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FIXED POINT RESULTS UNDER GENERALIZED c-DISTANCE WITH APPLICATION TO NONLINEAR FOURTH-ORDER DIFFERENTIAL EQUATION

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Abstract. We consider the notion of generalized c-distance in the setting of ordered cone b-metric spaces and obtain some new fixed point results. Our results provide a more general statement, under which can be unified some theorems of the existing literature. In particular, we refer to the results of Sintunavarat et al. [W. Sintunavarat, Y.J. Cho, P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, Comput. Math. Appl. 62 (2011) 1969-1978]. Some examples and an application to nonlinear fourth-order differential equation are given to support the theory.
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1. Introduction and preliminaries

It is well-known that ordered normed spaces and cones have many applications in applied mathematics and related topics. Briefly, we point out the setting in which our finding is developed. Hence, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century (see [7, 18]). In 2007, Huang and Zhang [9] reintroduced such spaces under the name of cone metric spaces by substituting an ordered normed space for the real numbers and obtained some fixed point results.

On the other hand, the concept of metric type or b-metric space was introduced and studied by Bakhtin [2] and Czerwik [6]. After that, analogously with definition of a b-metric space and cone metric space, Hussain and Shah [11] and Ćvetković et al. [5]

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defined cone metric type spaces or cone b-metric spaces and proved several fixed and common fixed point theorems.

In 1996, Kada et al. [13] defined the concept of w-distance in metric spaces. In the sequel, Cho et al. [4] and Wang and Guo [19] defined the concept of the c-distance in a cone metric space, which is a cone version of the w-distance. Then some fixed point results under w-distance in metric spaces and under c-distance in cone metric spaces and tvs-cone metric spaces were proved in [8, 14, 15, 16, 17] and references therein. Recently, Hussain et al. [10] defined the concept of wt-distance on a metric type space and proved some fixed point theorems on wt-distance in a partially ordered b-metric space. Then Bao et al. [3] defined generalized c-distance in cone b-metric spaces and obtained several fixed point results in ordered cone b-metric spaces.

Let E be a real Banach space. Then a subset P of E is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{\theta\}$;
(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) if $x, -x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq K\|y\|$. The least positive number satisfying the above condition is called the normal constant of P.

Definition 1.1. [5, 11] Let X be a nonempty set, $s \geq 1$ be a real number and E be a real Banach space with cone P. Suppose that $d : X \times X \to E$ is a mapping satisfying the conditions:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \preceq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then, the pair (X, d) is called cone b-metric space (or cone metric type space).

Obviously, for $s = 1$, the cone b-metric space is a cone metric space. Also, for $E = \mathbb{R}$ and $P = [0, \infty)$, the cone b-metric space is a b-metric space.

Definition 1.2. [5, 11] Let (X, d) be a cone b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim\limits_{n\to\infty} x_n = x$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.
(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence in X is convergent.

Lemma 1.3. *Let E be a real Banach space with a cone P in E. Then, for all $u, c \in E$, the following assertions hold:*

(p_1) *if $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$;*
(p_2) *let $x_n \to \theta$ in E as $n \to \infty$, $\theta \preceq x_n$ and $\theta \ll c$. Then there exists a positive integer n_0 such that $x_n \ll c$ for all $n \geq n_0$.*

In this paper, we state and prove some fixed point theorems via the notion of generalized c-distance in ordered cone b-metric spaces. Our results extend, unify and generalize well-known results such as: Altun et al. [1], Bao et al. [3], Cho et al. [4], Sintunavarat et al. [17], Hussain et al. [10], Rahimi and Soleimani Rad [14]. To motivate this study, we give some illustrative examples and establish the existence of at least a solution for a fourth-order boundary value problem.

2. Main results

We start by recalling the following definition of generalized c-distance.

