(t),u(t)) \text{ for a.e. } t > 0\}.$$

Definition 3.1. A pair of continuous functions (x,u) in $BC(J;X) \times BC(J;U)$ is a mild solution of (1.1)-(1.3) on J if there exists $f \in Sel^p_F(x,u)$ such that

$$x(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds$$
$$\langle B(x(t),u(t)) - g(t,x(t),u(t), v - u(t)\rangle + h(x(t),u(t),v) \ge 0,\ \forall v \in K,$$

for any $t > 0$.

Now, for each $(x,z) \in X \times U^*$, consider the original form of (1.2)

$$\langle B(x,u) - z, v - u\rangle + h(x,u,v) \ge 0,\ \forall v \in K. \tag{3.1}$$

Lemma 3.1. *Let* $(\mathcal{H}_B)$ *and* $(\mathcal{H}_h)$ *hold. In addition, we suppose* $m_B > \kappa_h$. *Then for each* $(x,z) \in X \times U^*$, *there exists a unique solution* $u \in K$ *of* (3.1). *Moreover, the solution mapping*

$$\mathbb{VI} : X \times U^* \to K,$$
$$(x,z) \mapsto u,$$

is Lipschitzian, more precisely,

$$\|\mathbb{VI}(x_1,z_1) - \mathbb{VI}(x_2,z_2)\|_U \le \frac{L_B + m_h}{m_B - \kappa_h}\|x_1 - x_2\|_X + \frac{1}{m_B - \kappa_h}\|z_1 - z_2\|_{U^*}, \tag{3.2}$$

for all $x_1, x_2 \in X$ *and* $z_1, z_2 \in U^*$.

Proof. By Lemma 2.6, for each $(x, z) \in X \times U^*$, there exists a unique solution $u \in K$ of (3.1). Let $x_1, x_2 \in X, z_1, z_2 \in U^*$ and $\mathbb{VI}(x_1, z_1) = u_1, \mathbb{VI}(x_2, z_2) = u_2$, one has

$$\langle B(x_1, u_1) - z_1, v - u_1\rangle + h(x_1, u_1, v) \geq 0, \forall v \in K, \tag{3.3}$$

$$\langle B(x_2, u_2) - z_2, v - u_2\rangle + h(x_2, u_2, v) \geq 0, \forall v \in K, \tag{3.4}$$

Take $v = u_2 \in K$ in (3.3) and $v = u_1 \in K$ in (3.4), then combining them, we have

$$\langle B(x_1, u_1) - B(x_2, u_2), u_2 - u_1\rangle + \langle z_1 - z_2, u_1 - u_2\rangle + h(x_1, u_1, u_2) + h(x_2, u_2, u_1) \geq 0.$$

Thanks to the condition $(\mathcal{H}_B)$, the first term of left hand side could be estimated as follows

$$\begin{aligned}\langle B(x_1, u_1) - B(x_2, u_2), u_2 - u_1\rangle &= -\langle B(x_1, u_1) - B(x_2, u_1), u_1 - u_2\rangle \\ &\quad - \langle B(x_2, u_1) - B(x_2, u_2), u_1 - u_2\rangle \\ &\leq -m_B\|u_1 - u_2\|_U^2 + L_B\|x_1 - x_2\|_X\|u_1 - u_2\|_U,\end{aligned}$$

and for the second term, one has the simple estimation that

$$\langle z_1 - z_2, u_1 - u_2\rangle \leq \|z_1 - z_2\|_{U^*}\|u_1 - u_2\|_U.$$

By $(\mathcal{H}_h)$, we obtain

$$h(x_1, u_1, u_2) + h(x_2, u_2, u_1) \leq m_h\|x_1 - x_2\|_X\|u_1 - u_2\|_U + \kappa_h\|u_1 - u_2\|^2.$$

Therefore, we have

$$\|u_1 - u_2\|_U \leq \frac{L_B + m_h}{m_B - \kappa_h}\|x_1 - x_2\|_X + \frac{1}{m_B - \kappa_h}\|z_1 - z_2\|_{U^*},$$

to deduce the conclusion. □

In the sequel, we consider the following variational inequality

$$\langle B(x, u) - g(\tau, x, u), v - u\rangle + h(x, u, v) \geq 0,\ \forall v \in K, \tag{3.5}$$

for given $\tau \in \mathbb{R}^+$ and $x \in X$. We arrive at the characteristic property of the solution set to (3.5).

Lemma 3.2. *Let $(\mathcal{H}_B)$, $(\mathcal{H}_g)$ and $(\mathcal{H}_h)$ hold. In addition, suppose that $\eta_B > \kappa_h + \eta_{2h}$. Then for each $(\tau, x) \in \mathbb{R}^+ \times X$, there exists a unique solution $u \in U$ of (3.5). Moreover, the solution mapping*

$$\begin{aligned}\mathcal{VI} : \mathbb{R}^+ \times X &\to U,\\ (\tau, x) &\mapsto u,\end{aligned}$$

is Lipschitzian, more precisely

$$\|\mathcal{VI}(\tau, x_1) - \mathcal{VI}(\tau, x_2)\|_U \leq \frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}\|x_1 - x_2\|_X, \tag{3.6}$$

for all $\tau \in \mathbb{R}^+$ and $x_1, x_2 \in X$.

Proof. For each $\tau \in \mathbb{R}^+$ and $x \in X$, one has

$$\begin{aligned}\|\mathbb{VI}(x, g(\tau, x, u_1)) - \mathbb{VI}(x, g(\tau, x, u_2))\|_U &\le \frac{1}{m_B - \kappa_h}\|g(\tau, x, u_1) - g(\tau, x, u_2)\|_{U^*} \\ &\le \frac{\eta_{2g}}{m_B - \kappa_h}\|u_1 - u_2\|_U,\end{aligned}$$

thanks to Lemma 3.5 and $(\mathcal{H}_g)$. Thus, if $\eta_{2g} < m_B$, then $\mathbb{V}I(x, g(\tau, x, \cdot))$ is a contraction mapping. It is deduced that there exists a unique $u \in U$ satisfying $\mathbb{V}I(x, g(\tau, x, u)) = u$, which implies $u = \mathcal{VI}(\tau, x)$.

