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FIXED POINT THEOREMS IN GENERALIZED BANACH ALGEBRAS AND APPLICATIONS

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Abstract. In this paper, we prove some fixed point theorems in vector algebra Banach spaces. We establish the versions of Perov, Schauder and Krasnosel'skki type fixed point theorem for the sum of a contraction operator and a compact operator. The obtained results are applied to prove some theorems on the existence of solutions to nonlinear integral equations in Banach algebras. Finally, some example are given to illustrate the result.
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1. Introduction

Operator equations of various kind create the base of numerous considerations conducted in nonlinear analysis and in the theories of differential and integral equations. The existence of solutions of those operator equations is mostly proved with the aid of miscellaneous fixed point theorems. The existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications occur frequently in nonlinear analysis. Over time, many improvements of Krasnosel'skii's theorem in Banach algebras have appeared in the literature by modifying the above assumptions; see, for example, [3, 6, 5, 4, 9, 10, 13] and the references therein.

In recent years, many authors have focused on the resolution of the following abstract equations in suitable Banach algebras:

$$A.B(u) = u, \quad u \in M, \tag{1.1}$$

$$A.B(u) + C(u) = u, \quad u \in M, \tag{1.2}$$

or

$$u \in A.B(u) + G(u), \quad u \in M, \tag{1.3}$$

where A, B, C are maps, G is a multivalued map, and M is a closed convex subset of a Banach algebra space X.

For the result related to equations of the type (1.2), we combine the Banach contraction principle and Schauder's fixed point theorem.

Fixed point theory for multivalued mappings is an important topic in set-valued analysis. Several well-known fixed point theorems for single-valued mappings such as those of Banach and Schauder have been extended to multivalued mappings in Banach spaces; see, for example, the monographs of Górniewicz *et al.* [1, 17] and Ben Amar and D. O'Regan [7]. Similarly, for the fixed-point theorems for multivalued operators on Banach algebras, see [14, 15] and the references therein. Very recently, multivalued analogues of Krasnosel'skii's fixed point theorem in Banach algebras were obtained by Ben Amar and O'Regan [8].

The classical Banach contraction principle was extended for contractive maps on spaces endowed with a vector-valued metric by Perov in 1964 [25] and Perov and Kibenko [26]. Also Schauder's fixed point theorem was extended to generalized Banach spaces by Viorel [33]. Very recently, for the case of vector-valued Banach spaces, Krasnosel'skii type fixed point theorem of single and multivalued operators was studied by Petre [28], Petre and Petruşel [27], and Ouahab [24].

The main goal of this paper is to prove some new fixed point theorems in generalized Banach algebras and apply them to prove some existence results for a certain system of fractional integral equations under mixed Lipschitz and continuous conditions.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. If, $x, y \in \mathbb{R}^n$,

$$x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n),$$

by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also

$$|x| = (|x_1|, \dots, |x_n|),\ \max(x, y) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$$

and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \quad x_i > 0, \forall i = 1, \dots, n\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric on X if and only if $d_i,\ i = 1, \dots, n$, are metrics on X.

For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i, i = 1, \dots, n\}$$

the open ball centered in x_0 with radius r, and by

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\} = \{x \in X : d_i(x_0, x) \leq r_i, i = 1, \dots, n\},$$

the closed ball centered in x_0 with radius r. We mention that, for generalized metric spaces, the notions of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\| : E \to \mathbb{R}^n$ with the following properties:

(i) $\|x\| \geq 0$ for all $x \in E$; if $\|x\| = 0$ then $x = (0, \cdots, 0)$;
(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e. $d(x, y) = \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 2.2. [32] *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*
(ii) *$M^k \to 0$ as $k \to \infty$;*
(iii) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(iv) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.3. We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ convergent to zero that also satisfy the property $(I - A)^{-1}|I - A| \leq I$ are the following:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$.

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$.

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

Definition 2.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a matrix M convergent to zero such that

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } \ x, y \in X.$$

For $n = 1$, we recover the classical Banach's contraction fixed point result.

Theorem 2.5. [25] (see also, [29]) *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(x_0, N(x_0)) \quad \text{for all } k \in \mathbb{N}.$$

As a consequence of Perov's theorem, we have the following result.

Theorem 2.6. [27] *Let (X, d) be a generalized Banach space and $N : X \to X$ an M-contraction. Then $I_X - N$ is a homeomorphism, i.e., $I_X - N$ is bijective and its inverse is continuous, that is, $(I_X - N)^{-1}$ is continuous too.*

Theorem 2.7. [33] *Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N : C \to C$ be a continuous operator with relatively compact range. Then N has at least a fixed point in C.*

As a consequence of Schauder fixed point theorem, we present the nonlinear alternative Leray-Schauder type theorem in generalized Banach spaces. For the proofs and more information, we refer the reader to [18, 33, 23].