Definition 2.1. [3] Let (X, d) be a cone b-metric space and $s \geq 1$ be a given real number. A function $q : X \times X \to E$ is called a generalized c-distance on X if the following properties are satisfied:

- (q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
- (q_2) $q(x, z) \preceq s[q(x, y) + q(y, z)]$ for all $x, y, z \in X$;
- (q_3) for $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$ and all $n \geq 1$, then $q(x, y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
- (q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.2. *Each wt-distance in a b-metric space is a generalized c-distance with $E = \mathbb{R}$ and $P = [0, \infty)$; but the converse does not hold. Thus, the generalized c-distance is a generalization of wt-distance defined by Hussain et al.* [10]*. Also, for $s = 1$, the generalized c-distance is a c-distance of* [4]*. In this manner, if we consider $E = \mathbb{R}$ and $P = [0, \infty)$, then we obtain the definition of w-distance introduced by Kada et al.* [13]*. Moreover, for a generalized c-distance q, $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$. Moreover, $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.*

Several examples of generalized c-distances can be found in [3]. Now, we give more examples.

Example 2.3. Let (X, d) be a cone b-metric space with given real number $s \geq 1$ such that the cone b-metric $d(\cdot, \cdot)$ is a continuous function in its second variable. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a generalized c-distance. In fact, (q_1) and (q_2) are immediate. The property (q_3) is nontrivial and it follows from $q(x, y_n) = d(x, y_n) \preceq u$, passing to the limit when $n \to \infty$ and using the continuity of d in its second variable. Let $c \in E$ with $\theta \ll c$ be given and put $e = \frac{c}{2s}$. Suppose that $q(z, x) \ll e$ and $q(z, y) \ll e$. Then

$$\begin{aligned} d(x, y) &\preceq s[d(x, z) + d(z, y)] \\ &= sd(z, x) + sd(z, y) \\ &= sq(z, x) + sq(z, y) \\ &\ll s(e + e) = c. \end{aligned}$$

This shows that q also satisfies (q_4) and hence q is a generalized c-distance.

Example 2.4. Let $E = \mathbb{R}$, $X = [0, \infty)$ and $P = \{x \in E : x \geq 0\}$. Define a mapping $d : X \times X \to E$ by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then (X, d) is a cone b-metric space with $s = 2$. Define a mapping $q : X \times X \to E$ by $q(x, y) = x^2 + y^2$ for all

$x, y \in X$. Then q is a generalized c-distance. Indeed, $(q_1) - (q_3)$ are clear. To show (q_4), for $c \in E$ with $\theta \ll c$, put $e = \frac{c}{2}$. Now, we have

$$d(x, y) = (x - y)^2 \preceq x^2 + y^2 \preceq q(z, x) + q(z, y) \ll e + e = c.$$

Example 2.5. Let $E = C^1([0,1], \mathbb{R})$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the non-normal cone $P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0,1]\}$. Also, let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|^s \psi$ for all $x, y \in X$, where $\psi : [0,1] \to \mathbb{R}$ is defined by $\psi(t) = 2^t$ for all $t \in [0,1]$. Then (X, d) is a cone b-metric space with $s \in \{1, 2\}$. Define a mapping $q : X \times X \to E$ by $q(x, y) = y^s \psi$ for all $x, y \in X$ and $s \in \{1, 2\}$. Then q is a generalized c-distance.

Lemma 2.6. *Let (X, d) be a cone b-metric space and q be a generalized c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$, and $\{u_n\}$ and $\{v_n\}$ be two sequences in P converging to θ. Then the following conditions hold:*

- (qp_1) *if $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$;*
- (qp_2) *if $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*
- (qp_3) *if $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X;*
- (qp_4) *if $q(y, x_n) \preceq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. The proof is similar to c-distance in [4, 17]. □

A relation $\sqsubseteq$ on X is called:

- (i) reflexive if $x \sqsubseteq x$ for all $x \in X$;
- (ii) transitive if $x \sqsubseteq y$ and $y \sqsubseteq z$ imply $x \sqsubseteq z$ for all $x, y, z \in X$;
- (iii) antisymmetric if $x \sqsubseteq y$ and $y \sqsubseteq x$ imply $x = y$ for all $x, y \in X$;
- (iv) pre-order if it is reflexive and transitive.

