Fixed Point Theory, 20(2019), No. 2, 649-662
DOI: 10.24193/fpt-ro.2019.2.43
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

COMMON FIXED POINT THEOREMS IN C*-ALGEBRA-VALUED B-METRIC SPACES WITH APPLICATIONS TO INTEGRAL EQUATIONS

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Abstract. Based on the concepts of C*-algebra-valued b-metric space, we give some common fixed point results in C*-algebra-valued b-metric space. As an application, existence and uniqueness result for one type of integral equations is also discussed.

Key Words and Phrases: Fixed point, b-metric space, C*-algebra, common fixed point, compatible, weakly compatible.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The Banach Contraction Principle (BCP) which is one of the most important results of analysis was introduced by the Polish mathematician Stefan Banach in 1922. It is the main source of metric fixed point theory and the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. The BCP was used to study the existence of solutions for integral equations and differential equations. Therefore, because of its usefulness and simplicity, it has become a very popular tool in solving existence problems in many branches of mathematical analysis and scientific applications, and it has been generalized in many different branches.

One of the branches of this theory is related to the study of common fixed points. In 1966, Jungck [19] introduced common fixed points for commuting mappings in metric spaces. The concept of commuting mappings has been weakened in several ways over the years. One such notion which is the concept of compatibility introduced by Jungck [20]. Since then, several authors have investigated coincidence and common fixed point results for mappings and generalizations of this concept in different types

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of spaces, see [1, 5, 8, 18, 21, 28] For example, cone metric spaces [16], fuzzy metric spaces [2], uniform spaces [30], non commutative Banach spaces [32], and etc.

In 1993, Czerwik [10] introduced another axiom for semi metric spaces as a generalization of metric space, which is weaker than the triangle inequality. Subsequently, several papers have dealt with fixed point theory in such spaces [3, 5, 7, 8, 13, 15, 16, 18, 19, 20, 21]. Moreover, Fagin and Stockmeyer [14] discussed about the same relaxation of the triangle inequality and called this new distance measure nonlinear elastic matching (NEM). They remark that this measure has been used, for example, in [9] for trademark shapes and in [26] to measure ice floes. Since then, Xia [31] used this semi metric distance to study the optimal transport path between probability measures. Xia has chosen to call these spaces b-metric space (or quasi metric space). For details of b-metric space, see [22], and references therein.

In 2007, Huang and Zhang [16] firstly introduced cone metric spaces as a generalization of metric spaces, and proved some fixed point theorems for contractive mappings. The existence of a common fixed point on cone metric spaces was investigated recently in [1, 7, 18, 21].

In [25], the authors introduced the concept of C*-algebra-valued metric spaces. The main idea consists of using the set of all positive elements of a unital C*-algebra instead of the set of real numbers. Obviously such spaces generalize the concept of metric spaces.

In [24], authors, based on the concept of operator-valued metric spaces, introduced the definitions of operator valued contraction map and expansive and proved the corresponding fixed point theorems.

In [23], Ma and Jiang, based on the concept and properties of C*-algebras, introduced a concept of C*-algebra-valued b-metric spaces which generalizes the concept of C*-algebra-valued metric spaces and gives some basic fixed point theorems for self-map with contractive condition on such spaces.

In 2015, Xin, Jiang and Ma [33] studied common fixed points in the frame of C*-algebra-valued metric spaces. They proved some common fixed point theorems for two mappings under the different contractive conditions. Authors furnished suitable examples to demonstrate the validity of the hypotheses of their results. They presented theorems and improved some recent results given in [25].

In this paper, we study common fixed points in the frame of C*-algebra-valued b-metric spaces. More precisely, we prove some common fixed point theorems for two mappings under the different contractive conditions. The paper is organized as follows: Based on the concept and properties of C*-algebras, the paper presents some common fixed point theorems in C*-algebra-valued b-metric spaces. Finally, as an application, existence and uniqueness result for one type of integral equation is given.

2. Basic definitions

To begin with, we recall some basic definitions, notations, and facts on the theory of C*-algebras, which will be needed in the sequel.

Throughout this paper, suppose that $\mathcal{A}$ is an unital C*-algebra with the unit I. Set $\mathcal{A}_h = \{a \in \mathcal{A} \ : \ a = a^*\}$. We call an element $a \in \mathcal{A}$ a positive element, denoted it by $a \geq 0_{\mathcal{A}}$ if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is the zero element in $\mathcal{A}$ and $\sigma(a)$

is the spectrum of a.
There is a natural partial ordering on $\mathcal{A}_h$ given by $a \leq b$ if and only if $b - a \geq 0_{\mathcal{A}}$. From now on, $\mathcal{A}_+$ and $\mathcal{A}'$ will denote the set $\{a \in \mathcal{A} \ : \ a \geq 0_{\mathcal{A}}\}$ and the set $\{a \in \mathcal{A} \ : ab = ba\,, \ \forall b \in \mathcal{A}\}$, respectively.