Theorem 2.8. *Let E be a generalized Banach space, $U \subset E$ be a bounded, convex open neighborhood of zero and let $G : \overline{U} \to E$ be a continuous compact map. If G satisfies the boundary condition*

$$x \neq \lambda G(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $Fix(G) = \{x \in U : x = G(x)\}$ is nonempty.

Theorem 2.9. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $N : E \to E$ be a continuous compact mapping. Then either:*

a) *The set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \quad \text{for some } \lambda \in (0, 1)\}$$

is unbounded, or

b) *Then operator N has a fixed point.*

Denote by

$$\begin{aligned}
&\mathcal{P}(X) = \{Y \subset X\colon\ Y \neq \emptyset\},\\
&\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\},\\
&\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\}.
\end{aligned}$$

Let (X, d_*) be a metric space, we will denote by H_{d_*} the Hausdorff pseudo-metric distance on $\mathcal{P}(X)$, defined as

$$H_{d_*}\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\},\ H_{d_*}(A, B) = \max\left\{\sup_{a\in A} d_*(a, B), \sup_{b\in B} d_*(A, b)\right\},$$

where $d_*(A,b) = \inf_{a\in A} d_*(a,b)$ and $d_*(a,B) = \inf_{b\in B} d_*(a,b)$. Then $(\mathcal{P}_{cl,b}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.

Let (X,d) be a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

We have mentioned that d is a generalized metric space on X if and only if $d_i,\ i = 1,\dots,n$, are metrics on X. Consider the generalized Hausdorff pseudo-metric distance

$$H_d\colon\ \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \vdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Let (X,d) and (Y,ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. Then F is said to be *lower semi-continuous* (*l.s.c.*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X\ :\ F(x)\cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X\ :\ F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if, for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

Let (X,d) be a generalized metric space and define the following metric spaces. Let $X_i = X,\ i = 1,\dots,n$. Consider $\prod_{i=1}^{n} X_i$ with $\bar{d}$:

$$\bar{d}((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^{n} d_i(x_i,y_i).$$

The diagonal space of $\prod_{i=1}^{n} X_i$ defined by

$$\widetilde{X} = \{(x, \dots, x) \in \prod_{i=1}^{n} X_i : x \in X,\ i = 1, \dots, n\}.$$

Thus, $\widetilde{X}$ is a metric space with the following distance

$$d_*((x, \dots, x), (y, \dots, y)) = \sum_{i=1}^{n} d_i(x, y), \text{ for each } x, y \in X.$$

It is clear that $\widetilde{X}$ is a closed set in $\prod_{i=1}^{n} X_i$. This fact is showed in the following result.

Lemma 2.10. [30] *Let (X, d) be a generalized metric space. Then the map $h : X \to \widetilde{X}$ defined by*

$$h(x) = (x, \dots, x), \quad x \in X,$$

is a homeomorphism.

Definition 2.11. Let (X, d) be a generalized metric space. A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \text{ as } k \to \infty$$

and

$$H_d(N(u), N(v)) \leq Md(u, v), \text{ for all } u, v \in X.$$

Remark 2.12. In generalized metric spaces in Perov's sense, the notations of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Theorem 2.13. [24] *Let (X, d) be a generalized complete metric space, and let $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x, y) + Bd(y, F(x)) + Cd(x, F(x)), \tag{2.1}$$

where $A + C$ converges to zero. Then, there exists $x \in X$ such that $x \in F(x)$.

Finally, we recall two basic topological fixed point theorems for set-valued maps.

Theorem 2.14. [11] [see also [34], p. 452] *Let X be a Banach space, C be a nonempty compact convex subset of X, $F : C \to \mathcal{P}_{cp,cv}(C)$ be an u.s.c. multivalued map, then the operator inclusion F has at least one fixed point.*

Next, we present the version of Bohnenblust-Karlin fixed point theorem in vector Banach spaces.

Theorem 2.15. *Let X be a generalized Banach space, C be a nonempty compact convex subset of X, and $F : C \to \mathcal{P}_{cp,cv}(C)$ be an u.s.c. multivalued map, then the operator inclusion F has at least one fixed point, that is, there exists $x \in C$ such that $x \in G(x)$.*

Proof. Let $X_* = \{(x, \dots, x) : x \in X\}$, $\|x\|_* = \sum_{i=1}^{n} \|x\|_i$. It is clear that $(X, \|\cdot\|_*)$ is a Banach space. From lemma 2.10, the function $h : X \to X_*$ defined by

$$h(x) = (x, \dots, x), \quad x \in X,$$

is a homeomorphism. We consider $F_* : C_* \to \mathcal{P}(C_*)$ by

$$F_* = h \circ F \circ h^{-1}, \quad \text{defined for elements } (x, \dots, x) \in C_*,$$

where

$$C_* = h(C) = \{(x, \dots, x) : x \in C\}.$$

It is clear that C_* is a convex closed bounded subset in X_*. By theorem 2.14, there exists $(x, \dots, x) \in C_*$ with $x \in C$ such that

$$(x, \dots, x) \in F_*(x, \dots, x) \Rightarrow x \in F(x).$$

For further readings and details on multi-valued analysis and fixed point theory, we refer the reader to the books by Andres and Górniewicz [1], Aubin and Frankowska [2], Djebali *et al* [16], Deimling [12], Górniewicz [17], and Hu and Papageorgiou [20, 21], Kamenskii [22], and Tolstonogov [31].