A pre-order $\sqsubseteq$ is called partial order or an order relation if it is antisymmetric. Given a partially ordered set $(X, \sqsubseteq)$; that is, the set X equipped with a partial order $\sqsubseteq$, the notation $x \sqsubset y$ stands for $x \sqsubseteq y$ and $x \neq y$.

Definition 2.7. Let $(X, \sqsubseteq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be nondecreasing if $x \sqsubseteq y$ implies that $fx \sqsubseteq fy$ for all $x, y \in X$.

Our first result is the following fixed point theorem involving a generalized c-distance in an ordered cone b-metric space without normality condition of the cone.

Theorem 2.8. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with given real number $s \geq 1$. Also, let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x, y \in X$ with $y \sqsubseteq x$,*
$$q(fx, fy) \preceq \alpha(x) q(x, y) + \beta(x) q(x, fy) + \gamma(x) q(y, fx);$$
- (t_4) *for all $x, y \in X$ with $y \sqsubseteq x$,*
$$q(fy, fx) \preceq \alpha(x) q(y, x) + \beta(x) q(fy, x) + \gamma(x) q(fx, y).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $q(v,v) = \theta$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, let $fx_0 \neq x_0$. Since f is nondecreasing with respect to $\sqsubseteq$ and $x_0 \sqsubseteq fx_0$, we have by induction that

$$x_0 \sqsubseteq fx_0 \sqsubseteq \cdots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \cdots,$$

where $x_n = fx_{n-1} = f^n x_0$. Now, set $x = x_n$ and $y = x_{n-1}$ in (t_3). Then we have

$$\begin{aligned} q(x_{n+1}, x_n) &= q(fx_n, fx_{n-1}) \\ &\preceq \alpha(x_n)q(x_n, x_{n-1}) + \beta(x_n)q(x_n, fx_{n-1}) + \gamma(x_n)q(x_{n-1}, fx_n) \\ &= \alpha(x_n)q(x_n, x_{n-1}) + \beta(x_n)q(x_n, x_n) + \gamma(x_n)q(x_{n-1}, x_{n+1}) \\ &\preceq \alpha(fx_{n-1})q(x_n, x_{n-1}) + s\beta(fx_{n-1})[q(x_n, x_{n+1}) + q(x_{n+1}, x_n)] \\ &\quad + s\gamma(fx_{n-1})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\preceq \alpha(x_{n-1})q(x_n, x_{n-1}) + s(\beta+\gamma)(x_{n-1})q(x_n, x_{n+1}) \\ &\quad + s[\beta(x_{n-1})q(x_{n+1}, x_n) + \gamma(x_{n-1})q(x_{n-1}, x_n)] \\ &\ \ \vdots \\ &\preceq \alpha(x_0)q(x_n, x_{n-1}) + s(\beta+\gamma)(x_0)q(x_n, x_{n+1}) \\ &\quad + s[\beta(x_0)q(x_{n+1}, x_n) + \gamma(x_0)q(x_{n-1}, x_n)]. \end{aligned} \tag{2.1}$$

Similarly, set $x = x_n$ and $y = x_{n-1}$ in (t_4). Then we have

$$\begin{aligned} q(x_n, x_{n+1}) \preceq\ & \alpha(x_0)q(x_{n-1}, x_n) + s(\beta+\gamma)(x_0)q(x_{n+1}, x_n) \\ & + s[\beta(x_0)q(x_n, x_{n+1}) + \gamma(x_0)q(x_n, x_{n-1})]. \end{aligned} \tag{2.2}$$

Adding up (2.1) and (2.2), we get

$$\begin{aligned} q(x_{n+1}, x_n) + q(x_n, x_{n+1}) \preceq\ & (\alpha + s\gamma)(x_0)[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] \\ & + s(2\beta+\gamma)(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n+1})]. \end{aligned}$$