Now, let $u_1 = \mathcal{VI}(\tau, x_1)$ and $u_2 = \mathcal{VI}(\tau, x_2)$, here $\tau \in \mathbb{R}^+, x_1, x_2 \in X$ are given. Then by using Lemma 3.5 and $(\mathcal{H}_g)$ again, we obtain

$$\begin{aligned}\|u_1 - u_2\|_U &= \|\mathbb{VI}(x_1, g(\tau, x_1, u_1)) - \mathbb{VI}(x_2, g(\tau, x_2, u_2))\|_U \\ &\le \frac{L_B + m_h}{m_B - \kappa_h}\|x_1 - x_2\|_X + \frac{1}{m_B - \kappa_h}\|g(\tau, x_1, u_1) - g(\tau, x_2, u_2)\|_{U^*} \\ &\le \frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h}\|x_1 - x_2\|_X + \frac{\eta_{2g}}{m_B - \kappa_h}\|u_1 - u_2\|_U\end{aligned}$$

It is equivalent to

$$\|u_1 - u_2\|_U \le \frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}\|x_1 - x_2\|_X.$$

We obtain the conclusion of the lemma. □

Due to the above description, we convert (1.1)-(1.3) to the following system

$$^C D_t^\alpha x(t) - Ax(t) \in F(t, x(t), u(t)), t \in J, \tag{3.7}$$

$$u(t) = \mathcal{VI}(t, x(t)), t \in J, \tag{3.8}$$

$$x(0) = \xi. \tag{3.9}$$

Denote

$$\begin{aligned}&\mathcal{F} : \mathbb{R}^+ \times X \to \mathcal{P}(X), \\ &\mathcal{F}(t, x) = F(t, x, \mathcal{VI}(t, x)).\end{aligned}$$

Then, one has the differential inclusion

$$^C D_t^\alpha x(t) - Ax(t) \in \mathcal{F}(t, x(t)), t \in J, \tag{3.10}$$

$$x(0) = \xi. \tag{3.11}$$

We define

$$\begin{aligned}&Sel_{\mathcal{F}}^p : \quad C(J; X) \to \mathcal{P}(L_{loc}^p(\mathbb{R}^+; X)), \\ &Sel_{\mathcal{F}}^p(x) := \{f \in L_{loc}^p(\mathbb{R}^+; X) : f(t) \in \mathcal{F}(t, x(t)) \text{ for a.e. } t > 0\}.\end{aligned}$$

It is easily seen that a pair (x, u) is a mild solution of (1.1)-(1.3) iff there exists $f \in Sel_{\mathcal{F}}^p(x)$ such that

$$\begin{aligned}x(t) &= \mathcal{S}_\alpha(t)\xi + \int_0^t (t - s)^{\alpha - 1}\mathcal{P}_\alpha(t - s)f(s)ds \\ u(t) &= \mathcal{VI}(t, x(t)),\end{aligned}$$

for any $t > 0$.

We have the following estimates of $\chi(\mathcal{F}(t, D))$:

$$\begin{aligned}\chi(\mathcal{F}(t,D)) &= \chi(F(t,D,\mathcal{VI}(t,D)))\\ &\le p(t)\chi(D) + q(t)\mathcal{U}(\mathcal{VI}(t,D))\\ &\le \left[p(t) + \frac{q(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}}\right]\chi(D) := \tilde{p}(t)\chi(D). \qquad (3.12)\end{aligned}$$

and we have the estimate

$$\begin{aligned}\|\mathcal{F}(t,x)\| &= \|F(t,x,\mathcal{VI}(t,x))\|\\ &\le a(t)\|x\|_X + b(t)\|\mathcal{VI}(t,x)\|_U + c(t)\\ &\le a(t)\|x\|_X + \frac{b(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}}\|x\|_X + \|\mathcal{VI}(t,0)\|_U + c(t)\\ &\le \tilde{a}(t)\|x\|_X + \tilde{b}(t), \qquad (3.13)\end{aligned}$$

here

$$\tilde{p}(t) = p(t) + \frac{q(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}},$$
$$\tilde{a}(t) = a(t) + \frac{b(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}}$$

and

$$\tilde{b}(t) = \|\mathcal{VI}(t,0)\|_U + c(t).$$

Then, one has the properties of $Sel^p_{\mathcal{F}}$ due to [2, Lemma 4.1] as follows.

Lemma 3.3. *Assume that $(\mathcal{H}_B)$, $(\mathcal{H}_F)$, $(\mathcal{H}_g)$ and $(\mathcal{H}_h)$ hold. Then, $Sel^p_{\mathcal{F}}$ is well-defined and weakly u.s.c. with weakly compact and convex values. In particular, $Sel^p_{\mathcal{F}}$ is a weakly quasicompact multimap.*

Denote the solution operator

$$\begin{aligned}&\mathcal{Q}: \quad BC(J;X) \to \mathcal{P}(BC(J;X)),\\ &\mathcal{Q}(x)(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds,\ f \in Sel^p_{\mathcal{F}}(x).\end{aligned}$$

The closedness of $\mathcal{Q}$ is shown by the following lemma.

Lemma 3.4. *Assume that $(\mathcal{H}_A)$, $(\mathcal{H}_B)$, $(\mathcal{H}_F)$, $(\mathcal{H}_g)$ and $(\mathcal{H}_h)$ are satisfied. Then, the solution operator $\mathcal{Q}$ is closed.*

Proof. Let $\{y_n\} \subset BC(J;X)$ be a sequence converging to y^* and $z_n \in \mathcal{Q}(y_n)$ be such that $z_n \to z^*$. By the definition of $\mathcal{Q}$, one can take $f_n \in Sel^p_{\mathcal{F}}(y_n)$ such that

$$z_n(t) = \mathcal{S}_\alpha(t)\xi + \mathcal{W}_\alpha(f_n)(t), t > 0, \qquad (3.14)$$

where $\mathcal{W}_\alpha$ is defined in (2.6). By Lemma 3.3, $f_n \rightharpoonup f^*$ in $L^p_{loc}(\mathbb{R}^+;X)$ with $f^* \in Sel^p_{\mathcal{F}}(y^*)$. We will show that

$$z^*(t) = \mathcal{S}_\alpha(t)\xi + \mathcal{W}_\alpha(f^*)(t), t > 0. \qquad (3.15)$$

Let $t > 0$, take $T > 0$ such that $t \geq T$ and consider the sequence $\{f_n|_{[0,T]}\}$. As argued in the proof of Lemma 3.3, this sequence is semicompact by the growth property of $\mathcal{F}$. Then by Proposition 2.5, $\mathcal{W}_\alpha(f_n) \to \mathcal{W}_\alpha(f^*)$ in $C([0,T];X)$ and in particular, $\mathcal{W}_\alpha(f_n)(t) \to \mathcal{W}_\alpha(f^*)(t)$ as $n \to \infty$ in X. Then, one can pass to the limit in (3.14) to get (3.15). The proof is complete. □

Lemma 3.5. *Let the hypotheses of Lemma 3.4 hold. If*

$$\ell := 4m \sup_{t\in J} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \left[p(s) + \frac{q(s)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}} \right] ds < \infty,$$

then

$$\chi_\infty(\mathcal{Q}(D)) \leq \ell \cdot \chi_\infty(D),$$

for all bounded sets $D \subset BC(J;X)$, here m is given by (2.5).

