Fixed Point Theory, 24(2023), No. 2, 433-458
DOI: 10.24193/fpt-ro.2023.2.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A FIXED POINT THEOREM IN ULTRAMETRIC n-BANACH SPACES AND HYPERSTABILITY RESULTS

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Abstract. In this paper, we prove a general fixed point theorem for ultrametric n-Banach spaces. We also show its applications in proving the hyperstability, in the sense of Găvruţă, of the following monomial functional equation

$$\sum_{k=0}^{s}(-1)^{s-k}C_k^s f(kx+y)=s!f(x),$$

where $C_k^s=\frac{s!}{(s-k)!k!}$ and $k,s\in\mathbb{N}$ such that $s\geq k$.
Key Words and Phrases: Fixed point theorem, functional equations, ultrametric n-Banach space, stability, hyperstability.
2020 Mathematics Subject Classification: 39B52, 54E50, 47H10, 39B82.

1. Introduction

The famous talk of S. M. Ulam in 1940 [44] seems to be the starting point for studying the stability of functional equations, in which he discussed a number of important unsolved problems. Among these was the question of the stability of group homomorphisms.

Ulam problem:[44] *Given a group G_1, a metric group G_2 with metric $d(\cdot\ ,\ \cdot)$ and a positive number ε, does there exist a $\delta>0$ such that if $f:G_1\to G_2$ satisfies*

$$d\big(f(xy)\ ,\ f(x)f(y)\big)\leq\varepsilon$$

for all $x,y\in G_1$, then a homomorphism $\phi:G_1\to G_2$ exists with

$$d\big(f(x)\ ,\ \phi(x)\big)\leq\delta$$

for all $x\in G_1$?

These kinds of questions serve as the foundation for the theory of stability. Under the assumption that G_1 and G_2 are Banach spaces, the case of approximately additive mappings was solved by D. H. Hyers in 1941 [31].

Hyers' [31] and Ulam [44] referred to this property as the stability of the functional equation $f(x+y)=f(x)+f(y)$. Hyers work has initiated much of the current research in the theory of the stability of functional equations. In 1978, the theorem of Hyers was significantly generalized by Th. Rassias [41], taking into account cases where the relevant inequality is not bound. This property was called *the Hyers-Ulam-Rassias stability of the additive Cauchy functional equation* $f(x+y)=f(x)+f(y)$.
This terminology also applies to other functional equations. The result of Rassias [41] has been further generalized by Rassias [42], Th. Rassias and P. Šemrl [43], P. Găvruţă [28], and S. -M. Jung [33]. Simultaneously, a special kind of stability has emerged, which is called *the hyperstability of functional equations*. This kind states that if f satisfies a stability inequality related to the given equation, then it is also a solution to this equation. It seems that the first hyperstability result was published in [11] and concerned ring homomorphisms. The term "hyperstability", on the other hand, appeared for the first time in [36]. Hyperstability is frequently mistaken for superstability, which also admits bounded functions. Further, J. Brzdęk and K. Ciepliński [15] introduced the following definition which describes the main ideas of such a hyperstability notion for equations in several variables ($\mathbb{R}_+$ stands for the set of all nonnegative reals and C^D denotes the family of all functions mapping a set $D\neq\phi$ into a set $C\neq\phi$).
Definition 1.1. [15] Let S be a nonempty set, (Y,d) be a metric space, $\varepsilon\in\mathbb{R}_+^{S^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be two operators mapping a nonempty set $\mathcal{D}\subset Y^S$ into Y^{S^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1,\cdots,x_n)=\mathcal{F}_2\varphi(x_1,\cdots,x_n),\quad x_1,\cdots,x_n\in S, \tag{1.1}$$

is ε-hyperstable provided every $\varphi_0\in\mathcal{D}$ that satisfies the inequality

$$d\big(\mathcal{F}_1\varphi_0(x_1,\cdots,x_n)\ ,\ \mathcal{F}_2\varphi_0(x_1,\cdots,x_n)\big)\le\varepsilon(x_1,\cdots,x_n),\quad x_1,\cdots,x_n\in S, \tag{1.2}$$

fulfils the equation (1.1).
Brzdęk et al. [15] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:
Theorem 1.2. [15] *Let X be a nonempty set, (Y,d) be a complete metric space, and $\Lambda: Y^X\to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty}\Lambda\delta_n=0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty}\delta_n=0$$

Suppose that $\mathcal{T}: Y^X\to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x),\mathcal{T}\mu(x)\big)\le\Lambda\big(\Delta(\xi,\mu)\big)(x),\ \ \xi,\mu\in Y^X,\ x\in X, \tag{1.3}$$

where $\Delta: Y^X\times Y^X:\to\mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi,\mu)(x):=d\big(\xi(x),\mu(x)\big),\ \ \xi,\mu\in Y^X,\ x\in X. \tag{1.4}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\big((\mathcal{T}\varphi)(x), \varphi(x)\big) \le \varepsilon(x) \tag{1.5}$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n \varepsilon\big)(x) < \infty \tag{1.6}$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.7}$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.8}$$

is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(x)\ ,\ \psi(x)\big) \le \varepsilon^*(x) \tag{1.9}$$

for all $x \in X$.

In 2013, Brzdęk [12] gave the fixed point result by applying Theorem 1.2 as follows:

Theorem 1.3. [12] *Let X be a nonempty set, (Y,d) be a complete metric space, $f_1,\cdots,f_r : X \to X$ and $L_1,\ldots,L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$d\big(\mathcal{T}\xi(x)\ ,\ \mathcal{T}\mu(x)\big) \le \sum_{i=1}^{r} L_i(x) d\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big), \tag{1.10}$$

for all $\xi, \mu \in Y^X, x \in X$ and

$$\Lambda\delta(x) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big), \quad \delta \in \mathbb{R}_+^X, x \in X. \tag{1.11}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\Big(\mathcal{T}\varphi(x)\ ,\ \varphi(x)\Big) \le \varepsilon(x) \tag{1.12}$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \left(\Lambda^n \varepsilon\right)(x) < \infty \tag{1.13}$$

for all $x \in X$, then the limit (1.7) exists for each $x \in X$. Moreover, the function (1.8) is a fixed point of $\mathcal{T}$ with (1.9) for all $x \in X$.

Brzdęk [12] then used this theorem to improved, extended, and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular, Theorem 1.2). Many papers on the stability and hyperstability of functional equations were published thanks to this important achievement. For example, we refer to [1]-[10], [18]-[20], and [40]. Another point worth noting is that there were other versions of Theorem 1.3 in ultrametric space [3], in 2-Banach space [4], [18], and in n-Banach space [19] that helped to discuss many results on the stability of functional equations. For more details on the stability and hyperstability in 2-Banach spaces and n-Banach spaces, we refer the reader to seeing the surveys [7] and [25].

Let X and Y be two linear spaces, $s \in \mathbb{N}$, and $f : X \to Y$ a given mapping. The functional equation

$$\sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx+y) = s! f(x), \quad x, y \in X, \tag{1.14}$$

where $C_k^s := \frac{s!}{(s-k)!k!}$, is called *an s-monomial functional equation* and every solution of the functional equation (1.14) is said to be *a monomial mapping of degree s*. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := ax^s$ is a particular solution of the functional equation (1.14) with $a \in \mathbb{R}$. In particular, the functional equation (1.14) is called *an additive (quadratic, cubic, quartic, and quintic, respectively) functional equation* for the case $n = 1 (n = 2, n = 3, n = 4$, and $n = 5$, respectively) and every solution of the functional equation (1.14) is said to be *an additive (quadratic, cubic, quartic, and quintic, respectively) mapping* for the case $n = 1 (n = 2, n = 3, n = 4$, and $n = 5$, respectively). The stability of the equation (1.14) has been investigated by many authors, for example, [21], [22], [29], [32], [34], [35], and [37]. Moreover, the stability and hyperstability of an equation more general than (1.14) have been investigated in [8], [9], [39], and [45].

Throughout this paper, $\mathbb{Q}$ stands for the set of all rational numbers, $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N}_{m_0}$ the set of all integers greater than or equals $m_0 \, (m_0 \in \mathbb{N}), \mathbb{R}_+ = [0, \infty)$ and we use the notation X_0 for the set $X \backslash \{0\}$.

2. Preliminaries

The concept of n-normed space was given by A. Misiak [38] as a generalization of the notions of classical normed space and of a 2-normed space introduced by S. Gähler [26], [27]. We need to recall some basic facts concerning n-normed spaces and some preliminary results.

Definition 2.1. [38] Let $n \in \mathbb{N}_2, X$ be a real linear space with $\dim X \geq n$. An n-norm on X is a real function $\|\cdot, \cdots, \cdot\| : X^n \to [0, \infty)$ satisfies the following conditions:

(1) $\|x_1, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(2) $\|x_1, \cdots, x_n\| = \|x_{i_1}, \cdots, x_{i_n}\|$ for every permutaion $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$,
(3) $\|\alpha x_1, \cdots, x_n\| = |\alpha| \, \|x_1, \cdots, x_n\|$,
(4) $\|x_1 + y, x_2, \cdots, x_n\| \leq \|x_1, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

for all $\alpha \in \mathbb{R}$, and all $x, y, x_1, \cdots, x_n \in X$. The pair $\left(X, \|\cdot, \cdots, \cdot\|\right)$ is called *an n-normed space.*
We note that $\|x_1, \cdots, x_n\| \geq 0$ for all $x_1, \cdots, x_n \in X$ because

$$\begin{aligned} 2\|x_1, \cdots, x_n\| &\geq \|x_1 - x_1, \cdots, x_n\| \\ &= \|0, \cdots, x_n\| \\ &= 0. \end{aligned}$$

Example 2.2. $\mathbb{R}^n$ equipped with the function $\left\|\cdot,\cdots,\cdot\right\|_E$ defined by

$$\left\|x_1,\cdots,x_n\right\|_E = \left|\det\left(x_{ij}\right)\right| = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots,x_{in}) \in \mathbb{R}^n$ for $i \in \{1,\cdots,n\}$, is n-normed space.