Let $u_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. We get that

$$u_n \preceq (\alpha + s\gamma)(x_0)u_{n-1} + s(2\beta+\gamma)(x_0)u_n.$$

Thus, we have $u_n \preceq hu_{n-1}$, where $0 \leq h = \dfrac{(\alpha+s\gamma)(x_0)}{1 - s(2\beta+\gamma)(x_0)} < \dfrac{1}{s}$ by (t_2).
By repeating the procedure, we get $u_n \preceq h^n u_0$ for all $n \in \mathbb{N}$ and hence

$$q(x_n, x_{n+1}) \preceq u_n \preceq h^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{2.3}$$

Let $m > n$. It follows from (2.3) and $0 \le sh < 1$ that

$$\begin{aligned} q(x_n, x_m) &\preceq s[q(x_n, x_{n+1}) + q(x_{n+1}, x_m)] \\ &\preceq sq(x_n, x_{n+1}) + s[sq(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_m)] \\ &\ \ \vdots \\ &\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} q(x_{m-1}, x_m)] \\ &\preceq (sh^n + s^2 h^{n+1} \cdots + s^{m-n} h^{m-1})[q(x_1, x_0) + q(x_0, x_1)] \\ &\preceq \frac{sh^n}{1 - sh}[q(x_1, x_0) + q(x_0, x_1)]. \end{aligned}$$

By using (qp_3) of Lemma 2.6, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. By using the continuity of f and since the limit of a sequence is unique, we get $fz = z$. Moreover, let $fv = v$ for $v \in X$. Then (t_3) implies that

$$\begin{aligned} q(v, v) = q(fv, fv) &\preceq \alpha(v)q(v, v) + \beta(v)q(v, fv) + \gamma(v)q(v, fv) \\ &= (\alpha + \beta + \gamma)(v)q(v, v). \end{aligned}$$

Since $0 \le (\alpha+\beta+\gamma)(v) < \big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(v)$ and $\big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(v) < 1$ by (t_2), we get that $q(v, v) = \theta$ by (p_1) of Lemma 1.3. This completes the proof. $\square$

We give an illustrative example.

Example 2.9. Let $E = C^1([0, 1], \mathbb{R})$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the non-normal cone $P = \{x \in E : x(t) \ge 0 \text{ for all } t \in [0, 1]\}$. Also, let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y)(t) = (x - y)^2 \psi(t)$ for all $x, y \in X$, where $\psi : [0, 1] \to \mathbb{R}$ is defined by $\psi(t) = 2^t$ for all $t \in [0, 1]$. Then (X, d) is a complete cone b-metric space with $s = 2$. Define a function $q : X \times X \to E$ by $q(x, y)(t) = d(x, y)(t)$ for all $x, y \in X$. Then q is a generalized c-distance. Let an order relation $\sqsubseteq$ be defined by $x \sqsubseteq y$ if and only if $x \preceq y$. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{4}$ for all $x \in X$. Define the mappings $\alpha(x) = \frac{(x+1)^2}{16}$ and $\beta(x) = \gamma(x) = 0$ for all $x \in X$. Next, note that

(i) $\alpha(fx) = \frac{1}{16}\big(\frac{x^2}{4} + 1\big)^2 \le \frac{1}{16}\big(x^2 + 1\big)^2 \le \frac{(x+1)^2}{16} = \alpha(x)$ for all $x \in X$;
(ii) $\beta(fx) = 0 \le 0 = \beta(x)$ and $\gamma(fx) = 0 \le 0 = \gamma(x)$ for all $x \in X$;
(iii) $\big(2(\alpha + 2\beta) + (2^2 + 2)\gamma\big)(x) = 2\frac{(x+1)^2}{16} = \frac{(x+1)^2}{8} < 1$ for all $x \in X$;
(iv) for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$\begin{aligned} q(fx, fy)(t) &= \left(\frac{x^2}{4} - \frac{y^2}{4}\right)^2 \psi(t) \\ &= \frac{(x + y)^2 (x - y)^2}{16} 2^t \\ &\preceq \frac{(x + 1)^2}{16}(x - y)^2 2^t \\ &\preceq \alpha(x)q(x, y)(t) + \beta(x)q(x, fy)(t) + \gamma(x)q(y, fx)(t); \end{aligned}$$

(v) similarly, for all $x, y \in X$ with $y \sqsubseteq x$, we have

$$q(fy, fx)(t) \preceq \alpha(x)q(y,x)(t) + \beta(x)q(fy,x)(t) + \gamma(x)q(fx,y)(t).$$

Moreover, f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Therefore, all the conditions of Theorem 2.8 are satisfied and hence f has a fixed point $x = 0$ with $q(0,0)(t) = 0$.