Proof. Let $D \subset BC(J;X)$ be a bounded set. For $y \in D$, one has

$$\mathcal{Q}(y)(t) = \mathcal{S}_\alpha(t)\xi + \mathcal{W}_\alpha \circ Sel^p_{\mathcal{F}}(y)(t).$$

Setting $\Omega = Sel^p_{\mathcal{F}}(D)|_{[0,T]}$. We observe that Ω is bounded in $L^p(0,T;X)$, so

$$\pi_T(\mathcal{W}_\alpha \circ Sel^p_{\mathcal{F}}(D)) = \mathcal{W}_\alpha(\Omega)$$

obeys the following estimate

$$\chi(\Omega(s)) \leq 4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\|_\chi \cdot \chi(\Omega(s)) ds, \tag{3.16}$$

due to Proposition 2.5. Now deploying (3.12), one has

$$\begin{aligned} \chi\left(\Omega(s)\right) &\leq \chi(\mathcal{F}(s, D(s))) \\ &\leq \tilde{p}(s)\chi(\pi_T(D(s))) \\ &\leq \tilde{p}(s)\chi_T(\pi_T(D)). \end{aligned}$$

Putting the last estimate in (3.16) and noting that

$$\|\mathcal{P}_\alpha(t-s)\|_\chi \leq \|\mathcal{P}_\alpha(t-s)\|_{\mathcal{L}(X)}$$

then using Lemma 2.4(4), we yield

$$\chi_\infty(\mathcal{Q}(D)) \leq 4m \sup_{t\in J} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \tilde{p}(s) ds \chi_\infty(D).$$

Therefore

$$\chi_\infty(\mathcal{Q}(D)) \leq 4m \sup_{t\in J} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \left(p(s) + \frac{q(s)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}} \right) ds \chi_\infty(D),$$

or equivalently

$$\chi_\infty(\mathcal{Q}(D)) \leq \ell \chi_\infty(D).$$

The proof is complete. □

Lemma 3.6. *Let the hypotheses of Lemma 3.5 hold. If*

$$\zeta_1 := \lim_{t\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}(\|\mathcal{V}\mathcal{I}(s,0)\|_U + c(s))ds = 0; \tag{3.17}$$

$$\zeta_2 := \sup_{t>0}\int_0^{\sigma t} \frac{1}{(1+t-s)^{2\alpha}}\left[a(s) + \frac{b(s)(L_B+m_h+\eta_{1g})}{m_B-\kappa_h-\eta_{2g}}\right]ds < \infty; \tag{3.18}$$

$$\zeta_3 := \sup_{t>0}\int_{\sigma t}^{t} \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\left[a(s) + \frac{b(s)(L_B+m_h+\eta_{1g})}{m_B-\kappa_h-\eta_{2g}}\right]ds < \infty. \tag{3.19}$$

for some $\sigma \in (0,1)$, then

$$d_\infty(\mathcal{Q}(D)) \le \zeta_3 d_\infty(D),$$

for all bounded sets $D \subset BC(J;X)$.

Proof. Let $D \subset BC(J;X)$ be a bounded set. Then for each $y \in D$, there exists $f \in Sel^p_{\mathcal{F}}(y)$ such that

$$\mathcal{Q}(y) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds.$$

Let $\epsilon > 0$ be any arbitrary small number. By Lemma 2.4, we have

$$\lim_{t\to\infty}\|\mathcal{S}_\alpha(t)\xi\|_X = 0,$$

thus,

$$d_\infty(\mathcal{Q}(D)) = d_\infty(\mathcal{W}_\alpha \circ Sel^p_{\mathcal{F}}(D)).$$

Now, we are in a position to evaluate $d_\infty(\mathcal{W}_\alpha \circ Sel^p_{\mathcal{F}}(D))$. Let $z \in \mathcal{W}_\alpha \circ Sel^p_{\mathcal{F}}(y), y \in D$. Taking $f \in Sel^p_{\mathcal{F}}(y)$ such that

$$z(t) = \mathcal{W}_\alpha f(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds.$$

We have

$$\begin{aligned}
\|z(t)\|_X &\le \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\|f(s)\|_X ds\\
&\le \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}(\tilde{a}(s)\|y(s)\|_X + \tilde{b}(s))ds,\\
&\le m\int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\tilde{b}(s)ds + m\left(\int_0^{\sigma t} + \int_{\sigma t}^t\right)\frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\tilde{a}(s)\|y(s)\|_X ds\\
&:= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

thanks to (3.13). By using hypothesis (3.17), we have

$$\lim_{t\to\infty} I_1(t) = 0. \tag{3.20}$$

For $I_2(t)$, using the boundedness of D, take $R > 0$ such that

$$\sup\{\|y\|_{BC} : y \in D\} \le R.$$

Then

$$\begin{aligned}
I_2(t) &\le mR\int_0^{\sigma t}\frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\tilde{a}(s)ds\\
&\le m\frac{R}{[(1-\sigma)t]^{1-\alpha}}\int_0^{\sigma t}\frac{\tilde{a}(s)}{(1+t-s)^{2\alpha}}ds\\
&\le m\frac{R}{[(1-\sigma)t]^{1-\alpha}}\int_0^{\sigma t}\frac{1}{(1+t-s)^{2\alpha}}\left[a(s)+\frac{b(s)(L_B+m_h+\eta_{1g}}{m_B-\kappa_h-\eta_{2g}}\right]ds\\
&\le \frac{mR\zeta_2}{[(1-\sigma)t]^{1-\alpha}},
\end{aligned}\tag{3.21}$$

here ζ_2 is given in (3.18). On the other hand, we have

$$\begin{aligned}
I_3(t) &\le m\left(\int_{\sigma t}^{t}\frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\tilde{a}(s)ds\right)\sup_{r\ge\sigma t}\|y(r)\|_X\\
&\le \zeta_3\sup_{r\ge\sigma t}\|y(r)\|_X,
\end{aligned}\tag{3.22}$$

where ζ_3 is given in (3.19). Combining (3.20), (3.21) and (3.22), we obtain

$$\|z(t)\|_X \le I_1(t)+\frac{mR\zeta_2}{[(1-\sigma)t]^{1-\alpha}}+\zeta_3\sup_{r\ge\sigma t}\|y(r)\|_X,$$

for all $t>0, y\in D, z\in \mathcal{W}_\alpha Sel^p_{\mathcal{F}}(y)$. The last inequality implies

$$d_\infty(\mathcal{Q}(D))\le\zeta_3 d_\infty(D).$$

The proof is complete. □

Theorem 3.7. *Let the hypotheses of Lemma 3.6 hold. Assume that*

$$\ell<1.$$

Then, problem (1.1)-(1.3) *has at least one mild solution in* $C(J;X)$ *provided that*

$$\varkappa := \sup_{t>0}\int_0^t\frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\left[a(s)+\frac{b(s)(L_B+m_h+\eta_{1g})}{m_B-\kappa_h-\eta_{2g}}\right]ds<1. \tag{3.23}$$

Proof. By Lemma 3.4, $\mathcal{Q}$ is closed. Moreover, $\mathcal{Q}$ is χ^*-condensing due to Lemma 3.5 and 3.6. Indeed, from (3.23) one has $\zeta_3<\varkappa<1$, so

$$\chi^*(\mathcal{Q}(D))\le\ell\chi_\infty(D)+\zeta_3 d_\infty(D)\le\max\{\ell;\varkappa\}\cdot\chi^*(D).$$

Therefore, if D is a bounded subset in $BC(J;X)$ and satisfies $\chi^*(\mathcal{Q}(D))\ge\chi^*(D)$, then $\chi^*(D)\le\max\{\ell;\varkappa\}\cdot\chi^*(D)$ implies that $\chi^*(D)=0$. Thus, D is a relatively compact. We get the χ^*-condensing property of $\mathcal{Q}$ as desired.