Lemma 2.3. [20]*Let* $\left(X, \left\|\cdot,\cdots,\cdot\right\|\right)$ *be an n-normed space. If* $\left\{x_k\right\}_{k\in\mathbb{N}}$ *is a convergent sequence of elements of X, then*

$$\lim_{k\to\infty}\left\|x_k, y_2,\cdots,y_n\right\| = \left\|\lim_{k\to\infty} x_k, y_2,\cdots,y_n\right\|, \text{ for every } y_2,\cdots,y_n \in X.$$

K. Hensel [30] has presented a normed space which does not have the Archimedean property. The non-Archimedean framework is of particular relevance since the theory of non-Archimedean spaces has piqued the interest of physicists for their research, particularly in quantum physics difficulties, p-adic strings, and superstrings.

In the following, we present some basic concepts on the non-Archimedean normed spces (for more details, we refer to [24]).

Definition 2.4. [24] Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $\left|\cdot\right| : \mathbb{K} \to \mathbb{R}$ such that for some real number $C \geq 1$, the following hold:

(1) $\left|x\right| \geq 0$ for any $x \in \mathbb{K}$ with equality only for $x = 0$,
(2) $\left|xy\right| = \left|x\right| \cdot \left|y\right|$ for any $x, y \in \mathbb{K}$,
(3) For any $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x+1| \leq C$.

The valuation $\left|\cdot\right|$ such that $\left|x\right| = 1$ for every non zero x and $\left|0\right| = 0$ is called *the trivial valuation.*

Definition 2.5. [24] A valuation $\left|\cdot\right|$ on $\mathbb{K}$ satisfies the ultrametric inequality if for any $x, y \in \mathbb{K}$

$$\left|x+y\right| \leq \max\left\{\left|x\right|, \left|y\right|\right\}.$$

Such valuation is called *a non-Archimedean valuation.*

Proposition 2.6. [24]*A valuation $|.|$ on $\mathbb{K}$ satisfies the ultrametric inequality if and only if one can take $C = 1$ in Definition 2.4.*

Example 2.7. (Non-Archimedean valued field)

Let p be a fixed prime number. Because of the unique fraction in $\mathbb{Z}$, every non-zero rational number x can be written as

$$x = \frac{a}{b}p^n$$

where n, a, and b are integers and $\gcd(p, ab) = 1$. We can define a valuation on $\mathbb{Q}$ as follows:

$$\left|x\right|_p = \begin{cases} p^{-n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\left|\cdot\right|_p$ is called *the p-adic valuation.* The completion of $\mathbb{Q}$ with respect to $\left|\cdot\right|_p$ is called *the field of p-adic numbers* and is denoted $\mathbb{Q}_p$.

By the trivial valuation we mean the function $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}$.

Definition 2.8. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A non-Archimedean norm on X is a map $\|\cdot\|_* : X \to \mathbb{R}_+$ satisfying the following conditions:

(1) $\|x\|_* = 0$ if and only if $x = 0$,
(2) $\|\lambda x\|_* = |\lambda|\, \|x\|_*$, for any $x \in X$ and any $\lambda \in \mathbb{K}$,
(3) $\|x + y\|_* \leq \max\{\|x\|_*,\ \|y\|_*\}$, for any $x, y \in X$.

Condition (3) of Definition 2.8 is referred to as the ultrametric or strong triangle inequality. The paire $(X, \|\cdot\|_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space*. For example, the paire $(\mathbb{Q}_p, |\cdot|_p)$ is a non-Archimedean normed space.

Definition 2.9. [23] Let X be a vector space with $\dim X \geq n$ over a valued field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot, \cdots, \cdot\|_* : X^n \to [0, \infty)$ is said to be *a non-Archimedean n-norm* if

(1) $\|x_1, \cdots, x_n\|_* = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(2) $\|x_1, \cdots, x_n\|_* = \|x_{i_1}, \cdots, x_{i_n}\|_*$ for every permutation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$,
(3) $\|\alpha x_1, \cdots, x_n\|_* = |\alpha|\, \|x_1, \cdots, x_n\|_*$,
(4) $\|x + y, x_2, \cdots, x_n\|_* \leq \max\{\|x, x_2, \cdots, x_n\|_*,\ \|y, x_2, \cdots, x_n\|_*\}$

for all $\alpha \in \mathbb{K}$, and all $x, y, x_1, \cdots, x_n \in X$. Then $(X, \|\cdot, \cdots, \cdot\|_*)$ is called *a non-Archimedean n-normed space* or *an ultrametric n-normed space*.

Example 2.10. Let p be a fixed prime number. We defined an ultrametric n-norm on $\mathbb{Q}_p^n$ by

$$\|x_1, \cdots, x_n\|_* = |\det(x_{ij})|_p,$$

where $x_i = (x_{i1}, \cdots, x_{in}) \in \mathbb{Q}_p^n$ for $i \in \{1, \cdots, n\}$.
If $(X, \|\cdot, \cdots, \cdot\|_*)$ is an ultrametric n-normed space, then

$$\left\|\sum_{i=1}^{k} y_i, x_2, \cdots, x_n\right\|_* \leq \max_{1 \leq i \leq k}\left\{\|y_i, x_2, \cdots, x_n\|_*\right\}$$

for any $k \in \mathbb{N}_2, x_2, \cdots, x_n \in X$ and all $y_i \in X$ for $i \in \{1, \cdots, n\}$.
According to the conditions in Definition 2.9, the next lemma follows from Remark 1 in [20].

Lemma 2.11. *Let $(X, \|\cdot, \cdots, \|_*)$ be an ultrametric n-normed space. If $z_1, \cdots, z_n \in X$ are linearly independent, $x \in X$ and*

$$\|x, w_2, \cdots, w_n\|_* = 0 \quad \text{for every } w_2, \cdots, w_n \in \{z_1, \cdots, z_n\},$$

then $x = 0$.

Definition 2.12. A sequence $(y_k)_{k \in \mathbb{N}}$ of elements of an ultrametric n-normed space $(X, \|\cdot, \cdots, \cdot\|_*)$ is called *a Cauchy sequence* if there are linearly independent points

$z_1, \cdots, z_n \in X$ such that

$$\lim_{k,\ell\to\infty} \left\|y_k - y_\ell, w_2, \cdots, w_n\right\|_* = 0 \quad \text{for every } w_2, \cdots, w_n \in \{z_1, \cdots, z_n\}.$$

Definition 2.13. A sequence $\{y_k\}_{k\in\mathbb{N}}$ is said to be *convergent* if there exists a $y \in X$ with

$$\lim_{k\to\infty} \left\|y_k - y, x_2, \cdots, x_n\right\|_* = 0 \quad \text{for every } x_2, \cdots, x_n \in X.$$

In this case, we call that $\{y_k\}_{k\in\mathbb{N}}$ converges to y or that y is the limit of $\{y_k\}_{k\in\mathbb{N}}$ and we write $\{y_k\}_{k\in\mathbb{N}} \to y$ as $k \to \infty$.

By condition (4) in Definition 2.9, we have

$$\left\|y_k - y_\ell, x_2, \cdots, x_n\right\|_* \le \max_{\ell\le j\le k-1} \left\{\left\|y_{j+1} - y_j, x_2, \cdots, x_n\right\|_*\right\}, \quad (\ell < k)$$

for all $x_2, \cdots, x_n \in X$. Therefore, a sequence $\{y_k\}_{k\in\mathbb{N}}$ is Cauchy in $\left(X, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ if and only if $\{y_{k+1} - y_k\}_{k\in\mathbb{N}}$ converges to zero in an ultrametric n-normed space $\left(X, \left\|\cdot, \cdots, \cdot\right\|_*\right)$.

Definition 2.14. If every Cauchy sequence in an ultrametric n-normed space $\left(X, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ converges to some $y \in X$, then $\left(X, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ is said to be *complete*. Any complete ultrametric n-normed space is said to be *an ultrametric n-Banach space*.

Now we state the following results as a lemma.

Lemma 2.15. *Let $\left(X, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ be an ultrametric n-normed space. Then the following conditions hold:*

(1) $\left|\left\|x, x_2, \cdots, x_n\right\|_* - \left\|y, x_2, \cdots, x_n\right\|_*\right| \le \left\|x - y, x_2, \cdots, x_n\right\|_*$ *for all $x, y, x_2, \cdots, x_n \in X$,*
(2) *if $x \in X$ and $\left\|x, x_2, \cdots, x_n\right\|_* = 0$ for all $x_2, \cdots, x_n \in X$, then $x = 0$,*
(3) *if $\{x_k\}_{k\in\mathbb{N}}$ is a convergent sequence of elements of X, then*

$$\lim_{k\longrightarrow\infty} \left\|x_k, y_2, \cdots, y_n\right\|_* = \left\|\lim_{k\to\infty} x_k, y_2, \cdots, y_n\right\|_* \quad \textit{for all } y_2, \cdots, y_n \in X.$$

3. Fixed point theorem

In this section, we assume $m \in \mathbb{N}$ and $\left(Y, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ is an ultrametric $(m+1)$-Banach space. Next theorem gives us an extension of the results of the fixed point given by J. Brzdęk and K. Ciepliński [17, Theorem 1] in ultrametric $(m+1)$-Banach spaces.