An immediate consequence of Theorem 2.8 can be stated in the form of corollary, as follows.

Corollary 2.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space. Also, let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $s(\alpha + 2\beta) + (s^2 + s)\gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x,y) + \beta q(x, fy) + \gamma q(y, fx),$$
$$q(fy, fx) \preceq \alpha q(y,x) + \beta q(fy, x) + \gamma q(fx, y),$$

for all $x, y \in X$ with $y \sqsubseteq x$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$, then $q(v,v) = \theta$.

Proof. We can prove this result by applying Theorem 2.8 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. □

Moreover, by setting $s = 1$ in Theorem 2.8 and Corollary 2.10, we obtain the following results of [14].

Theorem 2.11. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

(t_1) *$\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$;*

(t_2) *$(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*

(t_3) *for all $x, y \in X$ with $y \sqsubseteq x$,*

$$q(fx, fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x, fy) + \gamma(x)q(y, fx);$$

(t_4) *for all $x, y \in X$ with $y \sqsubseteq x$,*

$$q(fy, fx) \preceq \alpha(x)q(y,x) + \beta(x)q(fy, x) + \gamma(x)q(fx, y).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $q(v,v) = \theta$.

Corollary 2.12. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x,y) + \beta q(x, fy) + \gamma q(y, fx),$$
$$q(fy, fx) \preceq \alpha q(y,x) + \beta q(fy, x) + \gamma q(fx, y),$$

for all $x, y \in X$ with $y \sqsubseteq x$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$, then $q(v,v) = \theta$.

The following theorem is a second main result.

Theorem 2.13. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space with given real number $s \geq 1$. Also, let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha + \beta) + \gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x, y \in X$ with $x \sqsubseteq y$,*

$$q(fx, fy) \preceq \alpha(x)q(x, y) + \beta(x)q(x, fx) + \gamma(x)q(y, fy).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $q(v, v) = \theta$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, let $fx_0 \neq x_0$. As in the proof of Theorem 2.8, we get

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{n-1} \sqsubseteq x_n \sqsubseteq \cdots ,$$

where $x_n = fx_{n-1} = f^n x_0$. Now, set $x = x_{n-1}$ and $y = x_n$ in (t_3). Then we have

$$\begin{aligned}
q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\
&\preceq \alpha(x_{n-1})q(x_{n-1}, x_n) + \beta(x_{n-1})q(x_{n-1}, fx_{n-1}) + \gamma(x_{n-1})q(x_n, fx_n) \\
&= \alpha(fx_{n-2})q(x_{n-1}, x_n) + \beta(fx_{n-2})q(x_{n-1}, x_n) + \gamma(fx_{n-2})q(x_n, x_{n+1}) \\
&\preceq \big(\alpha(x_{n-2}) + \beta(x_{n-2})\big)q(x_{n-1}, x_n) + \gamma(x_{n-2})q(x_n, x_{n+1}) \\
&= \big(\alpha(fx_{n-3}) + \beta(fx_{n-3})\big)q(x_{n-1}, x_n) + \gamma(fx_{n-3})q(x_n, x_{n+1}) \\
&\preceq \big(\alpha(x_{n-3}) + \beta(x_{n-3})\big)q(x_{n-1}, x_n) + \gamma(x_{n-3})q(x_n, x_{n+1}) \\
&\;\;\vdots \\
&\preceq \big(\alpha(x_0) + \beta(x_0)\big)q(x_{n-1}, x_n) + \gamma(x_0)q(x_n, x_{n+1}).
\end{aligned} \tag{2.4}$$