In addition, $\mathcal{Q}$ has compact values. In fact, for $y\in BC(J;X)$, we have

$$\chi^*(\mathcal{Q}(y))\le\ell\cdot\chi^*(y)=0.$$

It follows that $\chi^*(\mathcal{Q}(y))=0$ and then $\mathcal{Q}(y)$ is a relatively compact. Thanks to the closedness of $\mathcal{Q}$, $\mathcal{Q}(y)$ is compact.

To apply Theorem 2.3, it suffices to show that there exists $R>0$ such that

$$\mathcal{Q}(\mathcal{B}_R)\subset\mathcal{B}_R,$$

where $\mathcal{B}_R$ is the closed ball in $BC(J;X)$, centered at origin with radius R.
Suppose that $y \in BC(J;X)$ and $z \in \mathcal{Q}(y)$. Then there exists $f \in Sel^p_{\mathcal{F}}(y)$ such that

$$z(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds.$$

By using the estimates (2.4) and (3.13), one has

$$\begin{aligned}\|z(t)\|_X &\le \frac{m\|\xi\|}{(1+t)^\alpha} + \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}}\left[a(s) + \frac{b(s)(L_B+m_h+\eta_{1g})}{m_B-\kappa_h-\eta_{2g}}\right]ds\|y\|_{BC}\\ &\le M_1 + \varkappa\|y\|_{BC}.\end{aligned}$$

Therefore, if we choose

$$R > \frac{M_1}{1-\varkappa},$$

then for any $y \in \mathcal{B}_R$, for any $z \in \mathcal{Q}(y)$, we have $z \in \mathcal{B}_R$. It is equivalent to that $\mathcal{Q}(\mathcal{B}_R) \subset \mathcal{B}_R$ as desired. The proof is complete.

□

Remark 3.2. (1) We can remove the condition $\ell < 1$ if the operator A generates a compact semigroup $S(t), t > 0$.
(2) The solution $x(\cdot)$ obtained from Theorem 3.7 is a decay solution of (1.1)-(1.3) due to $d_\infty(x(\cdot)) = 0$. As a special case, this solution is also an asymptotically periodic solution of problem (1.1)-(1.3), which can be discussed in thereafter.

4. Asymptotically periodic solutions

We begin this section by recalling a definition of asymptotically periodic functions.

Definition 4.1. [14, Definition 3.1] Let X be a Banach space. A function $f \in BC(J;X)$ is called S-asymptotically T-periodic if there exists $T > 0$ such that

$$\lim_{t\to\infty}(f(t+T) - f(t)) = 0.$$

In this case, we say that T is an asymptotic period of f.

Let $SAP_T(X)$ represent the space formed by all the X-valued S-asymptotically T-periodic functions endowed with the uniform convergence norm. Then $SAP_T(X)$ is a Banach space (see [14, Proposition 3.5]).

From now on, we fix $T > 0$ as an asymptotic period of mild solutions. We suppose that A, F and g satisfy the following assumptions:

($\mathcal{H}_A^*$) *A satisfies ($\mathcal{H}_A$) with $S(t)$ is compact for every $t > 0$.*

($\mathcal{H}_F^*$) *F satisfies ($\mathcal{H}_F$)(1)-(3). The assumption ($\mathcal{H}_F$)(4) is replaced by the stronger one. That is, $\|F(t,0,0)\| \in L^p_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ and there is a function $\vartheta(\cdot) \in L^p_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ such that for all $t \in \mathbb{R}^+, y_1, y_2 \in X$ and $v_1, v_2 \in U$, the following estimate holds*

$$\|\xi_1 - \xi_2\|_X \le \vartheta(t)(\|y_1\|_X + \|y_2\|_X + \|v_1\|_U + \|v_2\|_U + 1),$$

for all $\xi_1 \in F(t+T, y_1, v_1)$ and $\xi_2 \in F(t, y_2, v_2)$.

($\mathcal{H}_g^*$) *g satisfies* ($\mathcal{H}_g$) *and* $\langle g(t,0,0), v\rangle \geq h(0,0,v)$, *for all* $v \in U$ *and* $t \geq 0$. *In addition, g satisfies the estimate*

$$\|g(t+T, x_1, u_1) - g(t, x_2, u_2)\|_{U^*} \leq \kappa_{1g}\|x_1 - x_2\|_X + \kappa_{2g}\|u_1 - u_2\|_U,$$

for all $t \geq 0, x_1, x_2 \in X, u_1, u_2 \in U$.

Remark 4.1. (1) Under assumption ($\mathcal{H}_{\mathrm{F}}^*$), it follows ($\mathcal{H}_F$)(4) with

$$a(t) = b(t) = \vartheta(t),$$
$$c(t) = \|F(t+T, 0, 0)\| + \vartheta(t).$$

(2) The assumption ($\mathcal{H}_g^*$) ensures that $\mathcal{VI}(t, 0) = 0, \forall t \geq 0$.

Furthermore, we have some estimations given by the following lemma.

Lemma 4.1. *Suppose that the assumptions* ($\mathcal{H}_A^*$), ($\mathcal{H}_F^*$), ($\mathcal{H}_g^*$), ($\mathcal{H}_B$) *and* ($\mathcal{H}_h$) *hold. If* $m_B - \kappa_h > \max\{\eta_{2g}; \kappa_{2g}\}$, *then the following estimates are satisfied*

$$\|\mathcal{VI}(t+T, x_1) - \mathcal{VI}(t, x_2)\|_U \leq \frac{L_B + m_h + \kappa_{1g}}{m_B - \kappa_h - \kappa_{2g}}\|x_1 - x_2\|_X, \tag{4.1}$$

$$\|\mathcal{F}(t+T, x_1) - \mathcal{F}(t, x_2)\| \leq \gamma(t)(\|x_1\|_X + \|x_2\|_X) + \vartheta(t), \tag{4.2}$$

for all $t \geq 0, x_1, x_2 \in X$, *where*

$$\gamma(t) = \vartheta(t) + \frac{\vartheta(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \kappa_{2g}}.$$

Proof. Let $x_1, x_2 \in X$ and $u_1 = \mathcal{VI}(t+T, x_1), u_2 = \mathcal{VI}(t, x_2)$. Then it is easily seen that $u_1 = \mathbb{VI}(x_1, g(t+T, x_1, u_1))$ and $u_2 = \mathbb{VI}(x_2, g(t, x_2, u_2))$. One has

$$\begin{aligned}
\|u_1 - u_2\|_U &= \|\mathbb{VI}(x_1, g(t+T, x_1, u_1)) - \mathbb{VI}(x_2, g(t, x_2, u_2))\|_U \\
&\leq \frac{L_B + m_h}{m_B - \kappa_h}\|x_1 - x_2\|_X \\
&\quad + \frac{1}{m_B - \kappa_h}(\|g(t+T, x_1, u_1)\|_{U^*} - g(t, x_2, u_2)\|_{U^*}) \\
&\leq \frac{L_B + m_h + \kappa_{1g}}{m_B - \kappa_h}\|x_1 - x_2\|_X + \frac{\kappa_{2g}}{m_B - \kappa_h}\|u_1 - u_2\|_U,
\end{aligned}$$