Theorem 3.1. *Supposing that:*

(1) *X is a nonempty set, $\left(Y, \left\|\cdot, \cdots, \cdot\right\|_*\right)$ is an ultrametric $(m+1)$-normed space on a non-Archimedean field and $g : X \to Y$ is a surjective mapping,*
(2) *The operators $\mathcal{T} : Y^X \to Y^X$, $\Gamma : X^{m+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ and mapping $f : X \to X$ are such that $\Gamma(x, z_1, \cdots, z_m, \cdot)$ is nondecreasing for every $x, z_1, \cdots, z_m \in X$*

and

$$\Big\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x),g(z_1),\cdots,g(z_m)\Big\|_* \\ \leq \Gamma\Big(x,z_1,\cdots,z_m,\Big\|\xi(f(x))-\mu(f(x)),g(z_1),\cdots,g(z_m)\Big\|_*\Big), \tag{3.1}$$

for all $\xi,\mu\in Y^X$ *and all* $x,z_1,\cdots,z_m\in X$*,*

(3) *The functions* $\varepsilon: X^{m+1}\to\mathbb{R}_+,\varphi:X\to Y$ *are such that*

$$\big\|\mathcal{T}\varphi(x)-\varphi(x),g(z_1),\cdots,g(z_m)\big\|_*\leq\varepsilon\big(x,z_1,\cdots,z_m\big), \tag{3.2}$$

for all $x,z_1,\cdots,z_m\in X$*,*

(4) *The operator* $\mathcal{L}_f^\Gamma:\mathbb{R}_+^{X^{m+1}}\to\mathbb{R}_+^{X^{m+1}}$ *is defined by*

$$\mathcal{L}_f^\Gamma(\sigma)(x,z_1,\cdots,z_m):=\Gamma\Big(x,z_1,\cdots,z_m,\sigma\big(f(x),z_1,\cdots,z_m\big)\Big),$$

for all $\sigma\in\mathbb{R}_+^{X^{m+1}}$*, and all* $x,z_1,\cdots,z_m\in X$*, and satisfies the following two conditions*

$$\lim_{n\to\infty}\left(\mathcal{L}_f^\Gamma\right)^n(\varepsilon)\left(x,z_1,\cdots,z_m\right)=0 \tag{3.3}$$

and, for any $n\in\mathbb{N}_0$*,*

$$\Gamma\Big(x,z_1,\cdots,z_m,\sup_{i\geq n}\big\{\big(\mathcal{L}_f^\Gamma\big)^i(\varepsilon)\big(f(x),z_1,\cdots,z_m\big)\big\}\Big) \\ \leq\sup_{i\geq n+1}\Big\{\big(\mathcal{L}_f^\Gamma\big)^i(\varepsilon)(x,z_1,\cdots,z_m)\Big\}, \tag{3.4}$$

for all $x,z_1,\cdots,z_m\in X$*.*

Then for each $x\in X$*, the limit*

$$\psi(x):=\lim_{n\to\infty}\mathcal{T}^n\varphi(x) \tag{3.5}$$

exists and the function $\psi:X\to Y$*, defined in this way, is the unique fixed point of* $\mathcal{T}$ *with*

$$\big\|\varphi(x)-\psi(x),g(z_1),\cdots,g(z_m)\big\|_*\leq\sup_{n\in\mathbb{N}_0}\Big\{\big(\mathcal{L}_f^\Gamma\big)^n(\varepsilon)(x,z_1,\cdots,z_m)\Big\} \\ :=H(x,z_1,\cdots,z_m) \tag{3.6}$$

for all $x,z_1,\cdots,z_m\in X$*.*

Proof. It is easy to show by induction that, for each $n\in\mathbb{N}_0$,

$$\big\|\mathcal{T}^{n+1}\varphi(x)-\mathcal{T}^n\varphi(x),g(z_1),\cdots,g(z_m)\big\|_*\leq\big(\mathcal{L}_f^\Gamma\big)^n(\varepsilon)(x,z_1,\cdots,z_m), \tag{3.7}$$

for all $x,z_1,\cdots,z_m\in X$. Indeed, the case $n=0$ is just (3.2). Moreover, assuming that (3.7) holds for an $n\in\mathbb{N}_0$, by (3.1) and the fact that $\Gamma(x,z_1,\cdots,z_m,\cdot)$ is nondecreasing for every $x, z_1, \cdots, z_m \in X$, we obtain

$$\begin{aligned}
&\left\|\mathcal{T}^{n+2}\varphi(x) - \mathcal{T}^{n+1}\varphi(x), g(z_1), \cdots, g(z_m)\right\|_* \\
&\qquad \le \Gamma\Big(x, z_1, \cdots, z_m, \left\|\mathcal{T}^{n+1}\varphi\big(f(x)\big) - \mathcal{T}^{n}\varphi\big(f(x)\big), g(z_1), \cdots, g(z_m)\right\|_*\Big) \\
&\qquad \le \Gamma\Big(x, z_1, \cdots, z_m, \big(\mathcal{L}_f^\Gamma\big)^n(\varepsilon)\big(f(x), z_1, \cdots, z_m\big)\Big) \\
&\qquad = \Big(\mathcal{L}_f^\Gamma\Big)^{n+1}(\varepsilon)(x, z_1, \cdots, z_m), \quad x, z_1, \cdots, z_m \in X.
\end{aligned}$$

Clearly, by (3.3) and (3.7), for each $x, z_1, \cdots, z_m \in X$ the sequence $\big\{\mathcal{T}^n\varphi(x)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence, which means that the limit $\psi(x)$ exists.
Next, (3.7) yields that, for any $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $x, z_1, \cdots, z_m \in X$,

$$\begin{aligned}
&\left\|\mathcal{T}^{n}\varphi(x) - \mathcal{T}^{n+k}\varphi(x), g(z_1), \cdots, g(z_m)\right\|_* \\
&\qquad = \left\|\sum_{i=0}^{k}\Big(\mathcal{T}^{n+i}\varphi(x) - \mathcal{T}^{n+i+1}\varphi(x)\Big), g(z_1), \cdots, g(z_m)\right\|_* \\
&\qquad \le \max_{0\le i\le k-1}\Big\{\left\|\mathcal{T}^{n+i}\varphi(x) - \mathcal{T}^{n+i+1}\varphi(x), g(z_1), \cdots, g(z_m)\right\|_*\Big\} \\
&\qquad \le \sup_{i\ge n}\left\{\Big(\mathcal{L}_f^\Gamma\Big)^{i}(\varepsilon)(x, z_1, \cdots, z_m)\right\},
\end{aligned}$$

then

$$\left\|\mathcal{T}^{n}\varphi(x) - \mathcal{T}^{n+k}\varphi(x), g(z_1), \cdots, g(z_m)\right\|_* \le \sup_{i\ge n}\Big\{\Big(\mathcal{L}_f^\Gamma\Big)^{i}(\varepsilon)(x, z_1, \cdots, z_m)\Big\}, \tag{3.8}$$

for all $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $x, z_1, \cdots, z_m \in X$. Letting $k \longrightarrow \infty$ in (3.8), we get

$$\left\|\mathcal{T}^{n}\varphi(x) - \psi(x), g(z_1), \cdots, g(z_m)\right\|_* \le \sup_{i\ge n}\Big\{\Big(\mathcal{L}_f^\Gamma\Big)^{i}(\varepsilon)(x, z_1, \cdots, z_m)\Big\}, \tag{3.9}$$

for all $x, z_1, \cdots, z_m \in X$ and all $n \in \mathbb{N}_0$. Notice that the inequality (3.9) becomes (3.6) when $n = 0$. Furthermore, by (3.1), (3.4) and the fact that $\Gamma\big(x, z_1, \cdots, z_m, \cdot\big)$ is nondecreasing, for every $x, z_1, \cdots, z_m \in X$, we obtain

$$\begin{aligned}
&\left\|\mathcal{T}^{n+1}\varphi(x) - \mathcal{T}\psi(x), g(z_1), \cdots, g(z_m)\right\|_* \\
&\qquad \le \Gamma\Big(x, z_1, \cdots, z_m, \left\|\mathcal{T}^{n}\varphi\big(f(x)\big) - \psi\big(f(x)\big), g(z_1), \cdots, g(z_m)\right\|_*\Big) \\
&\qquad \le \Gamma\left(x, z_1, \cdots, z_m, \sup_{i\ge n}\Big\{\big(\mathcal{L}_f^\Gamma\big)^{i}(\varepsilon)\big(f(x), z_1, \cdots, z_m\big)\Big\}\right) \\
&\qquad \le \sup_{i\ge n+1}\left\{\Big(\mathcal{L}_f^\Gamma\Big)^{i}(\varepsilon)(x, z_1, \cdots, z_m)\right\}
\end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}_0$ and all $x, z_1, \cdots, z_m \in X$. In view of (3.3), the inequality (3.10) gives

$$\mathcal{T}\psi(x) = \lim_{k\longrightarrow\infty}\mathcal{T}\big(\mathcal{T}^k(\varphi)\big)(x) = \psi(x), \quad x \in X.$$

To show the uniqueness of ψ, we suppose that $\chi \in Y^X$ is also a fixed point of $\mathcal{T}$ with

$$\left\|\varphi(x)-\chi(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \leq H\left(x, z_1, \cdots, z_m\right), \quad x, z_1, \cdots, z_m \in X.$$

We prove that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned}\left\|\psi(x)-\chi(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* &= \left\|\mathcal{T}^n\psi(x)-\mathcal{T}^n\chi(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \\ &\leq \sup_{i\geq n}\left\{\left(\mathcal{L}_f^\Gamma\right)^i(\varepsilon)\left(x, z_1, \cdots, z_m\right)\right\},\end{aligned} \tag{3.11}$$

for all $x, z_1, \cdots, z_m \in X$. The case $n=0$ is trivial. So, assume that (3.11) holds for an $n \in \mathbb{N}_0$. Then, by (3.1), (3.4) and the fact that $\Gamma\left(x, z_1, \cdots, z_m, \cdot\right)$ is nondecreasing, for every $x, z_1, \cdots, z_m \in X$, we have

$$\begin{aligned}&\left\|\mathcal{T}^{n+1}\psi(x)-\mathcal{T}^{n+1}\chi(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \\ &\qquad\leq \Gamma\Big(x, z_1, \cdots, z_m, \left\|\mathcal{T}^n\psi\big(f(x)\big)-\mathcal{T}^n\chi\big(f(x)\big), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_*\Big) \\ &\qquad\leq \Gamma\Big(x, z_1, \cdots, z_m, \sup_{i\geq n}\Big\{\left(\mathcal{L}_f^\Gamma\right)^i(\varepsilon)\big(f(x), z_1, \cdots, z_m\big)\Big\}\Big) \\ &\qquad\leq \sup_{i\geq n+1}\Big\{\left(\mathcal{L}_f^\Gamma\right)^i(\varepsilon)\left(x, z_1, \cdots, z_m\right)\Big\}\end{aligned}$$

Thus (3.11) holds for each $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (3.11) and using (3.3), we get $\chi=\psi$. □

Let $r \in \mathbb{N}$. Given functions $f_1, \cdots, f_r : X \to X$ and $L_1, \cdots, L_r : X \to \mathbb{R}_+$, let's define the operators $\Gamma : X^{m+1} \times \mathbb{R}_+$and $\mathcal{L}_{f_i}^\Gamma : \mathbb{R}_+^{X^{m+1}} \to \mathbb{R}_+^{X^{m+1}}$ by

$$\Gamma\big(x, z_1, \cdots, z_m, L_i(x)\big) := \max_{1\leq i\leq r}\Big\{L_i(x)\sigma\big(f_i(x), z_1, \cdots, z_m\big)\Big\}$$

and

$$\mathcal{L}_{f_i}^\Gamma\sigma\left(x, z_1, \cdots, z_m\right) := \Gamma\Big(x, z_1, \cdots, z_m, L_i(x)\Big)$$

for all $\sigma \in X^{m+1}$ and all $x, z_1, \cdots, z_m \in X$. Theorem 3.1 with the above operators yields at once the following theorem concerning an analog of the fixed point theorem [19, Theorem 4] in ultrametric $(m+1)$-Banach spaces. We will use the symbol Λ instead of $\mathcal{L}_{f_i}^\Gamma$.