Now, from (2.4), we get

$$q(x_n, x_{n+1}) \preceq \frac{\alpha(x_0) + \beta(x_0)}{1 - \gamma(x_0)} q(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Repeating this process, we get

$$q(x_n, x_{n+1}) \preceq k^n q(x_0, x_1) \tag{2.5}$$

for all $n \in \mathbb{N}$, where $0 \leq k = \frac{\alpha(x_0)+\beta(x_0)}{1-\gamma(x_0)} < \frac{1}{s}$ by (t_2).
Let $m > n$. It follows from (2.5) and $0 \leq sk < 1$ that

$$\begin{aligned} q(x_n, x_m) &\preceq s[q(x_n, x_{n+1}) + q(x_{n+1}, x_m)] \\ &\preceq sq(x_n, x_{n+1}) + s[sq(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_m)] \\ &\ \ \vdots \\ &\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} q(x_{m-1}, x_m)] \\ &\preceq (sk^n + s^2 k^{n+1} \cdots + s^{m-n} k^{m-1}) q(x_0, x_1) \\ &\preceq \frac{sk^n}{1-sk} q(x_0, x_1). \end{aligned}$$

By using (qp_3) of Lemma 2.6, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. By using the continuity of f and since the limit of a sequence is unique, we get $fz = z$. Moreover, suppose that $fv = v$ for $v \in X$. Finally, one can prove that $q(v, v) = v$ by following arguments similar to those in the final part of the proof of Theorem 2.8. □

Corollary 2.14. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone b-metric space. Also, let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $s(\alpha + \beta) + \gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fx) + \gamma q(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$, then $q(v, v) = \theta$.

Proof. We can prove this result by applying Theorem 2.13 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. □

Example 2.15. Let $E = C^1([0, 1], \mathbb{R})$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the non-normal cone $P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$. Also, let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y)(t) = (x - y)^2 \psi(t)$ for all $x, y \in X$, where $\psi : [0, 1] \to \mathbb{R}$ is defined by $\psi(t) = 2^t$ for all $t \in [0, 1]$. Then (X, d) is a complete cone b-metric space with $s = 2$. Define a function $q : X \times X \to E$ by $q(x, y)(t) = (x^2 + y^2)\psi(t)$ for all $x, y \in X$. Then q is a generalized c-distance. Let an order relation $\sqsubseteq$ be defined by $x \sqsubseteq y$ if and only if $x \preceq y$. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{4}$ for all $x \in X$. Take $\alpha = \frac{1}{16}$, $\beta = \frac{1}{5}$ and $\gamma = 0$. Now, we have

(i) $s(\alpha + \beta) + \gamma = 2(\frac{1}{16} + \frac{1}{5}) = \frac{21}{40} < 1$;
(ii) for all comparable $x, y \in X$,

$$\begin{aligned} q(fx, fy)(t) &= ((fx)^2 + (fy)^2)\psi(t) \\ &= \left(\frac{x^4}{16} + \frac{y^4}{16}\right) 2^t \\ &\preceq \alpha q(x, y)(t) + \beta q(x, fx)(t) + \gamma q(y, fy)(t). \end{aligned}$$

Moreover, f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Therefore, all the conditions of Corollary 2.14 are satisfied and hence f has a fixed point $x = 0$ with $q(0,0) = 0$.

Remark 2.16. *For Banach-type and Kannan-type fixed point results, we need*

$$q(fx, fy) \preceq \alpha q(x,y), \qquad \alpha \in \left[0, \frac{1}{s}\right) \tag{2.6}$$

and

$$q(fx, fy) \preceq \lambda(q(x, fx) + q(y, fy)), \qquad \lambda \in \left[0, \frac{1}{s+1}\right),$$

respectively.