We yield

$$\|\mathcal{VI}(t+T, x_1) - \mathcal{VI}(t, x_2)\|_U \leq \frac{L_B + m_h + \kappa_{1g}}{m_B - \kappa_h - \kappa_{2g}}\|x_1 - x_2\|_X.$$

which leads to (4.1). For the second estimate, we have

$$\begin{aligned}
\|\mathcal{F}(t+T, x_1) - \mathcal{F}(t, x_2)\| &= \|F(t+T, x_1, \mathcal{VI}(t+T, x_1)) - F(t, x_2, \mathcal{VI}(t, x_2))\| \\
&\leq \vartheta(t)(\|x_1\|_X + \|x_2\|_X + \|\mathcal{VI}(t+T, x_1)\|_U + \|\mathcal{VI}(t, x_2)\|_U + 1) \\
&\leq \vartheta(t)\left[1 + \frac{(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \kappa_{2g}}\right](\|x_1\|_X + \|x_2\|_X) + \vartheta(t) \\
&\leq \gamma(t)(\|x_1\|_X + \|x_2\|_X) + \vartheta(t),
\end{aligned}$$

where

$$\gamma(t) = \vartheta(t) + \frac{\vartheta(t)(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \kappa_{2g}}.$$

The proof is complete. □

Lemma 4.2. *Suppose that the conditions of Lemma 4.1 are satisfied. If (x, u) is a mild solution to* (1.1)-(1.3) *and $x \in SAP_T(X)$, then $u \in SAP_T(U)$.*

Proof. We obtain the assertion of lemma by using (4.1) and some straightforward computations. □

Theorem 4.3. *Assume that all the conditions in Lemma 4.1 are satisfied. Then* (1.1)-(1.3) *has at least one S-asymptotically T-periodic solution (x, u) on $[0, \infty)$ provided that*

$$\lim_{t\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s)ds = 0, \tag{4.3}$$

$$\mu := \sup_{t>0} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s)ds < \frac{m_B - \kappa_h - \eta_{2g}}{m(m_B + L_B + m_h + \eta_{1g})}. \tag{4.4}$$

Proof. Let $\mathcal{Q} : SAP_T(X) \to \mathcal{P}(BC(J; X))$ be the operator defined by

$$\mathcal{Q}(x)(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds, f \in Sel^p_{\mathcal{F}}(x).$$

First, we prove that $\mathcal{Q}(SAP_T(X)) \subset SAP_T(X)$.

Indeed, if $y \in SAP_T(X)$ then for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\sup_{t\geq t_\epsilon} \|y(t+T) - y(t)\|_X \leq \epsilon$. Then for $t > t_\epsilon$, there is $f \in Sel^p_{\mathcal{F}}(y)$ such that

$$\begin{aligned}
\mathcal{Q}(y)(t+T) - \mathcal{Q}(y)(t) &= \mathcal{S}_\alpha(t+T)\xi + \int_0^{t+T} (t+T-s)^{\alpha-1}\mathcal{P}_\alpha(t+T-s)f(s)ds \\
&\quad - \mathcal{S}_\alpha(t)\xi - \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds \\
&= \mathcal{S}_\alpha(t+T)\xi - \mathcal{S}_\alpha(t)\xi + \int_{-T}^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s+T)ds \\
&\quad - \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds \\
&\leq \mathcal{S}_\alpha(t+T)\xi - \mathcal{S}_\alpha(t)\xi + \int_{-T}^0 (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s+T)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)[f(s+T) - f(s)]ds \\
&:= I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4.5}$$

For the estimate of $I_1(t)$, we have

$$\begin{aligned}
\|I_1(t)\|_X &\leq \|\mathcal{S}_\alpha(t+T)\xi\|_X + \|\mathcal{S}_\alpha(t)\xi\|_X \\
&\leq \frac{2m\|\xi\|_X}{(1+t)^\alpha},
\end{aligned}$$

thanks to (2.4). Then, $\lim_{t\to\infty} \|I_1(t)\|_X = 0$.

For the second term $I_2(t)$, we use the growth property of F. Indeed, one has

$$\begin{aligned}\|I_2(t)\|_X &\le \int_{-T}^{0} (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s) f(s+T)\|_X ds \\ &\le M \int_{-T}^{0} (t-s)^{\alpha-1} \frac{m}{(1+t-s)^{2\alpha}} ds \le \frac{mM((t+T)^\alpha - t^\alpha)}{\alpha(1+t)^{2\alpha}} \\ &\le \frac{mMT^\alpha}{\alpha(1+t)^{2\alpha}},\end{aligned}$$

then we yields that $I_2(t)$ tends to zero as $t \to \infty$, here

$$M := \sup_{t\in[0,T], \|y\|_X \le \|x\|_{BC}} \|\mathcal{F}(s,y)\|$$

For the last term of the right hand side of (4.5), we have

$$\begin{aligned}\|I_3(t)\|_X &\le \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \|f(s+T) - f(s)\|_X ds \\ &\le \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} [\gamma(s)(\|x(s)\|_X + \|x(s+T)\|_X) + \vartheta(s)] ds \\ &\le 2M_x \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \gamma(s) ds + \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s) ds \\ &\le \left(2M_x + \frac{2M_x(L_B + m_h + \eta_{1g})}{m_B - \kappa_h - \eta_{2g}} + 1\right) \int_0^t \frac{m(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s) ds\end{aligned}$$

where $M_x := \sup_{t>0} \|x(t)\|$. Then, according to (4.3), one has

$$\lim_{t\to\infty} \|I_3(t)\|_X = 0.$$

Now, we consider the solution multivalied map $\mathcal{Q} : SAP_T(X) \to \mathcal{P}(SAP_T(X))$. Using the same argument as in the proof of Theorem 3.7 together with the conditions $\ell < 1$ and (4.4), we obtain that $\mathcal{Q}$ has the following properties:

(i) $\mathcal{Q} : SAP_T(X) \to \mathcal{P}(SAP_T(X))$ is closed.
(ii) $\mathcal{Q} : SAP_T(X) \to \mathcal{P}(SAP_T(X))$ is compact due to the compactness of $S(t), S_\alpha(t), P_\alpha(t), t > 0$.
(iii) There exists $R > 0$ such that $\mathcal{Q}(\mathbb{B}_R) \subset \mathbb{B}_R$, where $\mathbb{B}_R$ is the closed ball in $SAP_T(X)$, centered at origin with radius R due to the estimation (4.4).