Theorem 3.2. *Supposing that:*

(1) X *is a nonempty set,* $\big(Y, \|\cdot, \cdots, \cdot\|_*\big)$ *is an ultrametric* $(m+1)$*-normed space on a non-Archimedean field and* $g : X \to Y$ *is a surjective mapping,*
(2) *The mappings* $f_1, \ldots, f_r : X \to X$, $L_1, \ldots, L_r : X \to \mathbb{R}_+$ *and the operator* $\mathcal{T} : Y^X \to Y^X$ *are such that*

$$\begin{aligned}&\Big\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x), g\left(z_1\right), \cdots, g\left(z_m\right)\Big\|_* \\ &\qquad\leq \max_{1\leq i\leq r}\Big\{L_i(x)\Big\|\xi\left(f_i(x)\right)-\mu\big(f_i(x)\big), g\left(z_1\right), \cdots, g\left(z_m\right)\Big\|_*\Big\}\end{aligned} \tag{3.12}$$

for all $\xi, \mu \in Y^X$ *and all* $x, z_1, \cdots, z_m \in X$,

(3) *The functions* $\varepsilon : X^{m+1} \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *are such that*

$$\left\| \mathcal{T}\varphi(x) - \varphi(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \varepsilon(x, z_1, \cdots, z_m) \tag{3.13}$$

and

$$\lim_{n\to\infty} \Lambda^n \varepsilon(x, z_1, \cdots, z_m) = 0 \tag{3.14}$$

for all $x, z_1, \cdots, z_m \in X$*, where* $\Lambda : \mathbb{R}_+^{X^{m+1}} \to \mathbb{R}_+^{X^{m+1}}$ *is given by*

$$\Lambda\delta(x, z_1, \cdots, z_m) := \max_{1\leq i\leq r} \Big\{ L_i(x)\delta\big(f_i(x), z_1, \cdots, z_m\big) \Big\}, \tag{3.15}$$

for all $\delta \in \mathbb{R}_+^{X^{m+1}}$ *and all* $x, z_1, \cdots, z_m \in X$.

Then we have:

(1) *For each* $x, z_1, \cdots, z_m \in X$*, the limit*

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x) \tag{3.16}$$

exists and the function $\psi : X \to Y$*, defined in this way, is the unique fixed point of* $\mathcal{T}$ *with*

$$\left\| \varphi(x) - \psi(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \sup_{n\in\mathbb{N}_0} \Big\{ \Lambda^n \varepsilon(x, z_1, \cdots, z_m) \Big\}, \tag{3.17}$$

(2) *If*

$$\Lambda\left(\sup_{n\in\mathbb{N}_0} \Big\{ \Lambda^n \varepsilon(x, z_1, \cdots, z_m) \Big\} \right) \leq \sup_{n\in\mathbb{N}_0} \Big\{ \Lambda^{n+1} \varepsilon(x, z_1, \cdots, z_m) \Big\}, \tag{3.18}$$

then ψ *is the unique fixed point of* $\mathcal{T}$ *satisfying (3.17).*

Proof. (1): We begin with proving, for each $n \in \mathbb{N}_0$, that

$$\left\| \mathcal{T}^n\varphi(x) - \mathcal{T}^{n+1}\varphi(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \Lambda^n \varepsilon(x, z_1, \cdots, z_m), \tag{3.19}$$

for all $x, z_1, \cdots, z_m \in X$. In view of the inequality (3.12), it easy to shaw that (3.19) holds for $n = 0$. Now, for $n \in \mathbb{N}_0$, suppose that (3.19) holds. Then, using (3.12) and inductive assumption, for every $x, z_1, \cdots, z_m \in X$, we obtain that

$$\begin{aligned}
&\left\| \mathcal{T}^{n+1}\varphi(x) - \mathcal{T}^{n+2}\varphi(x), g(z_1), \cdots, g(z_m) \right\|_* \\
&\leq \max_{1\leq i\leq r} \left\{ L_i(x) \left\| \mathcal{T}^n\varphi\big(f_i(x)\big) - \mathcal{T}^{n+1}\varphi\big(f_i(x)\big), g(z_1), \cdots, g(z_m) \right\|_* \right\} \\
&\leq \max_{1\leq i\leq r} \Big\{ L_i(x)\Lambda^n\varepsilon\big(f_i(x), z_1, \cdots, z_m\big) \Big\} \\
&= \Lambda^{n+1}\varepsilon(x, z_1, \cdots, z_m).
\end{aligned}$$

This concludes the proof of (3.19). Now, for $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we observe

$$\begin{aligned}\Big\|\mathcal{T}^n\varphi(x)-\mathcal{T}^{n+k}\varphi(x), g(z_1), \cdots, g(z_m)\Big\|_* &\\ \leq \max_{0\leq i\leq k-1}\left\{\Big\|\mathcal{T}^{n+i}\varphi(x)-\mathcal{T}^{n+i+1}\varphi(x), g(z_1), \cdots, g(z_m)\Big\|_*\right\} &\\ \leq \max_{n\leq i\leq n+k-1}\left\{\Lambda^i\varepsilon(x, z_1, \cdots, z_m)\right\}, \quad x, z_1, \cdots, z_m \in X.&\end{aligned}$$

Therefore,

$$\Big\|\mathcal{T}^n\varphi(x)-\mathcal{T}^{n+k}\varphi(x), g(z_1), \cdots, g(z_m)\Big\|_* \leq \sup_{n\in\mathbb{N}_0}\left\{\Lambda^n\varepsilon(x, z_1, \cdots, z_m)\right\}, \tag{3.20}$$

for all $x, z_1, \cdots, z_m \in X$. From the above estimation, we conclude that $\{\mathcal{T}^n\varphi(x)\}_{n\in\mathbb{N}}$, for all $x \in X$, is a Cauchy sequence and this sequence is convergence because Y is an ultrametric $(m+1)$-Banach space. Thus, the limit

$$\lim_{n\to\infty}\mathcal{T}^n\varphi(x)$$

exists for all $x \in X$ and defines a function $\psi : X \to Y$ as (3.16). Letting $k \to \infty$ in (3.20), we get that

$$\Big\|\mathcal{T}^n\varphi(x)-\psi(x), g(z_1), \cdots, g(z_m)\Big\|_* \leq \sup_{i\geq n}\left\{\Lambda^i\varepsilon(x, z_1, \cdots, z_m)\right\}, \tag{3.21}$$

for all $n \in \mathbb{N}_0$ and all $x, z_1, \cdots, z_m \in X$. Letting $n = 1$ in (3.21), we obtain that (3.17) holds for all $x, z_1, \cdots, z_m \in X$. In view of (3.12) and (3.21), we conclude that

$$\begin{aligned}\Big\|\mathcal{T}^{n+1}\varphi(x)-\mathcal{T}\psi(x), g(z_1), \cdots, g(z_m)\Big\|_* &\\ \leq \max_{1\leq i\leq j}\left\{L_i(x)\Big\|\psi\big(f_i(x)\big)-\mathcal{T}^n\varphi\big(f_i(x)\big), g(z_1), \cdots, g(z_m)\Big\|_*\right\} &\\ \leq \Lambda\left(\Big\|\mathcal{T}^n\varphi(x)-\psi(x), g(z_1), \cdots, g(z_m)\Big\|_*\right) &\\ \leq \Lambda\left(\sup_{i\geq n}\left\{\Lambda^i\varepsilon(x, z_1, \cdots, z_m)\right\}\right), &\end{aligned} \tag{3.22}$$

for all $n \in \mathbb{N}_0$ and all $x, z_1, \cdots, z_m \in X$. Letting $n \to \infty$ in (3.22) and using (3.14), we get $\mathcal{T}(\psi) = \psi$.

(2): To prove the uniqueness of ψ, suppose that $\psi_1, \psi_2 \in Y^X$ are two fixed points of $\mathcal{T}$ such that

$$\Big\|\varphi(x)-\psi_i(x), g(z_1), \cdots, g(z_m)\Big\|_* \leq \sup_{n\in\mathbb{N}_0}\left\{\Lambda^n\varepsilon(x, z_1, \cdots, z_m)\right\},$$

for all $x, z_1, \cdots, z_m \in X$ with $i = 1, 2$. For every $m \in \mathbb{N}_0$, we show that

$$\begin{aligned}\Big\| \psi_1(x) - \psi_2(x), g(z_1), \cdots, g(z_m) \Big\|_* &= \Big\| \mathcal{T}^m \psi_1(x) - \mathcal{T}^m \psi_2(x), g(z_1), \cdots, g(z_m) \Big\|_* \\ &\le \sup_{j \ge m} \Big\{ \Lambda^j \varepsilon(x, z_1, \cdots, z_m) \Big\}, \end{aligned} \tag{3.23}$$

for all $x, z_1, \cdots, z_m \in X$. For the case $m = 0$, we have

$$\begin{aligned}&\Big\| \psi_1(x) - \psi_2(x)\ ,\ g(z_1), \cdots, g(z_m) \Big\|_* \\ &\le \max \Big\{ \Big\| \psi_1(x) - \varphi(x), g(z_1), \cdots, g(z_m) \Big\|_*, \Big\| \varphi(x) - \psi_2(x), g(z_1), \cdots, g(z_m) \Big\|_* \Big\} \\ &\le \sup_{n \in \mathbb{N}_0} \Big\{ \Lambda^n \varepsilon(x, z_1, \cdots, z_m) \Big\}, \quad x, z_1, \cdots, z_m \in X.\end{aligned}$$