3. Generalized Banach algebras

In this section, we give the basic concept of generalized Banach algebras.

Definition 3.1. An algebra E is a vector space endowed with an internal composition law noted by (.)

$$\begin{cases} (.) : & E \times E & \to & E \\ & (x, y) & \to & x.y, \end{cases}$$

which is associative and bilinear.

Definition 3.2. A generalized normed algebra E is an algebra endowed with a norm satisfying the following property

$$\text{for all } x, y \in E \quad \|x.y\| \leq \|x\|\|y\|,$$

where

$$\|x.y\| := \begin{pmatrix} \|x.y\|_1 \\ \vdots \\ \|x.y\|_n \end{pmatrix}$$

and

$$\|x\|\|y\| := \begin{pmatrix} \|x\|_1\|y\|_1 \\ \vdots \\ \|x\|_n\|y\|_n \end{pmatrix}.$$

A complete generalized normed algebra is called a generalized Banach algebra.

For any $A, B \in \mathcal{P}_{cl}(X)$, let us denote

$$A + B = \{a + b : \ a \in A, \ b \in B\}, \quad A.B = \{a.b : a \in A, \ b \in B\},$$

$$\|A\| = \sup\{\|a\| : a \in A\}, \quad H_d(0, A) = \|A\|,$$

and

$$\lambda A = \{\lambda a : a \in A\},\ \lambda \in \mathbb{R}.$$

We will now show a few elementary properties of Banach algebras.

Proposition 3.3. *Let E be a generalized Banach algebra. Then multiplication is continuous from $E \times E$ to E.*

We assume that E satisfies that there exists an element 1 in E such that $1x = x = x1$ and $\|1\| = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. We will typically write λ for $\lambda 1$.

Proposition 3.4. *Let E be a generalized Banach algebra. If $\|x\| < \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then $1 - x$ is invertible with*

$$\|(1-x)^{-1}\| \leq \begin{pmatrix} \frac{1}{1-\|x\|_1} \\ \vdots \\ \frac{1}{1-\|x\|_n} \end{pmatrix}.$$

Proof. Since $\|x\| < \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then, for every $i = 1, \ldots, n$, we have $\|x\|_i < 1$. Hence, if $p \geq q$, we have

$$\left\| \sum_{k=0}^{p} x^k - \sum_{k=0}^{q} x^k \right\|_i = \left\| \sum_{k=q+1}^{p} x^k \right\|_i \leq \frac{\|x\|_i^{q+1}}{1 - \|x\|_i},$$

and the sequence of partial sums $\sum_{k=0}^{p} x^k$ is seen to be a Cauchy sequence in $(E, \|\cdot\|_i)$, for $i = 1, \ldots, n$. Then $\sum_{k=0}^{p} x^k$ is a Cauchy sequence in $(E, \|\cdot\|)$. Let $y = \sum_{k=0}^{\infty} x^k$, then

$$\begin{aligned} x.y &= (1-(1-x)).\sum_{k=0}^{\infty} x^k \\ &= \lim_{p\to\infty} (1-(1-x)).\sum_{k=0}^{p} x^k \\ &= \lim_{p\to\infty} (1-x^p) = 1, \end{aligned}$$

since $\lim_{p\to\infty} \|x^p\|_i = 0$. Similarly, $y.x = 1$, so that x is invertible with

$$x^{-1} = \sum_{k=0}^{\infty} x^k.$$

Further, for every $i = 1, \ldots, n$,

$$\begin{aligned} \|(1-x)^{-1}\|_i &= \lim_{p\to\infty} \left\| \sum_{k=0}^{p} x^k \right\|_i \\ &\leq \lim_{p\to\infty} \sum_{k=0}^{p} \|x\|_i^k = \frac{1}{1-\|x\|_i}. \end{aligned}$$

Corollary 3.5. *Let E be a generalized Banach algebra. If $\|1-x\| < \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then x is invertible with*

$$\|x^{-1}\| \leq \begin{pmatrix} \frac{1}{1-\|1-x\|_1} \\ \vdots \\ \frac{1}{1-\|1-x\|_n} \end{pmatrix}.$$