Then, by using the fixed point argument given in Theorem 2.3, there exists one S-asymptotically T-periodic solution of the converting differential inclusion. By Lemma 4.2, there exists one S-asymptotically T-periodic solution of (1.1)-(1.3). The proof is complete. □

As previous argument (Remark 2.1(2)), the obtained theoretical results are useful to a class of differential variational inequalities of parabolic-elliptic type. In this case, we choose $h(x,u,v) := \phi(v) - \phi(u), \forall x \in X, u, v \in U$, here ϕ is a proper, convex and lower semicontinuous function. For instance, we take $\phi = I_K$, the indicator function of K with K being a closed convex subset in U, namely,

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

then, the problem (1.1)-(1.2) reads as follows

$$^{C}D_t^{\alpha} x(t) = Ax(t) + F(t, x(t), u(t)),\ t > 0, \tag{4.6}$$

$$u(t) \in K, \forall t \geq 0, \tag{4.7}$$

$$\langle B(x(t), u(t)), v - u(t)\rangle \geq \langle g(t, x(t), u(t)), v - u(t)\rangle, \forall v \in K, t > 0. \tag{4.8}$$

Then, $\kappa_h = m_h = 0$ and we immediately get the following result.

Corollary 4.4. *Assume that the assumptions $(\mathcal{H}_A^*)$, $(\mathcal{H}_F^*)$, $(\mathcal{H}_g^*)$ and $(\mathcal{H}_B)$ hold. In addition, we suppose*

$$m_B > \max\{\eta_{2g}; \kappa_{2g}\};$$
$$\lim_{t\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s) ds = 0;$$
$$\sup_{t>0} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \vartheta(s) ds < \frac{m_B - \eta_{2g}}{m(m_B + L_B + \eta_{1g})}.$$

Then, (4.6)-(4.8) *has at least one S-asymptotically T-periodic solution (x, u) on $[0, \infty)$.*

We end this section for a single-valued version case of our fractional differential variational inequalities, namely, we consider (1.1)-(1.3) when F is a single-valued function. For this situation, we intend to relax the compactness of $S(t), t > 0$ by using the argument of Banach contraction mapping principle. We need the following assumptions:

(F1) *there are nonnegative functions $\zeta_X, \zeta_U \in L^p_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that*

$$\|F(t, x, u) - F(t, y, v)\|_X \leq \zeta_X(t)\|x - y\|_X + \zeta_U(t)\|u - v\|_U;$$

for all $t \geq 0, x, y \in X, u, v \in U$;

(F2) *there are nonnegative functions $\theta \in L^p_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that*

$$\|F(t + T, x, u) - F(t, x, u)\|_X \leq \theta(t)(\|x\|_X + \|u\|_U + 1);$$

for all $t \geq 0, x \in X, u \in U$.

Theorem 4.5. *Assume that $(\mathcal{H}_A)$,* (F1)-(F2)*, $(\mathcal{H}_g^*)$, $(\mathcal{H}_B)$ and $(\mathcal{H}_h)$ hold. If $m_B - \kappa_h > \max\{\eta_{2g}; \kappa_{2g}\}$ and*

$$\sup_{t\in J} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \|F(s, 0, 0)\| ds < +\infty; \tag{4.9}$$

$$\lim_{t\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \theta(s) ds = 0; \tag{4.10}$$

$$\mu := m \sup_{t\in J} \int_0^t \frac{(t-s)^{\alpha-1}}{(1+t-s)^{2\alpha}} \left[\zeta_X(s) + \zeta_U(s) \frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}\right] ds < 1; \tag{4.11}$$

then (1.1)-(1.3) *has a unique S-asymptotically T-periodic solution* (x,u) *on* J.

Proof. We convert (1.1)-(1.3) to the following system

$$
{}^C D_t^{\alpha} x(t) - Ax(t) = \mathcal{F}(t, x(t)), \tag{4.12}
$$

$$
u(t) = \mathcal{VI}(t, x(t)), \tag{4.13}
$$

$$
x(0) = \xi, \tag{4.14}
$$

here, $\mathcal{F}(t,x) := F(t, x, \mathcal{VI}(t,x))$ satisfies

$$
\begin{aligned}
\|\mathcal{F}(t,x) - \mathcal{F}(t,y)\|_X &\le \zeta_X(t)\|x-y\|_X + \zeta_U(t)\|\mathcal{VI}(t,x) - \mathcal{VI}(t,y)\|_U \\
&\le \zeta_X(t)\|x-y\|_X + \zeta_U(t)\frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}\|x-y\|_X \\
&\le \left[\zeta_X(t) + \zeta_U(t)\frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}\right]\|x-y\|_X \\
&\le \zeta(t)\|x-y\|_X,
\end{aligned}
$$

where

$$
\zeta(t) = \zeta_X(t) + \zeta_U(t)\frac{L_B + m_h + \eta_{1g}}{m_B - \kappa_h - \eta_{2g}}.
$$

From Lemma 4.1, we have

$$
\mathcal{VI}(t+T, x) = \mathcal{VI}(t,x).
$$

Then, combining this with (F2), we obtain

$$
\begin{aligned}
\|\mathcal{F}(t+T,x) - \mathcal{F}(t,x)\|_X &= \|F(t+T, x, \mathcal{VI}(t+T,x)) - F(t,x,\mathcal{VI}(t,x))\|_X \\
&\le \theta(t)(\|x\|_X + \|\mathcal{VI}(t,x)\|_U + 1) \\
&\le \theta(t)\left[\|x\|_X + \frac{L_B + m_h + \kappa_{1g}}{m_B - \kappa_h - \kappa_{2g}}\|x\|_X + 1\right].
\end{aligned}
$$

Now using the same argument used in [25, Theorem 4.1], we easily derive our assertion. □

5. Application

Example 5.1. In this section, we consider fractional partial differential equations with obstacle constraints. For a concrete example, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary, we consider the fractional partial differential equations with obstacle constraints

$$
{}^C D_t^{\frac{1}{2}} Z(t,x) - \Delta_x Z(t,x) = f(t, x, Z(t,x), u(t,x)), \tag{5.1}
$$

$$
-\Delta_x u(t,x) + \beta(u(t,x) - \psi(x)) \ni g(t,x,Z(t,x),u(t,x)), \tag{5.2}
$$

$$
Z(t,x) = 0, \forall x \in \partial\Omega, \forall t \ge 0, \tag{5.3}
$$

$$
Z(0,x) = Z_0(x), \forall x \in \Omega. \tag{5.4}
$$

where the maps $f, h : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, ψ is in $H^2(\Omega)$ and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph

$$\beta(r) = \begin{cases} 0 & \text{if } r > 0, \\ \mathbb{R}^- & \text{if } r = 0, \\ \emptyset & \text{if } r < 0. \end{cases}$$

Note that, the elliptic variational inequality (5.2) reads as follows

$$\begin{aligned} -\Delta_x u(t,x) &= g(t,x,Z(t,x),u(t,x)) \ \text{ in } \ \{(t,x) \in Q := (0,T) \times \Omega : u(t,x) \geq \psi(x)\}, \\ -\Delta_x u(t,x) &\geq g(t,x,Z(t,x),u(t,x)), \ \text{ in } \ Q, \\ u(t,x) &\geq \psi(x), \forall (t,x) \in Q, \end{aligned}$$

which represents a rigorous and efficient way to treat diffusion problems with a free or moving boundary. This model is called the *obstacle elliptic problem* (see [8]). We suppose that

(A1) *there exist nonnegative functions $a(t,\cdot), b(t,\cdot) \in L^\infty(\Omega)$ for each $t \geq 0$ such that*

$$|f(t,x,p,q) - f(t,x,p',q')| \leq a(t,x)|p - p'| + b(t,x)|q - q'|,$$

and moreover, we suppose $f(t,x,p,q) = f(t+T,x,p,q)$, for all $t \geq 0$, $x \in \Omega$, $p, q \in \mathbb{R}$.