Next, we assume that (3.23) is valid for some $m \in \mathbb{N}_0$. Then, by using (3.12), for every $x, z_1, \cdots, z_m \in X$, we get that

$$\begin{aligned}&\Big\| \mathcal{T}^{m+1} \psi_1(x) - \mathcal{T}^{m+1} \psi_2(x), g(z_1), \cdots, g(z_m) \Big\|_* \\ &\qquad = \Big\| \mathcal{T}(\mathcal{T}^m \psi_1)(x) - \mathcal{T}(\mathcal{T}^m \psi_2)(x), g(z_1), \cdots, g(z_m) \Big\|_* \\ &\qquad \le \max_{1 \le i \le r} \Big\{ L_i(x) \Big\| \mathcal{T}^m \psi_1\big(f_i(x)\big) - \mathcal{T}^m \psi_2\big(f_i(x)\big), g(z_1), \cdots, g(z_m) \Big\|_* \Big\} \\ &\qquad \le \max_{1 \le i \le r} \Big\{ L_i(x) \Big(\max_{j \ge m} \Big\{ \Lambda^j \varepsilon\big(f_i(x), z_1, \cdots, z_m\big) \Big\} \Big) \Big\} \\ &\qquad = \max_{j \ge m} \Big\{ \max_{1 \le i \le r} \Big\{ L_i(x) \Lambda^j \varepsilon\big(f_i(x), z_1, \cdots, z_m\big) \Big\} \Big\} \\ &\qquad = \max_{j \ge m+1} \Big\{ \Lambda^j \varepsilon(x, z_1, \cdots, z_m) \Big\}.\end{aligned}$$

Therefore, we have proved that (3.23) holds for any $m \in \mathbb{N}_0$. Letting $m \to \infty$ and using (3.14) and Lemma 2.11, we get that $\psi_1 = \psi_2$. So the fixed point satisfying (3.17) of $\mathcal{T}$ is unique. □

4. Hyperstability results

Suppose that $(X, +)$ is a group and $(Y, \|\cdot, \cdots, \cdot\|_*)$ is an ultrametric $(m+1)$-Banach space on a non-Archimedean field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|_* : \mathbb{K} \to \mathbb{R}_+$. We denote by $\mathrm{Aut}(X)$ the family of all automorphisms of X. Moreover, for each $u : X \to X$, we write $ux := u(x)$ for all $x \in X$ and define u' by $u'x := x - ux$ for all

$x \in X$. Let

$$\ell(X) := \Bigg\{ u \in \mathrm{Aut}(X) : u', (ku' + u) \in \mathrm{Aut}(X),$$

$$\alpha_u := \frac{1}{\big|s\big|_*} \max_{2 \le k \le s} \Big\{ \lambda(u), \lambda\left(u'\right), \lambda(ku' + u) \Big\} < 1 \Bigg\} \neq \emptyset, \quad (4.1)$$

where $k, s \in \mathbb{N}$ and

$$\lambda(u) := \inf \Big\{ t \in \mathbb{R}_+ : h(ux, uy, z_1, \cdots, z_m) \le t\, h\left(x, y, z_1, \cdots, z_m\right),$$

$$\text{for all } x, y, z_1, \cdots, z_m \in X_0 \Big\} \quad (4.2)$$

with $h : X_0^{m+2} \to \mathbb{R}_+$. We say that $\mathcal{U} \subset \ell(X)$ is commutative provided

$$u \circ v = v \circ u, \quad u, v \in \mathcal{U}. \quad (4.3)$$

According to Theorem 3.2 and using the type of fixed point approach proposed for the first time in [13], we will present and prove our hyperstability results for Eq. (1.14) on X_0.

Theorem 4.1. *Let $h : X_0^{m+2} \to \mathbb{R}_+$ be a function and $f : X \to Y$ be a mapping satisfies the inequality*

$$\left\| \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx + y) - s! f(x), g\left(z_1\right), \cdots, g\left(z_m\right) \right\|_* \le h\left(x, y, z_1, \cdots, z_m\right) \quad (4.4)$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $kx + y \neq 0$ where $g : X \to Y$ is a surjective mapping. If the following conditions hold:

(1)

$$\inf \Big\{ h\left(u'x, uy, z_1, \cdots, z_m\right) : u \in \mathcal{U} \Big\} = 0, \quad x, y, z_1, \cdots, z_m \in X_0, \quad (4.5)$$

(2)

$$\sup \Big\{ \alpha_u : u \in \mathcal{U} \Big\} < 1, \quad (4.6)$$

where α_u is defined in (4.1), then f is a solution to Eq. (1.14).

Proof. According to the parity of $s \in \mathbb{N}$, we study the following two cases:
1^{st} **case:** s is even:
In this case, we can write the inequality (4.4) as follows

$$\left\| f(y) - s f(x + y) - s! f(x) + \sum_{k=2}^{s} (-1)^{s-k} C_k^s f(kx + y), g\left(z_1\right), \cdots, g\left(z_m\right) \right\|_* \le h\left(x, y, z_1, \cdots, z_m\right), \quad (4.7)$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $kx + y \neq 0$. Fix a nonempty and commutative $\mathcal{U} \subset \ell(X)$. By replace x by $u'x$ and y by ux in (4.7), we obtain that

$$\Big\| \frac{1}{s} f(ux) - f(x) - (s-1)! f(u'x) + \frac{1}{s} \sum_{k=2}^{s} (-1)^{s-k} C_k^s f\big((ku'+u)x\big) , g(z_1), \cdots, g(z_m) \Big\|_* \leq \frac{1}{|s|_*} h(u'x, ux, z_1, \cdots, z_m), \tag{4.8}$$

for all $x, z_1, \cdots, z_m \in X_0$. So, we can define, for each $u \in \mathcal{U}$, the operators

$$\mathcal{T}_u : Y^{X_0} \to Y^{X_0} \text{ and } \Lambda_u : \mathbb{R}_+^{X_0^{m+1}} \to \mathbb{R}_+^{X_0^{m+1}}$$

by

$$\mathcal{T}_u \xi(x) := \frac{1}{s} \xi(ux) - (s-1)! \xi(u'x) + \frac{1}{s} \sum_{k=2}^{s} (-1)^{s-k} C_k^s \xi\big((ku'+u)x\big) \tag{4.9}$$

and

$$\Lambda_u \delta(x, z_1, \cdots, z_m) := \max_{2 \leq k \leq s} \Big\{ \frac{1}{|s|_*} \delta(ux, z_1, \cdots, z_m), \frac{1}{|s|_*} \delta(u'x, z_1, \cdots, z_m) , \frac{1}{|s|_*} \delta\big((ku'+u)x, z_1, \cdots, z_m\big) \Big\} \tag{4.10}$$

for all $x, z_1, \cdots, z_m \in X_0$, $\xi \in Y^{X_0}$, and $\delta \in \mathbb{R}_+^{X_0^{m+1}}$. Note that, for every $u \in \mathcal{U}$, the operator $\Lambda := \Lambda_u$ has the form given in (3.15) with

$$X := X_0, \quad r = s+1,$$

$$L_k(x) = \frac{1}{|s|_*} \text{ for } k = 1, \cdots, s+1,$$

$f_1(x) = ux$, $f_2(x) = u'x$, and $f_{k+1}(x) = (ku'+u)x$ for $k = 2, 3, \cdots, s$.
Furthermore, when we put

$$\varepsilon_u(x, z_1, \cdots, z_m) := \frac{1}{|s|_*} h(u'x, ux, z_1, \cdots, z_m),$$

the inequality (4.8) becomes

$$\big\| \mathcal{T}_u f(x) - f(x), g(z_1), \cdots, g(z_m) \big\|_* \leq \varepsilon_u(x, z_1, \cdots, z_m), \tag{4.11}$$

for all $x, z_1, \cdots, z_m \in X_0$ and all $u \in \mathcal{U}$. From here till the end of the paper, we denote by f the restriction of $f : X \to Y$ to the set $X_0 \subset X$ unless we mention otherwise. Moreover, for every $\xi, \mu \in Y^{X_0}$, we have

$$\begin{aligned}
&\Big\|\mathcal{T}_u\xi(x) - \mathcal{T}_u\mu(x), g(z_1), \cdots, g(z_m)\Big\|_* = \Big\|\frac{1}{s}\xi(ux) - (s-1)!\xi(u'x) \\
&+ \frac{1}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^s\xi\big((ku'+u)x\big) - \frac{1}{s}\mu(ux) + (s-1)!\mu(u'x) \\
&-\frac{1}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^s\mu\big((ku'+u)x\big), g(z_1), \cdots, g(z_m)\Big\|_* \\
&\leq \max_{2\leq k\leq s}\Bigg\{\frac{1}{|s|_*}\Big\|\xi(ux) - \mu(ux), g(z_1), \cdots, g(z_m)\Big\|_* \\
&, |(s-1)!|_*\Big\|\xi(u'x) - \mu(u'x), g(z_1), \cdots, g(z_m)\Big\|_* \\
&, \left|\frac{C_k^s}{s}\right|_*\Big\|\xi\big((ku'+u)x\big) - \mu((ku'+u)x), g(z_1), \cdots, g(z_m)\Big\|_*\Bigg\} \\
&\leq \max_{2\leq k\leq s}\Bigg\{\frac{1}{|s|_*}\Big\|\xi(ux) - \mu(ux), g(z_1), \cdots, g(z_m)\Big\|_* \\
&, \frac{1}{|s|_*}\Big\|\xi(u'x) - \mu(u'x), g(z_1), \cdots, g(z_m)\Big\|_* \\
&, \frac{1}{|s|_*}\Big\|\xi\big((ku'+u)x\big) - \mu((ku'+u)x), g(z_1), \cdots, g(z_m)\Big\|_*\Bigg\} \\
&= \max_{1\leq i\leq (s+1)}\Bigg\{L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big), g(z_1), \cdots, g(z_m)\Big\|_*\Bigg\}
\end{aligned}$$

for all $x, z_1, \cdots, z_m \in X_0$ and all $u \in \mathcal{U}$. This means that (3.12) holds for $\mathcal{T} := \mathcal{T}_u$ for any $u \in \mathcal{U}$. Next, by the definition of $\lambda(u)$, we have

$$h(ux, uy, z_1, \cdots, z_m) \leq \lambda(u)\; h(x, y, z_1, \cdots, z_m), \tag{4.12}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $u \in Aut(X)$. Therefore, by induction, we directly obtain that

$$\Lambda_u^n\varepsilon_u(x, z_1, \cdots, z_m) \leq \frac{1}{|s|_*}\alpha_u^n h(u'x, ux, z_1, \cdots, z_m), \tag{4.13}$$

for all $x, z_1, \cdots, z_m \in X_0$, all $n \in \mathbb{N}_0$, and all $u \in \mathcal{U}$, where

$$\alpha_u = \frac{1}{|s|_*}\max_{2\leq k\leq s}\Big\{\lambda(u), \lambda(u'), \lambda(ku'+u)\Big\}, \quad u \in \mathcal{U}.$$