Definition 2.1. Let $\mathcal{X}$ be a nonempty set. Suppose that the mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is defined, with the following properties:

(1) $d(x, y) \geq 0_{\mathcal{A}}$ for all x and y in $\mathcal{X}$;
(2) $d(x, y) = 0_{\mathcal{A}}$ if and only if $x = y$;
(3) $d(x, y) = d(y, x)$ for all x and y in $\mathcal{X}$;
(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y and z in $\mathcal{X}$.

Then d is said to be a C*-algebra-valued metric on $\mathcal{X}$, and $(\mathcal{X}, \mathcal{A}, d)$ is said to be a C*-algebra-valued metric space.

Definition 2.2. Let $\mathcal{X}$ be a nonempty set, and $\mathcal{A}$ be a C*-algebra. Let $b \in \mathcal{A}'$ be such that $||b|| \geq 1$. A mapping $d_b : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is said to be a C*-algebra-valued b-metric on $\mathcal{X}$ if the following conditions hold for all $x, y, z \in \mathcal{A}$:

(1) $d_b(x, y) \geq 0_{\mathcal{A}}$ for all x and y in $\mathcal{X}$ and $d_b(x, y) = 0 \Leftrightarrow x = y$
(2) $d_b(x, y) = d_b(y, x)$
(3) $d_b(x, y) \leq b[d_b(x, z) + d_b(z, y)]$

The triplet $(\mathcal{X}, \mathcal{A}, d_b)$ is called a C*-algebra-valued b-metric space with coefficient b.

The following technical lemmas will be useful later in this paper.

Lemma 2.3. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a C*-algebra-valued b-metric space:

(1) If $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ and $\lim\limits_{n\to\infty} x_n = 0_{\mathcal{A}}$, then for any $x \in \mathcal{A}$, $\lim\limits_{n\to\infty} x^* x_n x = 0_{\mathcal{A}}$.
(2) If $x, y \in \mathcal{A}_h$ and $z \in \mathcal{A}'_+$, then $x \leq y$ deduces $zx \leq zy$, where $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.
(3) Limit of a convergent sequence in a C*-algebra-valued b-metric space is unique, i.e., if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{X}$ and converges to x and y, respectively, then $x = y$.

Proof.

(1) By taking the relation $||x^* x_n x - 0_{\mathcal{A}}|| \leq ||x||^2 ||x_n||$, we immediately get the result.
(2) It is well known that $x \leq y$ implies $y - x \in \mathcal{A}_+$, and then there is $d \in \mathcal{A}_+$ such that $y - x = d^2$. Again, $z \in \mathcal{A}'_+$, then $z = e^2$ for some $e \in \mathcal{A}_+$. Note that
$$zy - zx = z(y - x) = e^2 d^2 = eded = (ed)^* ed \in \mathcal{A}_+,$$
which shows $zx \leq zy$.
(3) By taking the triangle inequality and $c \in \mathcal{A}'$, such that $c \geq 1$, we get
$$d_b(x, y) \leq c(d_b(x_n, x) + d_b(x_n, y)),$$
which, together with $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$, implies that $d_b(x, y) \longrightarrow 0_{\mathcal{A}}(n \to \infty)$. Hence $d_b(x, y) = 0_{\mathcal{A}}$, which implies that $x = y$.

Lemma 2.4. ([12, 27]) Let $\mathcal{A}$ be a unital C*-algebra with unit I

(1) For any $x \in \mathcal{A}_+$, we have $x \leq I \Leftrightarrow ||x|| \leq 1$.
(2) If $a \in \mathcal{A}_+$ with $||a|| < \frac{1}{2}$, then $I - a$ is invertible and $||a(I - a)^{-1}|| < 1$.
(3) Suppose that $a, b \in \mathcal{A}$ with $a, b \geq 0$ and $ab = ba$, then $ab \geq 0$.

(4) Let $a \in \mathcal{A}'$, if $b, c \in \mathcal{A}$ with $b \geq c \geq 0$, and $I - a \in \mathcal{A}'_+$ is an invertible operator, then
$$(I - a)^{-1} b \geq (I - a)^{-1} c.$$

The concepts of compatible and weakly compatible were introduced in b-metric space by some authors. For more details one can see [34, 5, 29].

Now, we introduce these concepts in C*-algebra-valued b-metric spaces.

Definition 2.5. Let T and S be two self-mappings of the set $\mathcal{X}$.

(1) If $x = Tx = Sx$ for some $x \in \mathcal{X}$, then x is called a common fixed point of T and S.
(2) If $z = Tx = Sx$ for some $z \in \mathcal{X}$, then x is called a coincidence point of T and S, and z is called a point of coincidence of T and S.
(3) If T and S commute at all of their coincidence point, i.e., $TSx = STx$ for all $x \in \{x \in \mathcal{X} : \ Tx = Sx\}$, then T and S are called weakly compatible.