(A2) *the map $g : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies $g(t,x,p,q) = g(t+T,x,p,q)$, $\forall x \in \Omega$, $t \geq 0$, $p, q \in \mathbb{R}$ and*

$$|g(t,x,p,q) - g(\bar{t},x,p',q')| \leq \eta(t,\bar{t}) + c(x)|p - p'| + d(x)|q - q'|,$$

for all $x \in \Omega, p, q \in \mathbb{R}$, where $c(\cdot), d(\cdot)$ are the nonnegative functions in $L^\infty(\Omega)$ and $\eta(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative continuous function.

Let $X = L^2(\Omega)$ and $U = H_0^1(\Omega)$. We define the abstract function

$$\begin{aligned} &F : \mathbb{R}^+ \times X \times U \to \mathcal{P}(X) \\ &F(t,Z,u) = f(t,x,Z(x),u(x)), \end{aligned}$$

and the operator

$$A = \Delta : D(A) \subset X \to X; D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then (5.1) can be reformulated as

$$Z'(t) - AZ(t) = F(t,Z(t),u(t)),$$

where $Z(t) \in X, u(t) \in Y$ such that $Z(t)(x) = Z(t,x)$ and $u(t)(x) = u(t,x)$. It is known that ([33]), the semigroup $S(t)$ generated by A is compact and exponentially stable, that is,

$$\|S(t)\|_{\mathcal{L}(X)} \leq e^{-\lambda_1 t},$$

then the assumption $(\mathcal{H}_A)$ is satisfied.

By the setting of function F, it is easy to see that F is a continuous periodic functions of time and

$$\|F(t,Z,u) - F(t,\bar{Z},\bar{u})\| \leq \|a(t,\cdot)\|_\infty \|Z - \bar{Z}\|_X + \frac{\|b(t,\cdot)\|_\infty}{\sqrt{\lambda_1}} \|u - \bar{u}\|_U.$$

Thus, (F1)-(F2) hold. Furthermore,

$$\theta(t) \equiv 0, \quad \zeta_X(t) = \|a(t,\cdot)\|_\infty$$

and

$$\zeta_U(t) = \frac{\|b(t,\cdot)\|_\infty}{\sqrt{\lambda_1}}.$$

Consider the elliptic variational inequality (5.2), putting $B(z,\cdot) = -\Delta, \forall z \in X$ where $-\Delta$ stands for the Laplace operator

$$\langle u, -\Delta v\rangle := \int_\Omega \nabla u(x)\nabla v(x)dx,$$

then $\langle B(z,u), u\rangle = \|u\|_U^2$. So, the assumption $(\mathcal{H}_B)$ takes place with $L_B = 1$.

As far as the nonlinear function g is concerned, we assume that the map g : $\mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies $g(t,x,p,q) = g(t+T,x,p,q), \forall x \in \Omega, t \geq 0, p, q \in \mathbb{R}$ and

$$|g(t,x,p,q) - g(\bar{t},x,p',q')| \leq c(x)|p-p'| + d(x)|q-q'|, \forall x \in \Omega, p, q \in \mathbb{R},$$

where $c(\cdot), d(\cdot)$ are the nonnegative functions in $L^\infty(\Omega)$.

Let $g : \mathbb{R}^+ \times X \times Y \to L^2(\Omega), g(t,\bar{Z},\bar{u})(x) = g(t,x,\bar{Z}(x),\bar{u}(x))$, we obtain

$$|g(t,Z,u) - g(\bar{t},\bar{Z},\bar{u})| \leq \|c\|_\infty \|Z - \bar{Z}\|_X + \frac{\|d\|_\infty}{\sqrt{\lambda_1}}\|u - \bar{u}\|_U.$$

Then the EVI (5.2) reads as

$$Bu(t) + \partial I_K(u(t)) \ni h(t, Z(t), u(t)),$$

where

$$\begin{aligned} K &= \{u \in H_0^1(\Omega) : u(y) \geq \psi(x), \text{ for a.e. } x \in \Omega\}, \\ \partial I_K(u) &= \big\{u \in H_0^1(\Omega) : \int_\Omega u(x)(v(x) - z(x))dx \geq 0, \forall z \in K\big\}, \\ &= \{u \in H_0^1(\Omega) : u(x) \in \beta(v(x) - \psi(x)), \text{ for a.e. } x \in \Omega\}. \end{aligned}$$

It follows that the conditions of Corollary 4.4 are satisfied. We have the following result.

Theorem 5.1. *If* $\|d\|_\infty^2 < \lambda_1$ *and*

$$\sup_{t>0} \int_0^t \frac{\|a(s,\cdot)\|_\infty + \|b(s,\cdot)\|_\infty}{\sqrt{t-s}(1+t-s)} ds$$

is small enough, then the problem (5.1)-(5.2) *has a mild* S*-asymptotically* T*-periodic solution* $(\mathbf{Z}, \mathbf{u})$.

Example 5.2. Let Ω be a bounded domain with C^1-boundary in $\mathbb{R}^n, n \geq 2$ and let K be a nonempty closed convex subset of $H_0^1(\Omega)$. We study a fractional partial differential equation mixed a variational inequality:

$$^{C}D_t^{\frac{1}{2}} Z(t,x) - \Delta_x Z(t,x) = f(t,x,Z(t,x)) + \mathcal{B}(t,u(t,x)), \tag{5.5}$$

$$\int_\Omega \nabla u(t,\xi)(\nabla v(\xi) - \nabla u(t,\xi))d\xi + \int_\Omega Z(t,\xi)(v(\xi)-u(t,\xi))d\xi$$
$$+ \nu \int_\Omega (v^2(\xi) - u(t,\xi)v(\xi))d\xi \geq 0, \forall v \in K, \tag{5.6}$$

$$Z(t,x) = 0, \forall x \in \partial\Omega, \forall t \geq 0, \tag{5.7}$$

$$Z(0,x) = Z_0(x). \tag{5.8}$$

Then, we can transfer (5.5)-(5.8) to our abstract problem by the following setting:

(1) $X = L^2(\Omega)$, $U = H_0^1(\Omega)$;
(2) $A = \Delta_x$, $F(t,Z,u)(x) = f(t,x,Z(x)) + \mathcal{B}(t,u(x))$;
(3) $B(Z,u) = \Delta_x u$;
(4) $h(Z,u,v) = \nu \int_\Omega (v^2(\xi) - u(\xi)v(\xi))d\xi$;
(5) $g(t,Z,u) = Z$.