By (4.13), in view of the definition of $\ell(X)$, we get that

$$\lim_{n\to\infty}\Lambda_u^n\varepsilon_u(x, z_1, \cdots, z_m) = 0, \quad x, z_1, \cdots, z_m \in X_0, \quad u \in \mathcal{U}. \tag{4.14}$$

In addition, we note that

$$\sup_{n\in\mathbb{N}_0}\left\{\Lambda_u^n\varepsilon_u\left(x,z_1,\cdots,z_m\right)\right\}=\Lambda_u^0\varepsilon_u\left(x,z_1,\cdots,z_m\right)=\varepsilon_u\left(x,z_1,\cdots,z_m\right)$$

implies

$$\sup_{n\in\mathbb{N}_0}\left\{\Lambda_u^{n+1}\varepsilon_u\left(x,z_1,\cdots,z_m\right)\right\}=\Lambda_u^n\varepsilon_u\left(x,z_1,\cdots,z_m\right)=\Lambda_u\left(\sup_{n\in\mathbb{N}_0}\left\{\Lambda_u^n\varepsilon_u\left(x,z_1,\cdots,z_m\right)\right\}\right),$$

for all $x,z_1,\cdots,z_m\in X_0$ and all $u\in\mathcal{U}$. As described above, we deduce that all assumptions of Theorem 3 hold. Therefore, there is, for every $u\in\mathcal{U}$, a unique fixed point $P_u:X_0\to Y$ of the operator $\mathcal{T}_u$ defined by

$$P_u(x):=\lim_{n\to\infty}\mathcal{T}_u^nf(x),\quad x\in X_0,u\in\mathcal{U}$$

such that

$$\left\|f(x)-P_u(x),g\left(z_1\right),\cdots,g\left(z_m\right)\right\|_*\leq\sup_{n\in\mathbb{N}_0}\left\{\Lambda_u^n\varepsilon_u\left(x,z_1,\cdots,z_m\right)\right\},\tag{4.15}$$

for all $x,z_1,\cdots,z_m\in X_0$ and all $u\in\mathcal{U}$. It means that

$$\begin{aligned}P_u(x)=&\frac{1}{s}P_u(ux)-(s-1)!P_u\left(u'x\right)\\&+\frac{1}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^sP_u\big((ku'+u)x\big),\\&x\in X_0,u\in\mathcal{U}.\end{aligned}\tag{4.16}$$

Now, for each $u\in\mathcal{U}$ and $x,y,z_1,\cdots,z_m\in X_0$ such that $kx+y\neq0$, we prove that

$$\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^nf(kx+y)-s!\mathcal{T}_u^nf(x),g\left(z_1\right),\cdots,g\left(z_m\right)\right\|_*\leq\alpha_u^nh\left(x,y,z_1,\cdots,z_m\right),\tag{4.17}$$

for any $n\in\mathbb{N}_0$. It is clear that if $n=0$, then (4.17) holds by (4.4). Fix an $n\in\mathbb{N}_0$ and assume that (4.17) holds for any $u\in\mathcal{U}$ and $x,y,z_1,\cdots,z_m\in X_0$ such that $kx+y\neq0$. Then, in view of (4.17), we get

$$\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^{n+1}f(kx+y)-s!\mathcal{T}_u^{n+1}f(x),g\left(z_1\right),\cdots,g\left(z_m\right)\right\|_*$$

$$
\begin{aligned}
&= \Bigg\| \frac{1}{s}\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u(kx+y)\big) - (s-1)!\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u'(kx+y)\big) \\
&\qquad + \frac{1}{s}\sum_{k=0}^{s}(-1)^{s-k}C_k^s\left(\sum_{k=2}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big((ku'+u)(kx+y)\big)\right) \\
&\qquad - \frac{s!}{s}\mathcal{T}_u^n f(ux) + s!(s-1)!\mathcal{T}_u^n f\left(u'x\right) \\
&\qquad - \frac{s!}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big((ku'+u)x\big), g\left(z_1\right), \cdots, g\left(z_m\right)\Bigg\|_* \\
&\le \max_{2\le k\le s}\left\{\frac{1}{|s|_*}\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u(kx+y)\big) - s!\mathcal{T}_u^n f(ux), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_*\right. \\
&, \ |(s-1)!|_*\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u'(kx+y)\big) - s!\mathcal{T}_u^n f\left(u'x\right), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \\
&, \left|\frac{C_k^s}{s}\right|_*\Bigg\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big((ku'+u)(kx+y)\big) \\
&\qquad \left. - s!\mathcal{T}_u^n f\big((ku'+u)x\big), g\left(z_1\right), \cdots, g\left(z_m\right)\Bigg\|_*\right\} \\
&\le \max_{2\le k\le s}\left\{\frac{1}{|s|_*}\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u(kx+y)\big) - s!\mathcal{T}_u^n f(ux), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_*\right. \\
&, \ \frac{1}{|s|_*}\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big(u'(kx+y)\big) - s!\mathcal{T}_u^n f\left(u'x\right), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \\
&, \frac{1}{|s|_*}\Bigg\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s\mathcal{T}_u^n f\big((ku'+u)(kx+y)\big) \\
&\qquad \left. - s!\mathcal{T}_u^n f\big((ku'+u)x\big), g\left(z_1\right), \cdots, g\left(z_m\right)\Bigg\|_*\right\} \\
&\le \frac{1}{|s|_*}\max_{2\le k\le s}\Big\{\alpha_u^n h\left(ux, uy, z_1, \cdots, z_m\right), \alpha_u^n h\left(u'x, u'y, z_1, \cdots, z_m\right) \\
&\qquad\qquad , \alpha_u^n h\big((ku'+u)x, (ku'+u)y, z_1, \cdots, z_m\big)\Big\} \\
&\le \alpha_u^n\frac{1}{|s|_*}\max_{2\le k\le s}\Big\{\lambda(u), \lambda\left(u'\right), \lambda(ku'+u)\Big\}h\left(x, y, z_1, \cdots, z_m\right) \\
&= \alpha_u^{n+1}h\left(x, y, z_1, \cdots, z_m\right), \quad u\in\mathcal{U},\ x, y, z_1, \cdots, z_m\in X_0,\ kx+y\in X_0.
\end{aligned}
$$

By mathematical induction, we deduce that (4.17) holds for any $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (4.17) and using the surjectivity of g in view of Lemma 2.11, we obtain the equality

$$\sum_{k=0}^{s}(-1)^{s-k}C_k^s P_u(kx+y) = s!P_u(x), \quad x,y \in X_0, kx+y \in X_0, u \in \mathcal{U}. \tag{4.18}$$

In this way, for each $u \in \mathcal{U}$, we obtain a function P_u such that (4.18) holds for all $x, y, kx+y \in X_0$ and

$$\begin{aligned}\Big\|f(x)-P_u(x), g(z_1), \cdots, g(z_m)\Big\|_* &\leq H(x, z_1, \cdots, z_m) \\ &:= \sup_{n\in\mathbb{N}_0}\{\Lambda_u^n \varepsilon_u(x, z_1, \cdots, z_m)\} \\ &\leq \frac{1}{|s|_*}\alpha_u^n h(u'x, ux, z_1, \cdots, z_m),\end{aligned} \tag{4.19}$$

for all $x, z_1, \cdots, z_m \in X_0$ and all $u \in \mathcal{U}$. In view of the conditions (4.5) and (4.6), we deduce that $H(x, z_1, \cdots, z_m) = 0$ for all $x, z_1, \cdots, z_m \in X_0$, which implies that $f(x) = P_u(x)$ for all $x \in X_0$ and $u \in \mathcal{U}$. Thus, f is solution to Eq. (1.14).

2^{nd} **case:** s is odd:

In this case, we can write the inequality (4.4) as follows:

$$\begin{aligned}\Big\|sf(x+y)-f(y)-s!f(x)+\sum_{k=2}^{s}(-1)^{s-k}C_k^s f(kx+y), g(z_1), \cdots, g(z_m)\Big\|_* \\ \leq h(x, y, z_1, \cdots, z_m),\end{aligned} \tag{4.20}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $kx+y \neq 0$. Fix a nonempty and commutative $\mathcal{U} \subset \ell(X)$ and replace x by $u'x$ and y by ux in (4.20). Therefore, we get

$$\begin{aligned}\Big\|f(x)-\frac{1}{s}f(ux)-(s-1)!f(u'x)+\frac{1}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^s f\big((ku'+u)x\big) \\ , g(z_1), \cdots, g(z_m)\Big\|_* \leq \frac{1}{|s|_*}h(u'x, ux, z_1, \cdots, z_m),\end{aligned} \tag{4.21}$$

for all $x, z_1, \cdots, z_m \in X_0$. So, we can define, for each $u \in \mathcal{U}$, the operators $\mathcal{T}_u : Y^{X_0} \to Y^{X_0}$ and $\Lambda_u : \mathbb{R}_+^{X_0^{m+1}} \to \mathbb{R}_+^{X_0^{m+1}}$ by

$$\mathcal{T}_u\xi(x) := \frac{1}{s}\xi(ux)+(s-1)!\xi(u'x)-\frac{1}{s}\sum_{k=2}^{s}(-1)^{s-k}C_k^s\xi\big((ku'+u)x\big) \tag{4.22}$$

and

$$\begin{aligned}\Lambda_u\delta(x, z_1, \cdots, z_m) := \max_{2\leq k\leq s}\Big\{\frac{1}{|s|_*}\delta(ux, z_1, \cdots, z_m), \frac{1}{|s|_*}\delta(u'x, z_1, \cdots, z_m) \\ , \frac{1}{|s|_*}\delta\big((ku'+u)x, z_1, \cdots, z_m\big)\Big\}\end{aligned} \tag{4.23}$$

for all $x, z_1, \cdots, z_m \in X_0$, $\xi \in Y^{X_0}$, and $\delta \in \mathbb{R}_+^{X_0^{m+1}}$. Note that, for every $u \in \mathcal{U}$, the operator $\Lambda := \Lambda_u$ has the form given in (3.15) with $X := X_0$, $r = s+1$, $L_k(x) = \frac{1}{|s|_*}$

for $k = 1, 2, \ldots, s+1$, $f_1(x) = ux$, $f_2(x) = u'x$, and $f_{k+1}(x) = (ku' + u)x$ for $k = 2, 3, \ldots, s$.
Moreover, when we write

$$\varepsilon_u(x, z_1, \cdots, z_m) := \frac{1}{|s|_*} h(u'x, ux, z_1, \cdots, z_m),$$

the inequality (4.20) becomes

$$\left\| \mathcal{T}_u f(x) - f(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \varepsilon_u(x, z_1, \cdots, z_m), \tag{4.24}$$

for all $x, z_1, \cdots, z_m \in X_0$ and all $u \in \mathcal{U}$. By the similar steps in 1^{st} case, we obtain the same results. □

5. Some consequences

From Theorem 4.1, we can obtain the following corollaries as natural results.