Lemma 3.6. *Let X be a generalized Banach algebra space, then, for every $A, B \in \mathcal{P}_{cl,b}(X)$, we have*

$$H_d(AC, BC) \leq H_d(0, C) H_d(A, B).$$

Proof. Let $x \in AC$, then there exist $a \in A$ and $c \in C$ such that $x = ac$. Now

$$\begin{aligned} d(x, BC) &= \inf\{\|x - b\bar{c}\| : b \in B, \bar{c} \in C\} \\ &= \inf\{\|ac - b\bar{c}\| : b \in B, \bar{c} \in C\}, \end{aligned}$$

where

$$\|ac - b\bar{c}\| = \begin{pmatrix} \|ac - b\bar{c}\|_1 \\ \vdots \\ \|ac - b\bar{c}\|_n \end{pmatrix}.$$

For each $i = 1, \ldots, n$, we have

$$\begin{aligned} \|ac - b\bar{c}\|_i &= \inf\{\|ac - bc + bc - b\bar{c}\|_i : b \in B, \bar{c} \in C\} \\ &\leq \inf\{\|ac - bc\|_i + \|bc - b\bar{c}\|_i : b \in B, \bar{c} \in C\} \\ &\leq \inf\{\|a - b\|_i \|c\|_i + \|c - \bar{c}\|_i \|b\|_i : b \in B, \bar{c} \in C\} \\ &= \inf\{\|a - b\|_i \|c\|_i : b \in B\} \\ &= d_i(a, B)\|c\|_i \\ &\leq \sup\{d_i(a, B) : a \in A\}\|c\|_i \\ &\leq H_{d_i}(A, B) H_{d_i}(0, C). \end{aligned}$$

Hence

$$\sup\{d(x, BC) : x \in AC\} \leq H_d(A, B) H_d(0, C). \tag{3.1}$$

If we change the role between AC and BC, we obtain

$$\sup\{d(x, AC) : x \in BC\} \leq H_d(A, B) H_d(0, C). \tag{3.2}$$

By (3.1) and (3.2), we get

$$H_d(AC, BC) \leq H_d(A, B) H_d(0, C).$$

Proposition 3.7. *For a generalized Banach algebra E, the set of all invertible elements $\mathcal{G}$ is an open set.*

Proof. Let $x \in \mathcal{G}$, then x is invertible. We show that

$$U = B(x, \|x^{-1}\|^{-1}) = \{y \in X : \|y - x\| < \|x^{-1}\|^{-1}\} \subseteq \mathcal{G},$$

where

$$\|x^{-1}\| = \begin{pmatrix} \frac{1}{\|x\|_1} \\ \vdots \\ \frac{1}{\|x\|_n} \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

For y in U, we have

$$\|1 - x^{-1}y\|_i \leq \|x^{-1}\|_i \|x - y\|_i, \quad i = 1, \ldots, n.$$

Hence

$$\|1 - x^{-1}y\|_i < 1, \quad \text{for each } i = 1, \ldots, n.$$

Therefore,

$$\|1 - x^{-1}y\| < 1.$$

By proposition 3.4, $x^{-1}y$ is invertible. However, we then have that

$$x(x^{-1}y) = y$$

is invertible.

4. Fixed point theorems in generalized Banach algebras

We use the Perov's fixed point theorem combined with Schauder fixed point theorem in generalized Banach spaces to obtain a new fixed point result in generalized Banach algebras.

Theorem 4.1. *Let C be a nonempty closed convex and bounded subset of a generalized Banach algebra X and let $N, G : C \to X$ be two operators satisfying the following conditions:*

(i) *there exists $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+), M = (a_{ij})_{1\leq i,j\leq n}$ such that*

$$\|N(x) - N(y)\| \leq M\|x - y\| \text{ for all } x, y \in C.$$

(ii) *G is completely continuous.*

(iii) *$N(x)G(y) \in C$ for all $x, y \in C$.*

Then the operator equation

$$N(x)G(x) = x$$

has a solution, whenever $M_* \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix converging to zero, with

$$M_* := \begin{pmatrix} b_1a_{11} & \cdots & b_na_{1n} \\ \vdots & \ddots & \vdots \\ b_1a_{n1} & \cdots & b_na_{nn} \end{pmatrix}, \tag{4.1}$$

where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \sup\{\|G(x)\| : x \in C\}.$$

Proof. For $y \in C$, we define a mapping $N_y : C \to C$ by

$$N_y(x) = N(x)G(y).$$

Let $x_1, x_2 \in C$, then

$$\begin{aligned} \|N_y(x_1) - N_y(x_2)\| &= \|N(x_1)G(y) - N(x_2)G(y)\| \\ &\leq \|N(x_1) - N(x_2)\|\|G(y)\| \\ &\leq M\|x_1 - x_2\|\|G(y)\|. \end{aligned}$$