The assumptions imposed on $f, \mathcal{B}$ are given by

(H_f) *there are nonnegative functions $\nu_{1f}(\cdot)$ and $\nu_{2f}(\cdot)$ satisfying*

$$|f(t,x,p) - f(t,x,q)| \leq \nu_{1f}(t)|p-q|, \forall t \geq 0, x \in \Omega, p,q \in \mathbb{R};$$
$$|f(t+T,x,p) - f(t,x,p)| \leq \nu_{2f}(t)(1+|p|), \forall t \geq 0, x \in \Omega, p \in \mathbb{R},$$

$(H_\mathcal{B})$ *there are nonnegative functions $\nu_{1\mathcal{B}}(\cdot)$ and $\nu_{2\mathcal{B}}(\cdot)$ satisfying*

$$|\mathcal{B}(t,p) - \mathcal{B}(t,q)| \leq \nu_{1\mathcal{B}}(t)|p-q|, \forall t \geq 0, p,q \in \mathbb{R};$$
$$|\mathcal{B}(t+T,p) - \mathcal{B}(t,p)| \leq \nu_{2\mathcal{B}}(t)(1+|p|), \forall t \geq 0, p \in \mathbb{R},$$

By some concrete calculations, we obtain the coefficients of abstract given functions in Theorem 4.5.

(i) $\zeta_X(t) = \nu_{1f}(t)$;
$\zeta_U(t) = \nu_{1\mathcal{B}}(t)$;
$\theta(t) = \max\{\nu_{2f}(t)|\Omega|; \nu_{2\mathcal{B}}(t)|\Omega|; \nu_{2f}(t); \nu_{2\mathcal{B}}(t)\}$;
(ii) $m_B = 1;\ L_B = 0$;
$\kappa_h = \nu;\ m_h = 0$;
$\eta_{1g} = 1;\ \eta_{2g} = 0;\ \kappa_{1g} = \kappa_{2g} = 0$.

Theorem 5.2. *Assume that (H_f) and $(H_\mathcal{B})$ are satisfied. Then Problem* (5.5)-(5.8) *has a S-asymptotically T-periodic provided that $\nu < 1$ and*

$$\lim_{t\to\infty} \int_0^t \frac{\nu_{2f}(s) + \nu_{2\mathcal{B}}(s)}{\sqrt{t-s}(1+t-s)} ds = 0,$$
$$m \sup_{t>0} \int_0^t \frac{1}{\sqrt{t-s}(1+t-s)} \left[\nu_{1f}(s) + \frac{\nu_{1\mathcal{B}}(s)}{1-\nu}\right] ds < 1.$$

In case of multivalued version, we replace (5.5) with a partial differential inclusion as follows

$$\frac{\partial Z}{\partial t}(t,x) - \Delta_x Z(t,x) = f(t,x) + \mathcal{B}(t,u(t,x)), \tag{5.9}$$

$$f(t,x) = \lambda f_1(t,x,Z(t,x)) + (1-\lambda) f_2(t,x,Z(t,x)),$$
$$\lambda \in [0,1], t > 0, x \in \Omega, \tag{5.10}$$

where $f_1, f_2 : \mathbb{R}^+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ are T-periodic continuous functions with respect to their first variables. Define the multivalued function:

$$F : \mathbb{R}^+ \times X \times U \to \mathcal{P}(X),$$
$$F(t,\bar{Z},\bar{u})(x) = \{\lambda f_1(t,x,\bar{Z}(x)) + (1-\lambda) f_2(t,x,\bar{Z}(x))\} + \mathcal{B}(t,\bar{u}(x)).$$

It is easily seen that F is a multimap with closed convex, compact values. Suppose that there exist nonnegative T-periodic-in-time functions $a_1(t,\cdot), a_2(t,\cdot) \in L^\infty(\Omega)$ for each $t > 0$ such that

$$|f_1(t,x,p)| \le a_1(t,x)|p| + a_1(t,x),\ \forall t \ge 0, x \in \Omega, p, q \in \mathbb{R},$$

$$|f_2(t,x,p)| \le a_2(t,x)|p| + a_2(t,x)\ \forall t \ge 0, x \in \Omega, p, q \in \mathbb{R}.$$

Then for $\xi_1 \in F(t,\bar{Z}_1,\bar{u}_1)$ and $\xi_2 \in F(t+T,\bar{Z}_2,\bar{u}_2)$, one has

$$\begin{aligned}\|\xi_1 - \xi_2\|_X \le{}& \max\{\|a_1(t,\cdot)\|_\infty, \|a_2(t,\cdot)\|_\infty\}(\|\bar{Z}_1\|_X + \|\bar{Z}_2\|_X)\\ &+ \nu_{2\mathcal{B}}(t)(2 + \|\bar{u}_1\|_U + \|\bar{u}_2\|_U)\\ &+ 2\max\{\|a_1(t,\cdot)\|_X, \|a_2(t,\cdot)\|_X\}.\end{aligned}$$

Because f_1 and f_2 are continuous, the fact that F has a closed graph can be testified by a simple argument. Furthermore, if $\{\bar{Z}_n\} \subset X, \{\bar{u}_n\} \subset Y$ are convergent sequences, then one can find a sequence $\{f_n\}, f_n \in F(\cdot,\bar{Z}_n,\bar{u}_n)$ that is convergent in X by using the Lebesgue dominated convergence theorem. So F is quasi-compact. We now recall the following lemma to assure the u.s.c. property of F.

Lemma 5.3. [17, Theorem 1.1.12] *Let $G : Y \to \mathcal{P}(E)$ a closed quasi-compact multimap with compact values. Then G is u.s.c.*

Then by Lemma 5.3, F is a u.s.c. multimap. The condition $(\mathcal{H}_F^*)$ is testified.

Applying Theorem 4.3, we arrive at the last result related to the existence of periodic solutions of (5.9)-(5.10) mixed the variational inequality (5.6).

Theorem 5.4. *Assume that $(H_\mathcal{B})$ are satisfied. Then Problem* (5.6)-(5.9)-(5.10) *has a S-asymptotically T-periodic provided that $\nu < 1$ and*

$$\lim_{t\to\infty} \int_0^t \frac{\|a_1(s,\cdot)\|_\infty + \|a_2(s,\cdot)\|_\infty + \nu_{2\mathcal{B}}(s)}{\sqrt{t-s}(1+t-s)} ds = 0,$$
$$m \sup_{t>0} \int_0^t \frac{1}{\sqrt{t-s}(1+t-s)} \left[\max\{\|a_1(s,\cdot)\|_\infty + \|a_2(s,\cdot)\|_\infty\} + \frac{\nu_{1\mathcal{B}}(s)}{1-\nu}\right] ds < 1.$$

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