The next corollary corresponds to the results on the following inhomogeneous monomial functional equation

$$\sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx + y) = s!f(x) + F(x, y), \tag{5.1}$$

where $F : X^2 \to Y$.

Corollary 5.1. *Let $(X, +)$ and $(Y, \|\cdot, \cdots, \cdot\|_*)$ are group and ultrametric $(m+1)$-Banach space, respectively, and let $h : X_0^{m+2} \to Y$ and $F : X^2 \to Y$ be two mappings such that $F(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X_0$. Suppose that*

(1)

$$\left\| F(x, y), g(z_1), \cdots, g(z_m) \right\|_* \leq h(x, y, z_1, \cdots, z_m), \quad x, y, z_1, \cdots, z_m \in X_0, \tag{5.2}$$

where $g : X \to Y$ is a surjective mapping,

(2) *There exists a nonempty set $\mathcal{U} \subset l(X)$ such that (4.3), (4.5) and (4.6) hold.*

Then, for all $x, y \in X_0$, the inhomogeneous equation

$$\sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx + y) = s!f(x) + F(x, y) \tag{5.3}$$

has no solutions in the class of functions $f : X \to Y$.

Proof. Suppose that $f : X \to Y$ is a solution to (5.3). Then

$$\begin{aligned}\Bigg\| \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx+y) - s!f(x), g(z_1), \cdots, g(z_m) \Bigg\|_* \\ &= \Bigg\| s!f(x) + F(x,y) - s!f(x), g(z_1), \cdots, g(z_m) \Bigg\|_* \\ &= \Bigg\| F(x,y), g(z_1), \cdots, g(z_m) \Bigg\|_* \\ &\le h(x, y, z_1, \cdots, z_m), \quad x, y, z_1, \cdots, z_m \in X_0.\end{aligned}$$

Consequently, by Theorem 4.1, f is solution of (1.14), whence

$$F(x_0, y_0) = \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx_0 + y_0) - s!f(x_0) = 0$$

which is contradiction. □

In the rest of this section, E is a normed space, X is a subgroup of the commutative group $(E, +)$ and $(Y, \|\cdot, \cdots, \cdot\|_*)$ is an ultrametric $(m+1)$-Banach space.

Corollary 5.2. *Let $p, q \in \mathbb{R}$ such that $p < 0$ and $q < 0$, $g : X \to Y$ be a surjective mapping, and let $\theta, r \ge 0$. If $f : X \to Y$ satisfies*

$$\Bigg\| \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx+y) - s!f(x), g(z_1), \cdots, g(z_m) \Bigg\|_* \le \theta \Big(\|x\|^p + \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \tag{5.4}$$

for all $x, y, z_1, \cdots, z_m \in X_0$, then f satisfies the monomial functional equation (1.14) on X_0.

Proof. The proof follows from Theorem 4.1 by taking

$$h(x, y, z_1, \cdots, z_m) := \theta \Big(\|x\|^p + \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r, \quad x, y, z_1, \cdots, z_m \in X_0,$$

with some real numbers θ, $r \ge 0$, $p < 0$ and $q < 0$. For each $\ell \in \mathbb{N}$, we define $u_\ell : X \to X$ by $u_\ell x := -\ell x$ and $u'_\ell : X \to X$ by $u'_\ell x := (1+\ell)x$. Then

$$\begin{aligned} h(u_\ell x, u_\ell y, z_1, \cdots, z_m) &= h(-\ell x, -\ell y, z_1, \cdots, z_m) \\ &= \theta \Big(\| -\ell x\|^p + \| -\ell y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \Big(\ell^p \|x\|^p + \ell^q \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &\le \big(\ell^p + \ell^q \big) h(x, y, z_1, \cdots, z_m), \end{aligned}$$

$$\begin{aligned} h\left(u'_\ell x, u'_\ell y, z_1, \cdots, z_m\right) &= h\Big((1+\ell)x, (1+\ell)y, z_1, \cdots, z_m\Big) \\ &= \theta\Big(\big\|(1+\ell)x\big\|^p + \big\|(1+\ell)y\big\|^q\Big)\prod_{i=1}^{m}\big\|z_i\big\|^r \\ &= \theta\Big((1+\ell)^p\big\|x\big\|^p + (1+\ell)^q\big\|y\big\|^q\Big)\prod_{i=1}^{m}\big\|z_i\big\|^r \\ &\leq \Big((1+\ell)^p + (1+\ell)^q\Big)h\left(x, y, z_1, \cdots, z_m\right), \end{aligned}$$

and

$$\begin{aligned} &h\Big((ku'_\ell + u_\ell)x, (ku'_\ell + u_\ell)y, z_1, \cdots, z_m\Big) \\ &\qquad = h\Big(\big(k+k\ell-\ell\big)x, \big(k+k\ell-\ell\big)y, z_1, \cdots, z_m\Big) \\ &\qquad = \theta\Big(\big\|\big(k+k\ell-\ell\big)x\big\|^p + \big\|\big(k+k\ell-\ell\big)y\big\|^q\Big)\prod_{i=1}^{m}\big\|z_i\big\|^r \\ &\qquad = \theta\Big(\big|k+k\ell-\ell\big|^p\big\|x\big\|^p + \big|k+k\ell-\ell\big|^q\big\|y\big\|^q\Big)\prod_{i=1}^{m}\big\|z_i\big\|^r \\ &\qquad \leq \big(\big|k+k\ell-\ell\big|^p + \big|k+k\ell-\ell\big|^q\big)h\left(x, y, z_1, \cdots, z_m\right), \end{aligned}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $\ell \in \mathbb{N}$. Therefore, we deduce that $\lambda\left(u_\ell\right) = \ell^p + \ell^q$, $\lambda\left(u'_\ell\right) = (1+\ell)^p + (1+\ell)^q$, and $\lambda\big((ku'_\ell + u_\ell)\big) = \big|k+k\ell-\ell\big|^p + \big|k+k\ell-\ell\big|^q$ for $\ell \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $\ell \geq n_0$ and

$$\alpha_{u_\ell} = \frac{1}{\big|s\big|_*}\max_{2\leq k\leq s}\left\{\left(\ell^p + \ell^q\right), \Big((1+\ell)^p + (1+\ell)^q\Big), \Big(\big|k+k\ell-\ell\big|^p + \big|k+k\ell-\ell\big|^q\Big)\right\} < 1.$$

So, it's easily seen that (4.1) is fulfilled with

$$\mathcal{U} := \Big\{u_\ell \in \mathrm{Aut}(X) : \ell \in \mathbb{N}_{n_0}\Big\} \neq \phi.$$

In addition, we have

$$\begin{aligned} \lim_{\ell\to\infty} h\left(u'_\ell x, u_\ell y, z_1, \cdots, z_m\right) &\leq \lim_{\ell\to\infty}\big((1+\ell)^p + \ell^q\big)h\left(x, y, z_1, \cdots, z_m\right) \\ &= 0, \end{aligned}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ which means that (4.5) and (4.6) are valid. Therefore, by Theorem 4.1, every $f : X \to Y$ satisfying (5.4) is a solution of the functional equation (1.14) on X_0. □

Corollary 5.3. *Let $\theta, r \geq 0$, and $p < 0$. Assume that $f : X \to Y$ satisfies*

$$\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s f(kx+y) - s!f(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \leq \theta\left(\|x\|^p + \|y\|^p\right)\prod_{i=1}^{m}\|z_i\|^r, \tag{5.5}$$

for all $x, y, z_1, \cdots, z_m \in X_0$. Then f satisfies the monomial functional equation (1.14) on X_0.

Proof. By similar method in the proof of Corollary 5.2, it is easily seen that the function h defined by

$$h\left(x, y, z_1, \cdots, z_m\right) := \theta\left(\|x\|^p + \|y\|^p\right)\prod_{i=1}^{m}\|z_i\|^r, \quad x, y, z_1, \cdots, z_m \in X_0,$$

satisfies (4.5) and (4.6) because

$$\begin{aligned} h\left(u_\ell x, u_\ell y, z_1, \cdots, z_m\right) &= h\left(\ell x, \ell y, z_1, \cdots, z_m\right) \\ &= \theta\left(\|\ell x\|^p + \|\ell y\|^p\right)\prod_{i=1}^{m}\|z_i\|^r \\ &= \theta\ell^p\left(\|x\|^p + \|y\|^p\right)\prod_{i=1}^{m}\|z_i\|^r \\ &= \ell^p h\left(x, y, z_1, \cdots, z_m\right), \end{aligned}$$

$$h\left(u'_\ell x, u'_\ell y, z_1, \cdots, z_m\right) = (1+\ell)^p h\left(x, y, z_1, \cdots, z_m\right),$$

and

$$h\left((ku'_\ell + u_\ell)x, (ku'_\ell + u_\ell)y, z_1, \cdots, z_m\right) = \left|k + k\ell - \ell\right|^p h\left(x, y, z_1, \cdots, z_m\right)$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $\ell \in \mathbb{N}$. The remainder of the proof is similar to the proof of Corollary 5.2. $\square$

If $f : X \to Y$ satisfies (5.5) for $x, y, z_1, \cdots, z_m \in X_0$ with $p < 0$, then by Theorem 4 we know that f satisfies the monomial equation on X_0. It is not hard to show that if $f(0) = 0$, then f satisfies the monomial equation on the whole X. So we have the following corollary.