Example 2.17. Consider E, $||\cdot||$, P, d, s, ψ and order relation $\sqsubseteq$ as in Example 2.15. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{2}$ for all $x \in X$. Consider $X = [0,2]$. Since $d(f(0), f(2))(t) = d(0,2)(t)$, there is not $0 \le \alpha < \frac{1}{s}$ such that $d(fx, fy) \preceq \alpha d(x,y)$ for all $x, y \in X$. Hence, Banach-type result on cone b-metric space cannot be applied to this example. Now, define a function $q : X \times X \to E$ by $q(x,y)(t) = y^2\psi(t)$ for all $x, y \in X$ and $X = [0,1]$. Then q is a generalized c-distance. Moreover,

$$q(fx, fy)(t) = (fy)^2\psi(t) = \left(\frac{y^4}{4}\right) 2^t \preceq \frac{1}{4} q(x,y)(t).$$

Thus, (2.6) holds with $\alpha = \frac{1}{4} \in [0, \frac{1}{2})$. Hence, all the conditions of Banach-type fixed point results with respect to generalized c-distance on cone b-metric spaces are satisfied. Note that f has a (trivial) fixed point $0 \in [0,1] \subseteq [0,2]$ and $q(0,0) = 0$.

By setting $s = 1$ in Theorem 2.13, Corollary 2.14 and Remark 2.16, then we obtain the following results of [4, 17].

Theorem 2.18. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0,1)$ such that the following conditions hold:*

- (t_1) *$\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$;*
- (t_2) *$(\alpha + \beta + \gamma)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x, y \in X$ with $x \sqsubseteq y$,*

$$q(fx, fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x, fx) + \gamma(x)q(y, fy).$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $q(v,v) = \theta$.

Corollary 2.19. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x,y) + \beta q(x, fx) + \gamma q(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fv = v$, then $q(v,v) = \theta$.

Remark 2.20. *For Banach-type and Kannan-type fixed point results with respect to a c-distance on cone metric space, we need*

$$q(fx, fy) \preceq \alpha q(x, y), \qquad \alpha \in [0, 1) \tag{2.7}$$

and

$$q(fx, fy) \preceq \lambda(q(x, fx) + q(y, fy)), \qquad \lambda \in \left[0, \frac{1}{2}\right),$$

respectively.

Finally, we pose two open questions to the reader.

Question 1. Can the continuity condition of mapping f be replaced by another condition in the above mentioned fixed point results?

Question 2. Can one obtain the above mentioned fixed point results on generalized c-distance in cone b-metric spaces instead of ordered cone b-metric spaces?

3. Application to nonlinear fourth-order differential equation

It is well-known that fourth-order differential equations are useful tools for modeling the elastic beam deformation. Precisely, we refer to beams in equilibrium state, whose two ends are simply supported. Consequently, this study has applications in engineering and physical science. In this section, the existence of solutions of fourth-order boundary value problems is established as a consequence of Theorem 2.8 (Theorem 2.13). In particular, the focus is on the equivalent integral formulation of the boundary value problem below and the use of Green's functions. First, we introduce the mathematical background as follows, see also Jleli et al. [12].

Let $E = C^1([0,1], \mathbb{R})$, $P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0,1]\}$ and $X = C([0,1], \mathbb{R})$ be the set of all non-negative real-valued continuous functions on the interval $[0,1]$. Let X be endowed with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$, where

$$\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|,$$

and define a mapping $d : X \times X \to E$ by

$$d(x, y) = e^v \sup_{t\in[0,1]} (x(t) - y(t))^2 \text{ for all } x, y \in X, \quad v \in [0, 1].$$

Also, consider the partial order

$$(x, y) \in X \times X,\ x \sqsubseteq y \Longleftrightarrow x(t) \leq y(t), \text{ for all } t \in [0, 1].$$

Clearly, $(X, \sqsubseteq)$ is a partially ordered set and (X, d) is a complete cone b-metric space with $s = 2$. Finally, consider the generalized c-distance $q : X \times X \to E$ given by $q(x, y) = e^v \sup_{t\in[0,1]}(y(t))^2$ for all $x, y \in X$ and $v \in [0, 1]$. Thus, we study the following fourth-order two-point boundary value problem

$$\begin{cases} x^{iv}(t) = k(t, x(t)), \quad 0 < t < 1, \\ \\ x(0) = x'(0) = x''(1) = x'''(1) = 0, \end{cases} \tag{3.1}$$

with $k \in C([0,1] \times \mathbb{R}, \mathbb{R})$ nondecreasing.