Definition 2.6. The two mappings T and S on a C*-algebra-valued b-metric space $(\mathcal{X}, \mathcal{A}, d_b)$ is said to be compatible, if for arbitrary sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{X}$, such that $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Sx_n = t \in \mathcal{X}$, then $d_b(TSx_n, STx_n) \longrightarrow 0_{\mathcal{A}}\ (n \to \infty)$.

Example 2.7. ([23]) Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{A} = M_n(\mathbb{R})$. Define
$$d(x, y) = diag(c_1|x - y|^p, c_2|x - y|^p, \ldots, c_n|x - y|^p),$$
which "diag" denotes a diagonal matrix, and $x, y \in \mathbb{R}$, $c_i \geq 0$ $(i = 1, 2, \ldots, n)$ are constants and $p > 1$. It is easy to verify that $d(., .)$ is a complete C*-algebra-valued b-metric, for proving (3) of 2.2 we only need to use the following inequality:
$$|x - y|^p \leq 2^p(|x - z|^p + |z - y|^p),$$
which implies that $d(x, y) \leq A[d(x, z) + d(z, y)]$ for all $x, y, z \in \mathcal{X}$, where $A = 2^p I \in \mathcal{A}'$ and $A > I$ by $2^p > 1$. But $|x - y|^p \leq |x - z|^p + |z - y|^p$ is impossible for all $x > z > y$. Thus $(\mathcal{X}, M_n(\mathbb{R}), d)$ is not a C*-algebra-valued metric space.

Lemma 2.8. If the mappings T and S on the C*-algebra-valued b-metric space $(\mathcal{X}, \mathcal{A}, d_b)$ are compatible, then they are weakly compatible.

Proof. The proof is the same as [33].

In the following example, we show that the converse of the above lemma dose not hold.

Example 2.9. Let $\mathcal{X} = [0, 6]$ and $\mathcal{A} = M_2(\mathbb{C})$. Define $d_b : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ by
$$d_b(x, y) = \begin{bmatrix} c_1|x - y|^p & 0 \\ 0 & c_2|x - y|^p \end{bmatrix},$$
where $p > 1$ and $c_1, c_2 \geq 0$ are constant. Then $(\mathcal{X}, \mathcal{A}, d_b)$ is a C*-algebra-valued b-metric space. Set
$$Tx = \begin{cases} 5 - x & ; \quad x \in [0, \frac{5}{2}], \\ \quad 5 & ; \quad x \in (\frac{5}{2}, 6], \end{cases} \quad \text{and } Sx = \begin{cases} 4x & ; \quad x \in (1, 4], \\ \ x & ; \quad x \in [0, 1] \cup (4, 6]. \end{cases}$$

It is easy to compute that the set of their coincidence points is singleton set $\{5\}$, and it is clear that T and S commute at this point. Therefore, T and S are weakly compatible.

Now, we show that they are not compatible. In order to do this, suppose that $\{x_n\}$ is a sequence in $\mathcal{X}$ suth that $x_n = 1 + \frac{1}{n} \in \mathcal{X}$ for $n \in \mathbb{N}$ with $n \geq 2$. We get

$$Tx_n = 5 - (1 + \frac{1}{n}) = 4 - \frac{1}{n} \text{ and } Sx_n = 4(1 + \frac{1}{n}) = 4 + \frac{4}{n}.$$

Then $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Sx_n = 4$. In fact, we have

$$d_b(Tx_n, 4) = d_b(4 - \frac{1}{n}, 4) = \begin{bmatrix} c_1|\frac{1}{n}|^p & 0 \\ 0 & c_2|\frac{1}{n}|^p \end{bmatrix} \longrightarrow 0 \ (n \to \infty).$$

$$d_b(Sx_n, 4) = d_b(4 + \frac{4}{n}, 4) = \begin{bmatrix} c_1|\frac{4}{n}|^p & 0 \\ 0 & c_2|\frac{4}{n}|^p \end{bmatrix} \longrightarrow 0 \ (n \to \infty).$$

But

$$\begin{aligned} d_b(TSx_n, STx_n) &= d_b(T(4 + \frac{4}{n}), S(4 - \frac{1}{n})) = d_b(5, 16 - \frac{4}{n}) \\ &= \begin{bmatrix} c_1|11 - \frac{4}{n}|^p & 0 \\ 0 & c_2|11 - \frac{4}{n}|^p \end{bmatrix} \longrightarrow \begin{bmatrix} 11^p c_1 & 0 \\ 0 & 11^p c_2 \end{bmatrix}, \end{aligned}$$

which yields $d_b(TSx_n, STx_n) \nrightarrow 0$.

Lemma 2.10. ([1]) Let T and S be weakly compatible mappings of a set $\mathcal{X}$. If T and S have a unique point of coincidence, then it is the unique common fixed point of T and S.

3. Main results

By using the above results, we are now ready to prove the main theorem of this paper.