Corollary 5.4. *Let $\theta, r \geq 0$, and $p < 0$. Assume that $f : X \to Y$ satisfies $f(0) = 0$ and fulfills the inequality*

$$\left\|\sum_{k=0}^{s}(-1)^{s-k}C_k^s f(kx+y) - s!f(x), g\left(z_1\right), \cdots, g\left(z_m\right)\right\|_* \leq \theta\left(\|x\|^p + \|y\|^p\right)\prod_{i=1}^{m}\|z_i\|^r, \tag{5.6}$$

for all $x, y, z_1, \cdots, z_m \in X_0$. Then f satisfies the monomial functional equation (1.14) on the whole X.

Corollary 5.5. *Let $p, q \in \mathbb{R}$ such that $p+q<0$ and let $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx+y) - s! f(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \theta \|x\|^p \, \|y\|^q \prod_{i=1}^{m} \|z_i\|^r, \tag{5.7}$$

for all $x, y, z_1, \cdots, z_m \in X_0$. Then f satisfies the monomial functional equation (1.14) on X_0.

Proof. It is easily seen that the function h given by

$$h(x, y, z_1, \cdots, z_m) := \theta \|x\|^p \, \|y\|^q \prod_{i=1}^{m} \|z_i\|^r, \quad x, y, z_1, \cdots, z_m \in X_0$$

satisfies (4.5) and (4.6) because

$$\begin{aligned} h(u_\ell x, u_\ell y, z_1, \cdots, z_m) &= h(\ell x, \ell y, z_1, \cdots, z_m) \\ &= \theta \|\ell x\|^p \, \|\ell y\|^q \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \ell^{p+q} \|x\|^p \|y\|^q \prod_{i=1}^{m} \|z_i\|^r \\ &= \ell^{p+q} h(x, y, z_1, \cdots, z_m), \\ h(u'_\ell x, u'_\ell y, z_1, \cdots, z_m) &= \big(1+\ell\big)^{p+q} h(x, y, z_1, \cdots, z_m), \end{aligned}$$

and

$$h\Big((ku'_\ell + u_\ell)x, (ku'_\ell + u_\ell)y, z_1, \cdots, z_m\Big) = \big|k + k\ell - \ell\big|^{p+q} h(x, y, z_1, \cdots, z_m)$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $\ell \in \mathbb{N}$. The rest of the proof is similar to the proof of Corollary 5.2. □

By an analogous conclusion, the function h given by

$$h(x, y, z_1, \cdots, z_m) := \theta \Big(\|x\|^p + \|y\|^q + \|x\|^p \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r, \quad x, y, z_1, \cdots, z_m \in X_0,$$

where $p<0$ and $q<0$, satisfies (4.5) and (4.6) because

$$\begin{aligned} h(u_\ell x, u_\ell y, z_1, \cdots, z_m) &= h(\ell x, \ell y, z_1, \cdots, z_m) \\ &= \theta \Big(\|\ell x\|^p + \|\ell y\|^q + \|\ell x\|^p \|\ell y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \Big(\ell^p \|x\|^p + \ell^q \|y\|^p + \ell^{p+q} \|x\|^p \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &\leq \big(\ell^p + \ell^q + \ell^{p+q}\big) h(x, y, z_1, \cdots, z_m), \end{aligned}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $\ell \in \mathbb{N}$. So we have the following corollary.

Corollary 5.6. *Let $p < 0, q < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| \sum_{k=0}^{s} (-1)^{s-k} C_k^s f(kx+y) - s! f(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \theta \Big(\|x\|^p + \|y\|^q + \|x\|^p \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r, \quad (5.8)$$

for all $x, y, z_1, \cdots, z_m \in X_0$. Then f satisfies the monomial functional equation (1.14) on X_0.

References

[1] L. Aiemsomboon, W. Sintunavarat, *On generalized hyperstability of a general linear equation*, Acta. Math. Hungar., **149**(2016), no. 2, 413-422.

[2] L. Aiemsomboon, W. Sintunavarat, *On new approximations for generalized Cauchy functional equations using Brzdęk and Ciepliński's fixed point theorems in 2-Banach spaces*, Acta. Math. Sci., **40**(2020), no. 3, 824-834.

[3] M. Almahalebi, A. Chahbi, *Hyperstability of the Jensen functional equation in ultrametric spaces*, Aequat. Math., **91**(2017), no. 4, 647-661.

[4] M. Almahalebi, A. Chahbi, *Approximate solution of p-radical functional equation in 2-Banach spaces*, Acta. Math. Sci., **39**(2019), 551-566.

[5] M. Almahalebi, S. Kabbaj, G.H. Kim, *On the hyperstability of Jensen functional equation in 2-Banach spaces*, Fixed Point Theory, **20**(2019), no. 2, 417-430.

[6] M. Almahalebi, C. Park, *On the hyperstability of a functional equation in commutative groups*, J. Comput. Anal. Appl., **20**(2016), 826-833.

[7] A. Bahyrycz, J. Brzdęk, E.-S. El-Hady, Z. Leśniak, *On Ulam stability of functional equations in 2-normed spaces – A survey*, Symmetry, **13**(2021), no. 11, 2200.

[8] A. Bahyrycz, J. Olko, *On stability of the general linear equation*, Aequat. Math., **89**(2015), 1461-1474.

[9] A. Bahyrycz, J. Olko, *Hyperstability of general linear functional equation*, Aequat. Math., **90**(2016), 527-540.

[10] A. Bahyrycz, M. Piszczek, *Hyperstability of the Jensen functional equation*, Acta Math. Hungar., **142**(2014), 353-365.

[11] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J., **16**(1949), 385-397.

[12] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory and Applications, **2013**(2013), no. 1, 1-9.

[13] J. Brzdęk, *On Ulam's type stability of the Cauchy additive equation*, The Scientific World Journal, **2014**(2014), Article ID 540164.

[14] J. Brzdęk, J. Chudziak, Zs. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74**(2011), no. 17, 6728-6732.

[15] J. Brzdęk, K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces,* Nonlinear Anal., **74**(2011), no. 18, 6861-6867.

[16] J. Brzdęk, K. Ciepliński, *Hyperstability and superstability*, Abs. Appl. Anal., **2013**(2013), Article ID 401756, 13 pp.

[17] J. Brzdęk, K. Ciepliński, *Ulam stability in non-Archimedean spaces*, Journal of Mathematical Analysis and Applications, **400**(2013), 68-75.

[18] J. Brzdęk, K. Ciepliński, *On a fixed point theorem in 2-Banach spaces and some of its applications*, Acta. Math. Sci., **38**(2018), no. 2, 377-744.

[19] J. Brzdęk, K. Ciepliński, *A fixed point theorem in n-Banach spaces and Ulam stability*, Journal of Mathematical Analysis and Applications, **470**(2019), 632-646.

[20] J. Brzdęk, E.-S. El-Hady, *On hyperstability of the Cauchy functional equation in n-Banach Spaces*, Mathematics, **8**(2020), Art. no. 1886.

[21] L. Cădariu, V. Radu, *The fixed points method for the stability of some functional equations*, Carpathian J. Math., **23**(2007), 63-72.

[22] C.-K. Choi, *Stability of a monomial functional equation on restricted domains of Lebesgue measure zero*, Results Math., (2017), 1-11, doi:10.1007/s00025-017-0742-0.

[23] H.Y. Chu, S.H. Ku, *A Mazur-Ulam problem in non-Archimedean n-normed spaces*, J. Inequal. Appl., **34**(2013), 1-10.

[24] T. Diagana, F. Ramaroson, *Non-Archimedean Operator Theory*, Springer, 2016.

[25] E.-S. El-Hady, J. Brzdęk, *On Ulam stability of functional equations in 2-normed spaces – A survey II*, Symmetry, **14**(2022), no. 7, 1365.

[26] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr., **26**(1963), 115-148.

[27] S. Gähler, *Linear 2-normiete Räumen*, Math. Nachr., **28**(1964), 1-43.

[28] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[29] A. Gilànyi, *On the stability of monomial functional equations*, Publ. Math. Debrecen, **56**(2000), 201-212.

[30] K. Hensel, *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresber. Dtsch. Math.-Ver., **6**(1897), 83-88.

[31] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222-224.

[32] D.H. Hyers, *Transformations with bounded m-th differences*, Pacific J. Math., **11**(1961), 591-602.

[33] S.-M. Jung, *On the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **204**(1996), 221-226.

[34] Z. Kaiser, *On stability of the monomial functional equation in normed spaces over fields with valuation*, J. Math. Appl., **322**(2006), 1188-1198.

[35] Y.-H. Lee, *On the stability of the monomial functional equation*, Bull. Korean Math. Soc., **45**(2008), 397-403.

[36] Gy. Maksa, Zs. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedag. Nyíregyháziensis, **17**(2001), 107-112.

[37] A.K. Mirmostafaee, *Non-Archimedean stability of the monomial functional equations*, Tamsui Oxford Journal of Mathematical Sciences, **262**(2010), 221-235.

[38] A. Misiak, *n-inner product spaces*, Math. Nachr., **140**(1989), 299-319.

[39] P. Phochai, S. Saejung, *Hyperstability of generalised linear functional equations in several variables*, Bull. Austral. Math. Soc., **102**(2020), 293-302.

[40] M. Piszczek, J. Szczawińska, *Hyperstability of the Drygas functional equation*, Journal of Function Spaces and Applications, **2013**(2013), Article ID:912718.

[41] Th.M. Rassias, *On the stability of linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[42] Th.M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158**(1991), 106-113.

[43] Th.M. Rassias, P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl., **173**(1993), 325-338.

[44] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960. Reprinted as: Problems in Modern Mathematics, New York, 1964.

[45] D. Zhang, *On hyperstability of generalised linear functional equations in several variables*, Bull. Aust. Math. Soc., **92**(2015), 259-267.

Received: March 10, 2021; Accepted: September 6, 2022.