Therefore,

$$\|N_y(x_1) - N_y(x_2)\| \leq M_*\|x_1 - x_2\|.$$

By theorem 2.5, there is a unique point $x(y) \in C$ such that

$$x(y) = N(x(y))G(y).$$

Let $L : C \to C$ be defined by $L(y) = x(y)$. Next, we show that L is continuous and compact. Let y be fixed and $(y_p)_{p\in\mathbb{N}}$ be a sequence such that $y_p \to y$ as $p \to \infty$, thus

$$\begin{aligned} \|L(y_p) - L(y)\| &= \|N(x_p(y_p))G(y_p) - N(x(y))G(y)\| \\ &\leq \|N(x_p(y_p)) - N(x(y))\|\|G(y_p)\| \\ &\quad +\|N(x(y))\|\|G(y_p) - G(y)\| \\ &\leq M_*\|x_p(y_p) - x(y)\| + \|N(x(y))\|\|G(y_p) - G(y)\| \\ &= M_*\|L(y_p) - L(y)\| + \|N(x(y))\|\|G(y_p) - G(y)\|. \end{aligned}$$

Then

$$\|L(y_p) - L(y)\| \leq (I - M_*)^{-1}\|N(x(y))\|\|G(y_p) - G(y)\|.$$

Since G is continuous, then

$$\|L(y_p) - L(y)\| \leq (I - M_*)^{-1}\|N(x(y))\|\|G(y_p) - G(y)\| \to 0 \quad \text{as } p \to \infty.$$

This implies that $L : C \to C$ is continuous. By Schauder fixed point theorem (theorem 2.7), there exists $y \in C$ such that

$$y = L(y) \Rightarrow y = x(y) \Rightarrow y = N(y)G(y).$$

The following result can be viewed as the companion Leray-Schauder alternative of Theorem 4.1.

Theorem 4.2. *Let X be a generalized Banach algebra space, $0 \in U$ be an open bounded and convex subset of X, let $N : X \to X$ be a contractive operator and $G : \overline{U} \to X$ be a completely continuous map. Suppose that condition (i) of Theorem 4.1 is satisfied and also that $M_* \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix converging to zero, defined as in (4.1), where*

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \sup\{\|G(x)\| : x \in \overline{U}\},$$

and suppose also that

I) $\left(\frac{I_X}{N}\right)^{-1}$ *exists on* $G(\overline{U})$, I_X *being the identity operator on* X, *and the operator* $\frac{I_X}{N} : X \to X$ *being defined by* $\left(\frac{I_X}{N}\right)(x) = \frac{x}{N(x)}$.

Then either

(a) *the equation* $x = N(x)G(x)$ *has a solution in* $\overline{U}$, *or*

(b) *there exists a point* $x \in \overline{U}\backslash U$ *such that* $x = \lambda N(\frac{x}{\lambda})G(x)$ *for some* $\lambda \in (0,1)$.

Proof. For every $y \in \overline{U}$, we consider $N_y : X \to X$ by

$$N_y(x) = N(x)G(y).$$

As in theorem 4.1, we prove that there exists a unique $x(y) \in X$ such that

$$x(y) = N(x(y))G(y).$$

As a result,

$$\frac{x(y)}{N(x(y))} = G(y).$$

Next, we show that

$$\left(\frac{I_X}{N}\right)^{-1} : G(\overline{U}) \to X$$

is continuous. Let $(x_p)_{p\in\mathbb{N}}$ be a sequence converging to $x \in G(\overline{U})$. For each $p \in \mathbb{N}$, we set

$$\left(\frac{I_X}{N}\right)^{-1}(x_p) = y_p \quad \text{and} \quad \left(\frac{I_X}{N}\right)^{-1}(x) = y.$$

Then

$$y_p = N(y_p)x_p, \text{ for all } p \in \mathbb{N}, \text{ and } y = N(y)x.$$

Hence

$$\begin{aligned} \|y_p - y\| &= \|N(y_p)x_p - N(y)x\| \\ &\leq \|N(y_p) - N(y)\|\|x_p\| + \|N(y)\|\|x_p - x\| \\ &\leq M\|y_p - y\|\|x_p\| + \|N(y)\|\|x_p - x\| \\ &\leq M_*\|y_p - y\| + \|N(y)\|\|x_p - x\|. \end{aligned}$$

Therefore

$$\|y_p - y\| \leq (I - M_*)^{-1}\|N(y)\|\|x_p - x\| \to 0 \text{ as } p \to \infty.$$

Now, we consider $N_* = \left(\frac{I_X}{N}\right)^{-1} \circ G : \overline{U} \to X$, which is a completely continuous operator. Then, from Theorem 2.8, either N_* has a fixed point in $\overline{U}$, so the conclusion (a) holds, or there exists $x \in \partial U$ such that $x = \lambda N_*(x)$ for some $\lambda \in (0,1)$, this implies that the condition (b) holds.

Now, we formulate the Krasnosels'kĭi fixed point theorem type in generalized Banach algebras.