Theorem 3.1. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a compatible C*-algebra-valued b-metric space and let $T, S : \mathcal{X} \to \mathcal{X}$ be two self-mappings satisfy

$$d_b(Tx, Sy) \leq a^* d_b(x, y) a \text{ for any } x, y \in \mathcal{X}, \tag{3.1}$$

where $a \in \mathcal{A}$ with $||a|| < 1$. Then T and S have a unique fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ and $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{X}$ such that: $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$. From 3.1, we have

$$\begin{aligned} d_b(x_{2n+2}, x_{2n+1}) &= d_b(Sx_{2n+1}, Tx_{2n}) \\ &\leq a^* d_b(x_{2n+1}, x_{2n}) a \\ &\leq (a^*)^2 d_b(x_{2n}, x_{2n-1}) a^2 \\ &\vdots \\ &\leq (a^*)^{2n+1} d_b(x_1, x_0) a^{2n+1}, \end{aligned}$$

where we use the property: if $b, c \in \mathcal{A}_h$, then $b \leq c$ implies $a^*ba \leq a^*ca$. Similarly

$$\begin{aligned} d_b(x_{2n+1}, x_{2n}) &= d_b(Sx_{2n}, Tx_{2n-1}) \\ &\leq a^* d_b(x_{2n}, x_{2n-1})a \\ &\vdots \\ &\leq (a^*)^{2n} d_b(x_1, x_0)a^{2n}, \end{aligned}$$

In fact, we can easily obtain for any $n \in \mathbb{N}$,

$$d_b(x_{n+1}, x_n) \leq (a^*)^n d_b(x_1, x_0)a^n.$$

Now, by using the triangle inequality for any $p \in \mathbb{N}$, we have

$$\begin{aligned} d_b(x_{n+p}, x_n) &\leq b[d_b(x_{n+p}, x_{n+p-1}) + d_b(x_{n+p-1}, x_n)] \\ &\leq bd_b(x_{n+p}, x_{n+p-1}) + b^2[d_b(x_{n+p-1}, x_{n+p-2}) + d_b(x_{n+p-2}, x_n)] \\ &\leq bd_b(x_{n+p}, x_{n+p-1}) + b^2 d_b(x_{n+p-1}, x_{n+p-2}) + \dots \\ &+ b^{p-1} d_b(x_{n+2}, x_{n+1}) + b^{p-1} d_b(x_{n+1}, x_n) \\ &\leq b(a^*)^{n+p-1} d_b(x_1, x_0)a^{n+p-1} + b^2(a^*)^{n+p-2} d_b(x_1, x_0)a^{n+p-2} + \dots \\ &+ b^{p-1}(a^*)^{n+1} d_b(x_1, x_0)a^{n+1} + b^{p-1}(a^*)^n d_b(x_1, x_0)a^n \\ &= b(a^*)^{n+p-1} B_0 a^{n+p-1} + b^2(a^*)^{n+p-2} B_0 a^{n+p-2} + \dots \\ &+ b^{p-1}(a^*)^{n+1} B_0 a^{n+1} + b^{p-1}(a^*)^n B_0 a^n \\ &= \sum_{k=1}^{p-1} b^k (a^*)^{n+p-k} B_0 a^{n+p-k} + b^{p-1}(a^*)^n B_0 a^n \\ &= \sum_{k=1}^{p-1} ((a^*)^{n+p-k} b^{\frac{k}{2}} B_0^{\frac{1}{2}})(B_0^{\frac{1}{2}} b^{\frac{k}{2}} a^{n+p-k}) + ((a^*)^n b^{\frac{p-1}{2}} B_0^{\frac{1}{2}})(B_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^n) \\ &= \sum_{k=1}^{p-1} (B_0^{\frac{1}{2}} b^{\frac{k}{2}} a^{n+p-k})^* (B_0^{\frac{1}{2}} b^{\frac{k}{2}} a^{n+p-k}) + (B_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^n)^* (B_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^n) \\ &\leq \sum_{k=1}^{p-1} ||B_0^{\frac{1}{2}} b^{\frac{k}{2}} a^{n+p-k}||^2 1_{\mathcal{A}} + ||B_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^n||^2 1_{\mathcal{A}} \\ &\leq ||B_0^{\frac{1}{2}}||^2 \sum_{k=1}^{p-1} ||a||^{2(n+p-k)} ||b||^k 1_{\mathcal{A}} + ||B_0^{\frac{1}{2}}||^2 ||b^{\frac{p-1}{2}}||^2 ||a^n||^2 1_{\mathcal{A}} \\ &\leq ||B_0|| \frac{||b||^p ||a||^{2(n+1)}}{||b|| - ||a||^2} 1_{\mathcal{A}} + ||B_0|| ||b^{\frac{p-1}{2}}||^2 ||a||^{2n} 1_{\mathcal{A}} \\ &\longrightarrow 0 \ (n \to \infty), \end{aligned}$$

where $1_{\mathcal{A}}$ is the unit element in $\mathcal{A}$ and $d_b(x_1, x_0) = B_0$ for some $B_0 \in \mathcal{A}_+$, this can be done since $d_b(x_1, x_0) \in \mathcal{A}_+$.