It is well-known that the problem (3.1) may be equivalently expressed in integral form: find $x^* \in X$ solution of

$$x(t) = \int_0^1 G(t,\tau)k(\tau, x(\tau))\, d\tau, \quad t \in [0,1], \tag{3.2}$$

where the Green function $G(t,\tau)$ is given by

$$G(t,\tau) = \frac{1}{6}\begin{cases} \tau^2(3t-\tau), & 0 \le \tau \le t \le 1, \\ t^2(3\tau - t), & 0 \le t \le \tau \le 1. \end{cases}$$

Also, it is immediate to show that

$$0 \le G(t,\tau) \le \frac{1}{2}t^2\tau, \text{ for all } t,\tau \in [0,1]. \tag{3.3}$$

Next, we consider the hypotheses:

(I) There exists $\alpha : X \to [0, \frac{1}{2})$ such that

$$0 \le k(t, y(t)) \le 4\sqrt{\alpha(x)q(x,y)e^{-v}}, \tag{3.4}$$

for all $x, y \in X$, with $x \sqsubseteq y$ and for all $t \in [0,1]$ and

$$\alpha\left(\int_0^1 G(t,\tau)k(\tau,x(\tau))\, d\tau\right) \le \alpha(x), \text{ for all } x \in X. \tag{3.5}$$

(II) There exists $x_0 \in X$ such that

$$x_0(t) \le \int_0^1 G(t,\tau)k(\tau, x_0(\tau))\, d\tau, \quad t \in [0,1],$$

that is, the integral equation (3.2) admits a lower solution in X.

Now, we prove the existence of at least a solution of (3.1) in X.

Theorem 3.1. *The existence of at least a solution of problem* (3.1) *in X is established, provided that the function $k \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfies the hypotheses (I) and (II).*

Proof. The problem in study is equivalent to the fixed point problem obtained by introducing the continuous integral operator $T : X \to X$ given as

$$(Tx)(t) = \int_0^1 G(t,\tau)k(\tau, x(\tau))\, d\tau, \quad t \in [0,1], \quad x \in X.$$

Now, we show that the operator T satisfies all the conditions in Theorem 2.8 (Theorem 2.13) to conclude that there exists a fixed point of T in X. Since $k \in C([0,1] \times \mathbb{R}, \mathbb{R})$ is nondecreasing, we deduce that T is a nondecreasing mapping with respect to $\sqsubseteq$.

Also, by using (3.4), for all $t \in [0,1]$ and for all $x, y \in X$ with $x \sqsubseteq y$, we get

$$\begin{aligned}
q(Tx,Ty) &= e^v \sup_{t\in[0,1]} \left(\int_0^1 G(t,\tau)k(\tau,y(\tau))\,d\tau \right)^2 \\
&\le e^v \sup_{t\in[0,1]} \left(\int_0^1 G(t,\tau)4\sqrt{\alpha(x)q(x,y)e^{-v}}\,d\tau \right)^2 \\
&\le \sup_{t\in[0,1]} \left(\int_0^1 G(t,\tau)\,d\tau \right)^2 e^v 16 e^{-v}\alpha(x)q(x,y) \\
&\le \alpha(x)q(x,y) \quad \text{(from (3.3))}.
\end{aligned}$$

It follows that the conditions (t_3) and (t_4) of Theorem 2.8 ((t_3) of Theorem 2.13) hold true, with $\beta(x) = \gamma(x) = 0$ for all $x \in X$. By hypothesis (II) we get that there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$. Then, from (3.5) and the fact that the function α assumes values in the interval $[0, \frac{1}{2})$, we have

$$\alpha(Tx) \le \alpha(x) < \frac{1}{2} \quad \text{for all } x \in X;$$

that is, the conditions (t_1) and (t_2) of Theorem 2.8 (Theorem 2.13) hold true, again with $\beta(x) = \gamma(x) = 0$ for all $x \in X$. We conclude that all the conditions of Theorem 2.8 (Theorem 2.13) hold true and so we deduce the existence of a fixed point of T; that is, there exists a solution of problem (3.1) in X. □

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