Theorem 4.3. *Let* X *be a generalized Banach algebra,* C *be a nonempty closed convex subset of* X *and let* $N, G, K : C \to X$ *be three operators such that*

(i) N *and* K *are* M_1, M_2-*contractive respectively, where* $M_1, M_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_+), M_1 = (a_{ij})_{1\le i,j\le n},\ M_2 = (\bar{a}_{ij})_{1\le i,j\le n}$.

(ii) $\left(\frac{I_X}{N}\right)^{-1}$ *exists on* $G(C)$, I_X *being the identity operator on* X, *and the operator* $\frac{I_X}{N} : X \to X$ *being defined by* $\left(\frac{I_X}{N}\right)(x) = \frac{x}{N(x)}$.

(iii) G *is completely continuous.*

(iv) $\left(\frac{I_X}{N}\right)^{-1}(I_X - K)^{-1}$ *maps* $G(C)$ *into* C.

Then the operator equation

$$N(x)G(x) + K(x) = x$$

has a solution in C, *whenever*

$$M_* := \begin{pmatrix} b_1a_{11} & \cdots & b_na_{1n} \\ \vdots & \ddots & \vdots \\ b_1a_{n1} & \cdots & b_na_{nn} \end{pmatrix} + \begin{pmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \cdots & \bar{a}_{nn} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

is a matrix converging to zero, where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \ge \sup\{\|G(x)\| : x \in C\}. \tag{4.2}$$

Proof. For every $y \in C$, we consider $N_y : X \to X$ by

$$N_y(x) = N(x)G(y) + K(y).$$

As in theorem 4.1, we prove that there exists a unique $x(y) \in X$ such that

$$x(y) = N(x(y))G(y) + K(y).$$

As a result,

$$\frac{x(y) - K(y)}{N(x(y))} = G(y).$$

From hypothesis (i), it follows that K is an M_2-contractive operator and so by Theorem 2.6, $(I_X - K)^{-1}$ exists on X. Again, the operator $\left(\frac{I_X}{N}\right)^{-1}$ exists in view of hypothesis (ii). Therefore,

$$\left(\frac{I_X - K}{N}\right)^{-1} = \left(\frac{I_X}{N}\right)^{-1}(I_X - K)^{-1}.$$

Next, we show that

$$\left(\frac{I_X - K}{N}\right)^{-1} : G(C) \to X$$

is continuous. Let $(x_p)_{p\in\mathbb{N}}$ be a sequence converging to $x \in G(C)$. For each $p \in \mathbb{N}$, we set

$$\left(\frac{I_X - K}{N}\right)^{-1}(x_p) = y_p,$$

and

$$\left(\frac{I_X - K}{N}\right)^{-1}(x) = y.$$

Then

$$y_p = N(y_p)x_p + K(y_p), \text{ for all } p \in \mathbb{N}, \text{ and } y = N(y)x + K(y).$$

Thus

$$\begin{aligned}
\|y_p - y\| &\leq \|N(y_p)x_p - N(y)x\| + \|K(y_p) + K(y)\| \\
&\leq \|N(y_p) - N(y)\|\|x_p\| + \|N(y)\|\|x_p - x\| + M_2\|y_p - y\| \\
&\leq M_1\|y_p - y\|\|x_p\| + \|N(y)\|\|x_p - x\| + M_2\|y_p - y\| \\
&\leq M_*\|y_p - y\| + \|N(y)\|\|x_p - x\|.
\end{aligned}$$

Therefore,

$$\|y_p - y\| \leq (I - M_*)^{-1}\|N(y)\|\|x_p - x\| \to 0 \text{ as } p \to \infty.$$

Consider $N_* = \left(\frac{I_X - K}{N}\right)^{-1} \circ G : C \to X$, which is a completely continuous operator. Now we show that $N_*(C) \subseteq C$. Indeed, let $x \in C$, then

$$N_*(x) = \left(\left(\frac{I_X - K}{N}\right)^{-1} \circ G\right)(x),$$

or, equivalently,

$$N_*(x) = N(N_*(x))G(x) + K(N_*(x)).$$

By (iv), we deduce that $N_*(x) \in C$. Hence, $N_*(C) \subset C$. Then, from Theorem 2.7, N_* has a fixed point in C, which is a solution to the operator equation

$$x = N(x)G(x) + K(x).$$

A Leray-Schauder type fixed point theorem for the sum of two operators is obtained as follows.