By the definition of cauchy sequence, we get that $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in $\mathcal{X}$ and from the completeness of $\mathcal{X}$ it follows that there exists $x \in \mathcal{X}$ such that $\lim_{n\to\infty} x_n = x$.

In fact from the triangle inequality and 3.1, we have

$$\begin{aligned} d_b(x, Sx) &\le b[d_b(x, x_{2n+1}) + d_b(x_{2n+1}, Sx)] \\ &\le b[d_b(x, x_{2n+1}) + d_b(Tx_{2n}, Sx)] \\ &\le b[d_b(x, x_{2n+1}) + a^* d_b(x_{2n}, x)a]. \end{aligned}$$

Taking $n \to \infty$, the right hand side of the above inequality approaches $0_{\mathcal{A}}$, and then $Sx = x$. Again, nothing that

$$0_{\mathcal{A}} \le d_b(Tx, x) = d_b(Tx, Sx) \le a^* d_b(x, x)a = 0_{\mathcal{A}}.$$

That is $d_b(Tx, x) = 0_{\mathcal{A}}$, which means $Tx = x$.
To prove the uniqueness of common fixed point in $\mathcal{X}$, assume that there is another point $y \in \mathcal{X}$ such that $Ty = Sy = y$. From 3.1, we have

$$d_b(x, y) = d_b(Tx, Sy) \le a^* d_b(x, y)a,$$

The above inequality with $||a|| < 1$ yields that

$$0 \le ||d_b(x, y)|| \le ||a||^2 ||d_b(x, y)|| < ||d_b(x, y)||.$$

The above inequality holds only when $||d_b(x, y)|| = 0$ and $d_b(x, y) = 0_{\mathcal{A}}$, which gives $y = x$. Hence, T and S have a unique common fixed point in $\mathcal{X}$.

An easy consequence of Theorem 3.1 is the following result.

Corollary 3.2. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let $T, S : \mathcal{X} \to \mathcal{X}$ be two mappings such that,

$$||d_b(Tx, Sy)|| \le ||a|| ||d_b(x, y)|| \text{ for any } x, y \in \mathcal{X},$$

where $a \in \mathcal{A}$ with $||a|| < 1$. Then T and S have a unique common fixed point in $\mathcal{X}$.

Corollary 3.3. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let the mapping $T : \mathcal{X} \to \mathcal{X}$ satisfies

$$d_b(T^m x, T^n y) \le a^* d_b(x, y)a, \quad \text{for any } x, y \in \mathcal{X},$$

where $a \in \mathcal{A}$ with $||a|| < 1$, and m and n are fixed positive integers. Then T has a unique fixed point in $\mathcal{X}$.

Proof. Set $T = T^m$ and $S = T^n$ in 3.1, then the result follows from Theorem 3.1.

Remark 3.4. Note that in Theorem 3.1, if we take $S = T$, then 3.1 becomes:

$$d_b(Tx, Ty) \le a^* d_b(x, y)a, \text{ for any } x, y \in \mathcal{X},$$

where $a \in \mathcal{A}$ with $||a|| < 1$. Thus, we have the following corollary, for details one can see [25, Theorem 2.1].

Corollary 3.5. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space with coefficient b and let the mapping $T : \mathcal{X} \to \mathcal{X}$ satisfies

$$d_b(Tx, Ty) \le a^* d_b(x, y)a,$$

where $a \in \mathcal{A}$ with $||a|| < 1$, then T has a unique fixed point in $\mathcal{X}$.

Now, we have the following interesting theorem.

Theorem 3.6. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let $T, S : \mathcal{X} \to \mathcal{X}$ be two self-mappings such that

$$d_b(Tx, Ty) \leq a^* d_b(Sx, Sy)a \text{ for any } x, y \in \mathcal{X}, \tag{3.2}$$

where $a \in \mathcal{A}$ with $||a|| < 1$. If $R(T)$ is contained in $R(S)$, and $R(S)$ is complete in $\mathcal{X}$, then T and S have a unique point of coincidence in $\mathcal{X}$.
Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in $\mathcal{X}$.
Proof. Let $x_0 \in \mathcal{X}$ be an arbitrary point, choose $x_1 \in \mathcal{X}$ such that $Sx_1 = Tx_0$, this can be done since $R(T) \subseteq R(S)$. Let $x_2 \in \mathcal{X}$ such that $Sx_2 = Tx_1$. Continuing this process, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ satisfying $Sx_n = Tx_{n-1}$. Then from 3.2 we have

$$\begin{aligned} d_b(Sx_{n+1}, Sx_n) &= d_b(Tx_n, Tx_{n-1}) \\ &\leq a^* d_b(Sx_n, Sx_{n-1})a \\ &\vdots \\ &\leq (a^*)^n d_b(Sx_1, Sx_0)a^n, \end{aligned}$$

which shows that $\{Sx_n\}_{n=1}^{\infty}$ is a cauchy sequence in $R(S)$. By completion of $R(S)$ in $\mathcal{X}$, there exists $q \in \mathcal{X}$ such that $\lim_{n\to\infty} Sx_n = Sq$. Thus

$$d_b(Sx_n, Tq) = d_b(Tx_{n-1}, Tq) \leq a^* d_b(Sx_{n-1}, Sq)a.$$

From $\lim_{n\to\infty} Sx_n = Sq$ and Lemma 2.3 (1), we get $a^* d_b(Sx_{n-1}, Sq)a \to 0$ as $n \to \infty$, and then $\lim_{n\to\infty} Sx_n = Tq$. It follows from Lemma 2.3 (3) that $Tq = Sq$. If there is a point w in $\mathcal{X}$ such that $Tw = Sw$, 3.2 implies

$$d_b(Sq, Sw) = d_b(Tq, Tw) \leq a^* d_b(Sq, Sw)a.$$

Following an argument similar to that given in Theorem 3.1 we obtain $Sq = Sw$, Because

$$\begin{aligned} 0 &\leq ||d_b(Sq, Sw)|| \leq ||a||^2 ||d_b(Sq, Sw)|| \\ &\Rightarrow ||d_b(Sq, Sw)|| = 0 \Rightarrow d_b(Sq, Sw) = 0 \Rightarrow Sq = Sw. \end{aligned}$$

Hence, T and S have a unique point of coincidence in $\mathcal{X}$. From Lemma 2.10, T and S have a unique common fixed point in $\mathcal{X}$.
Theorem 3.7. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let two mappings $T, S : \mathcal{X} \to \mathcal{X}$ satisfy

$$d_b(Tx, Ty) \leq a d_b(Tx, Sx) + a d_b(Ty, Sy), \quad \text{for any } x, y \in \mathcal{X}. \tag{3.3}$$

where $a \in \mathcal{A}'_+$ with $||a|| < \frac{1}{2}$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in $\mathcal{X}$, then T and S have a unique point of coincidence in $\mathcal{X}$.
Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in $\mathcal{X}$.

Proof. Similar to Theorem 3.6, choose $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ and set $Sx_n = Tx_{n-1}$. Then from 3.3,

$$\begin{aligned} d_b(Sx_{n+1}, Sx_n) &= d_b(Tx_n, Tx_{n-1}) \\ &\preceq ad_b(Tx_n, Sx_n) + ad_b(Tx_{n-1}, Sx_{n-1}) \\ &= ad_b(Sx_{n+1}, Sx_n) + ad_b(Sx_n, Sx_{n-1}). \end{aligned}$$

Thus,

$$(1-a)d_b(Sx_{n+1}, Sx_n) \preceq ad_b(Sx_n, Sx_{n-1}).$$

Since $||a|| < \frac{1}{2}$, then $1-a$ is invertible, and furthermore $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$, which together with $a \in \mathcal{A}'_+$, implies $(1-a)^{-1} \in \mathcal{A}'_+$. By Lemma 2.3 (2), we have

$$d_b(Sx_{n+1}, Sx_n) \preceq bd_b(Sx_n, Sx_{n-1}), \tag{3.4}$$

where $b = (1-a)^{-1}a \in \mathcal{A}'_+$ with $||b|| < 1$. Now, by induction and Lemma 2.3 (2), we get

$$d_b(Sx_{n+1}, Sx_n) \preceq b^n d_b(Sx_1, Sx_0).$$

For any $m \geq 1$, $p \geq 1$, and $c \in \mathcal{A}'$ where $||c|| > 1$,

$$\begin{aligned} d_b(Sx_{m+p}, Sx_m) &\preceq c[d_b(Sx_{m+p}, Sx_{m+p-1}) + d_b(Sx_{m+p-1}, Sx_m)] \\ &= cd_b(Sx_{m+p}, Sx_{m+p-1}) + cd_b(Sx_{m+p-1}, Sx_m) \\ &\preceq cd_b(Sx_{m+p}, Sx_{m+p-1}) + c^2[d_b(Sx_{m+p-1}, Sx_{m+p-2}) \\ &+ d_b(Sx_{m+p-2}, Sx_m)] \\ &\preceq cd_b(Sx_{m+p}, Sx_{m+p-1}) + c^2 d_b(Sx_{m+p-1}, Sx_{m+p-2}) \\ &+ c^3 d_b(Sx_{m+p-2}, Sx_{m+p-3}) + \dots \\ &+ c^{p-1} d_b(Sx_{m+2}, Sx_{m+1}) + c^{p-1} d_b(Sx_{m+1}, Sx_m) \\ &\preceq cb^{m+p-1} d_b(Sx_1, Sx_0) + c^2 b^{m+p-2} d_b(Sx_1, Sx_0) \\ &+ c^3 b^{m+p-3} d_b(Sx_1, Sx_0) + \dots + c^{p-1} b^{m+1} d_b(Sx_1, Sx_0) \\ &+ c^{p-1} b^m d_b(Sx_1, Sx_0) \\ &= cb^{m+p-1} B_0 + c^2 b^{m+p-2} B_0 \\ &+ c^3 b^{m+p-3} B_0 + \dots + c^{p-1} b^{m+1} B_0 + c^{p-1} b^m B_0 \\ &= \sum_{k=1}^{p-1} c^k b^{m+p-k} B_0 + c^{p-1} b^m B_0 \\ &= \sum_{k=1}^{p-1} |B_0^{\frac{1}{2}} b^{\frac{m+p-k}{2}} c^{\frac{k}{2}}|^2 + |B_0^{\frac{1}{2}} c^{\frac{p-1}{2}} b^{\frac{m}{2}}|^2 \end{aligned}$$