Theorem 4.4. *Let X be a generalized Banach algebra space, $0 \in U$ be an open bounded and convex subset of X and let $N, K : X \to X$ be two contractive operators, $G : \overline{U} \to X$ be a completely continuous map. Suppose that the condition (i) of Theorem 4.3 holds, and assume also that*

I) $\left(\frac{I_X}{N}\right)^{-1}$ *exists on $G(\overline{U})$, I_X being the identity operator on X, and the operator $\frac{I_X}{N} : X \to X$ being defined by $\left(\frac{I_X}{N}\right)(x) = \frac{x}{N(x)}$.*

Suppose that the matrix M_* given in the statement of Theorem 4.3 is converging to zero, where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \sup\{\|G(x)\| : x \in \overline{U}\}.$$

Then either:

($\bar{a}$) *the equation $x = N(x)G(x) + K(x)$ has a solution in $\overline{U}$, or*

($\bar{b}$) *there exists a point $x \in \overline{U} \backslash U$ such that*

$$x = \lambda N\left(\frac{x}{\lambda}\right) G(x) + \lambda K\left(\frac{x}{\lambda}\right) \quad \text{for some } \lambda \in (0,1).$$

Proof. As in the proof of Theorem 4.3, we prove that the operator

$$N_* = \left(\frac{I_X - K}{N}\right)^{-1} \circ G : \overline{U} \to X$$

is completely continuous. Then from Theorem 2.8, either N_* has a fixed point in $\overline{U}$, so the conclusion ($\bar{a}$) holds, or there exists $x \in \partial U$ such that $x = \lambda N_*(x)$ for some $\lambda \in (0,1)$, which implies that the condition ($\bar{b}$) holds.

Analogously to Theorem 4.3 or Theorem 4.4, it is straightforward to prove the following global singlevalued version of Krasnosel'skii's fixed point theorem.

Theorem 4.5. *Let X be a generalized Banach algebra and let $N, G, K : X \to X$ be three operators such that the condition (i) of Theorem 4.3 is satisfied, and assume also that*

(ii) $\left(\frac{I_X}{N}\right)^{-1}$ *exists, I_X being the identity operator on X, and the operator $\frac{I_X}{N} : X \to X$ being defined by $\left(\frac{I_X}{N}\right)(x) = \frac{x}{N(x)}$.*

(iii) *G is completely continuous.*

Suppose also that the matrix M_* given in the statement of Theorem 4.3 is converging to zero, where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \sup\{\|G(x)\| : x \in X\}.$$

Then, either

(a) $x = \lambda N\left(\frac{x}{\lambda}\right) G(x) + \lambda K\left(\frac{x}{\lambda}\right)$ *has a solution for $\lambda = 1$,*

or

(b) *the set $\left\{x \in X : x = \lambda N\left(\frac{x}{\lambda}\right) G(x) + \lambda K\left(\frac{x}{\lambda}\right),\ \lambda \in (0,1)\right\}$ is unbounded.*

5. Multivalued fixed point theorems in Banach algebras

Before proving our main hybrid fixed point theorems for multi-valued operators in generalized Banach algebras, we give the following auxiliary result.

Theorem 5.1. [27] *Let X be a generalized Banach space and Y be a generalized metric space. Assume that $F : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is a multivalued map and that there exists $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ a matrix converging to zero such that*

$$H_d(F(x_1, y), F(x_2, y)) \leq M\|x_1 - x_2\| \quad \text{for each } x_1, x_2 \in X,\ y \in Y,$$

and for every $y \in Y$ the multifunction $F(x, \cdot)$ is l.s.c. Then there exists a continuous mapping $f : X \times Y \to Y$ such that $f(x, y) \in P_F(x)$ for every $(x, y) \in X \times Y$, where

$$P_F(x) = \{y \in Y : y \in F(x, y)\}.$$

Theorem 5.2. *Let X be a generalized Banach algebra space and let $N : X \to \mathcal{P}_{cv,cl,b}(X)$, $G : X \to \mathcal{P}_{cp}(X)$ be two multivalued operators such that*

(i) *N is an $M-$contraction, where $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+), M = (a_{ij})_{1\leq i,j\leq n}$.*
(ii) *G is l.s.c. and compact.*

Suppose that

$$M_* := \begin{pmatrix} b_1a_{11} & \cdots & b_na_{1n} \\ \vdots & \ddots & \vdots \\ b_1a_{n1} & \cdots & b_na_{nn} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

is a matrix converging to zero, where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \sup\{\|G(x)\| : x \in X\}.$$

Moreover, assume that the set

$$\mathcal{E} = \{x \in X : x \in \lambda N(x)G(x), 0 \leq \lambda < 1\}$$

is bounded. Then there exists $x \in X$ such that $x \in N(x)G(x)$.