$$\leq ||B_0|| \sum_{k=1}^{p-1} ||c||^k ||b||^{m+p-k} 1_{\mathcal{A}} + ||c||^{p-1} ||b||^m ||B_0|| 1_{\mathcal{A}}$$
$$\leq ||B_0|| \frac{||c||^p ||b||^{m+1}}{||c|| - ||b||} 1_{\mathcal{A}} + ||c||^{p-1} ||b||^m ||B_0|| 1_{\mathcal{A}}$$
$$\longrightarrow 0 \ (m \to \infty),$$

where $B_0 = d_b(Sx_1, Sx_0)$. Hence $\{Sx_n\}_{n=0}^{\infty}$ is a cauchy sequence in $R(S)$. By the completeness of $R(S)$, there exists $q \in \mathcal{X}$ such that $\lim_{n\to\infty} Sx_n = Sq$.

Again by 3.4, we have

$$d_b(Sx_n, Tq) = d_b(Tx_{n-1}, Tq) \leq b d_b(Sx_{n-1}, Sq).$$

This implies that $\lim_{n\to\infty} Sx_n = Tq$. By Lemma 2.3 (3), the uniqueness of a limit in C*-algebra-valued b-metric space tells us that $Tq = Sq$.

Therefore T and S have a point of coincidence in $\mathcal{X}$. Now, we will show the uniqueness of points of coincidence. To do this, suppose that there is $p \in \mathcal{X}$ such that $Tp = Sp$. Using 3.3, we have

$$d_b(Sp, Sq) = d_b(Tp, Tq) \leq a d_b(Tp, Sp) + a d_b(Tq, Sq).$$

This implies that $||d_b(Sp, Sq)|| = 0$, and then $Sp = Sq$. Hence by Lemma 2.10, T and S have a unique common fixed point in $\mathcal{X}$.

Theorem 3.8. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let $T, S : \mathcal{X} \to \mathcal{X}$ be two mappings satisfy

$$d_b(Tx, Ty) \leq a d_b(Tx, Sy) + a d_b(Sx, Ty) \ \text{ for any } \ x, y \in \mathcal{X}, \tag{3.5}$$

where $a \in \mathcal{A}'_+$ with $||ab|| < \frac{1}{2}$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in $\mathcal{X}$, then T and S have a unique point of coincidence in $\mathcal{X}$.
Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in $\mathcal{X}$.

Proof. Similar to Theorem 3.6, choose $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ and set $Sx_n = Tx_{n-1}$. Then from 3.5

$$\begin{aligned} d_b(Sx_{n+1}, Sx_n) &= d_b(Tx_n, Tx_{n-1}) \\ &\leq a d_b(Tx_n, Sx_{n-1}) + a d_b(Sx_n, Tx_{n-1}) \\ &= a d_b(Sx_{n+1}, Sx_{n-1}) + a d_b(Sx_n, Sx_n) \\ &\leq ab d_b(Sx_{n+1}, Sx_n) + ab d_b(Sx_n, Sx_{n-1}). \end{aligned}$$

Thus,

$$(1 - ab) d_b(Sx_{n+1}, Sx_n) \leq ab d_b(Sx_n, Sx_{n-1}).$$

So

$$d_b(Sx_{n+1}, Sx_n) \leq (1 - ab)^{-1} ab d_b(Sx_n, Sx_{n-1}),$$

and consequently

$$d_b(Sx_{n+1}, Sx_n) \leq t d_b(Sx_n, Sx_{n-1}),$$

where $t = (1 - ab)^{-1} ab \in \mathcal{A}'_+$ with $||t|| < 1$.

Following an argument similar to that given in Theorem 3.7, we obtain T and S have a point of coincidence Tq in $\mathcal{X}$. In the following we will show the uniqueness of points of coincidence. To do this, suppose that there is $p \in \mathcal{X}$ such that $Tp = Sp$. Using 3.5, we have

$$d_b(Sp, Sq) = d_b(Tp, Tq) \leq ad_b(Tp, Sq) + ad_b(Sp, Tq) = ad_b(Sp, Sq) + ad_b(Sp, Sq),$$

i.e.,

$$d_b(Sp, Sq) \leq (I - a)^{-1} ad_b(Sp, Sq).$$

Since $||(I - a)^{-1}a|| < 1$, implies that $||d_b(Sp, Sq)|| = 0$, and then $Sp = Sq$. Hence by Lemma 2.10, T and S have a unique common fixed point in $\mathcal{X}$.

In Theorem 3.8, by choosing $S = id_{\mathcal{X}}$, we have $R(S) = \mathcal{X}$, and T is weakly compatible with S. Furthermore, we have the following consequence. For more details see [23, Theorem 2.2].

Corollary 3.9. Let $(\mathcal{X}, \mathcal{A}, d_b)$ be a complete C*-algebra-valued b-metric space and let the mapping $T : \mathcal{X} \to \mathcal{X}$ satisfy

$$d_b(Tx, Ty) \leq ad_b(Tx, y) + ad_b(Ty, x) \quad \text{for any } \; x, y \in \mathcal{X},$$

where $a \in \mathcal{A}'_+$ with $||ba|| < \frac{1}{2}$, then T have a unique fixed point in $\mathcal{X}$.

4. Application

Fixed point theorems for operators in b-metric spaces are widely used and have found various applications in differential and integral equations. As an application, let us consider the following system of integral equations

$$x(t) = \int_E K_1(t, s, x(s))ds + g(t) \quad t \in E, \tag{4.1}$$
$$x(t) = \int_E K_2(t, s, x(s))ds + g(t) \quad t \in E,$$

where E is a Lebesgue measurable set and $m(E) < \infty$.

Theorem 4.1. Suppose that the following conditions hold

(1) $K_1 : E \times E \times \mathbb{R} \to \mathbb{R}$, $K_2 : E \times E \times \mathbb{R} \to \mathbb{R}$ are integrable, and $g \in L^\infty(E)$;

(2) there exist $k \in (0, 1)$ and a continious function $\varphi : E \times E \to \mathbb{R}^+$ such that

$$|K_1(t, s, u) - K_2(t, s, v)| \leq k|\varphi(t, s)||u - v|,$$

for $t, s \in E$ and $u, v \in \mathbb{R}$.

(3) $\sup_{t \in E} \int_E |\varphi(t, s)|ds \leq 1$.

Then the integral equations 4.1 have a unique common solution in $L^\infty(E)$.

Proof. Let $\mathcal{X} = L^\infty(E)$ be the set of essentially bounded measurable functions on E and $B(L^2(E))$ be the set of bounded linear operators on a Hilbert space $L^2(E)$. Suppose $d : \mathcal{X} \times \mathcal{X} \to B(L^2(E))$ defined by $d(f, g) = \pi_{|f-g|^p}$ where $\pi_h : L^2(E) \to L^2(E)$ is the multiplication operator defined by

$$\pi_h(\psi) = h.\psi \quad ; \quad \psi \in L^2(E).$$

Hence $(\mathcal{X}, B(L^2(E)), d)$ is a complete C*-algebra-valued b-metric space, Example 2.7.

Define $T, S : L^{\infty}(E) \to L^{\infty}(E)$ by

$$T(x(t)) = \int_E K_1(t,s,x(s))ds + g(t) \quad t \in E \text{ and}$$

$$S(x(t)) = \int_E K_2(t,s,x(s))ds + g(t) \quad t \in E.$$

Set $B = kI$, then $B \in L(L^2(E))_+$ and $||B|| = k < 1$. For any $h \in L^2(E)$ we have

$$\begin{aligned}
||d(Tx,Sy)|| &= \sup_{||h||=1} (\pi_{|Tx-Sy|^p} h, h) \\
&= \sup_{||h||=1} \int_E (|\int_E K_1(t,s,x(s)) - K_2(t,s,y(s))|^p) h(t)\bar{h(t)} dt \\
&\le \sup_{||h||=1} \int_E (|\int_E K_1(t,s,x(s)) - K_2(t,s,y(s))|)^p h(t)\bar{h(t)} dt \\
&\le \sup_{||h||=1} \int_E (\int_E |k\varphi(t,s)(x(s)-y(s))|ds)^p h(t)\bar{h(t)} dt \\
&\le k^p \sup_{||h||=1} \int_E (\int_E |\varphi(t,s)|ds)^p |h(t)|^2 dt ||x-y||_{\infty}^p \\
&\le k \sup_{||h||=1} \int_E |\varphi(t,s)|ds . \sup_{||h||=1} \int_E |h(t)|^2 dt . ||x-y||_{\infty}^p \\
&\le k||x-y||_{\infty}^p \\
&= ||B||||d(x,y)||.
\end{aligned}$$

Hence the mappings T and S satisfy all the conditions of Corollary 3.2, and then T and S have a unique common fixed point, which is equivalent to that the integral equation 4.1, have a unique common solution in $L^{\infty}(E)$.

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Received: June 7, 2017; Accepted: March 9, 2018.