Proof. We consider the following multivalued operator $N_* : X \times X \to \mathcal{P}_{cl,b}(X)$ by

$$N_*(x, y) = N(x)G(y), \quad \text{for all } x, y \in X.$$

For each fixed $y \in X$, the multivalued operator $N_*^y(\cdot) = N_*(\cdot, y)$ is an M_*-contraction. Indeed, let $x, \bar{x} \in X$, then

$$\begin{aligned} H_d(N_*^y(x), N_*^y(\bar{x})) &= H_d(N_*(x, y), N_*(\bar{x}, y)) \\ &= H_d(N(x)G(y), N(\bar{x})G(y)) \\ &\leq H_d(N(x), N(\bar{x}))H_d(0, G(y)) \\ &\leq M_*\|x - \bar{x}\|. \end{aligned}$$

Since M_* is a matrix converging to zero, by Theorem 2.13, there exists $x(y) \in X$ such that $x(y) \in N(x(y))G(y)$. Hence,

$$Fix(N_*^y) = \{x \in X : x \in N(x)G(y)\}$$

is nonempty and closed subset of X for each $y \in X$. Now, the operator N_* satisfies all the conditions of Theorem 5.1, this implies that there exists a continuous mapping $f : X \times X \to X$ such that

$$f(x, y) \in N(x)G(y) \quad \text{for all } x, y \in X.$$

Let us consider the single-valued operator $A : X \to X$ defined by

$$A(x) = f(x, x), \quad \text{for each } x \in X.$$

Then, A is continuous. By the compactness of G, we can easily prove that A is completely continuous. Since

$$\mathcal{E}_* = \{x \in X : x = \lambda A(x), \quad \lambda \in (0, 1)\} \subseteq \mathcal{E}.$$

Then from Theorem 2.9 either A has a fixed point in X, or the set

$$\mathcal{E}_* = \{x \in X : x = \lambda A(x), \quad \lambda \in (0, 1)\}$$

is unbounded, which implies that $\mathcal{E}$ is unbounded.

Theorem 5.3. *Let X be a generalized Banach algebra, C be a closed convex and bounded subset of X and let $N, G : X \to \mathcal{P}_{cl,cv,b}(X)$ and $K : C \to \mathcal{P}_{cp,cv}(X)$ be three multi-valued operators such that*

(i) *There exist $M_1, M_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_+), M_1 = (a_{ij})_{1\le i,j\le n}, M_2 = (\bar{a}_{ij})_{1\le i,j\le n}$ such that N and K are multi-valued M_1, M_2-Lipschitz operators, respectively.*

(ii) *G is l.s.c. and compact.*

(iii) *$N(x)G(y) + K(x) \in \mathcal{P}_{cl,cv}(C)$ for all $x, y \in C$ and $M_* + M_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix converging to zero, where*

$$M_* := \begin{pmatrix} b_1a_{11} & \cdots & b_na_{1n} \\ \vdots & \ddots & \vdots \\ b_1a_{n1} & \cdots & b_na_{nn} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

and

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \ge \sup\{\|G(x)\|_{\mathcal{P}} : x \in C\}.$$

Then there exists $x \in C$ such that

$$x \in N(x)G(x) + K(x).$$

Proof. Consider the following operator $N_* : C \times C \to \mathcal{P}_{cl,cv}(C)$ defined by

$$N_*(x, y) = N(x)G(y) + K(x), \quad x, y \in C.$$

We show that, for every $y \in C$, the operator $N_*(\cdot, y)$ is a contraction. Indeed, let $x, \bar{x} \in C$, then

$$\begin{aligned} H_d(N_*(x, y), N_*(\bar{x}, y)) &= H_d(N(x)G(y) + K(x), N(\bar{x})G(y) + K(\bar{x})) \\ &\le H_d(N(x), N(\bar{x}))H_d(0, G(y)) + H_d(K(x), K(\bar{x})) \\ &\le (M_* + M_2)\|x - \bar{x}\|. \end{aligned}$$

Since $M_* + M_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix converging to zero, by Theorem 2.13, there exists $x(y) \in X$ such that

$$x(y) \in N(x(y))G(y) + K(x(y)).$$

Hence,

$$Fix(N_*(\cdot, y)) = \{x \in C : x \in N(x)G(y) + K(x)\}$$

is a nonempty and closed subset of C for each $y \in C$. Now, the operator N_* satisfies all the conditions of Theorem 5.1, this implies that there exists a continuous mapping $f : X \times X \to X$ such that

$$f(x, y) \in N(x)G(y) + K(x) \quad \text{for all } x, y \in C.$$

Let us consider the single-valued operator $A : C \to C$ defined by

$$A(x) = f(x, x), \quad \text{for each } x \in X.$$

Then, A is continuous. By the compactness of G, we can easily prove that A is completely continuous. Then, from Theorem 2.7 the operator A has a fixed point in C, which is a solution to the following operator equation

$$x \in N(x)G(x) + K(x).$$

Next, we present our main result when G is *u.s.c.*

Theorem 5.4. *Let C be a closed, convex and bounded subset of the generalized Banach algebra X and let $G : C \to \mathcal{P}_{cp,cv}(X)$ be a compact u.s.c. multi-valued map, $N : C \to X$ operators satisfying that:*

i) $\left(\frac{I_X}{N}\right)^{-1}$ *exists on $G(C)$, I_X being the identity operator on X, and the operator* $\frac{I_X}{N} : X \to X$ *being defined by* $\left(\frac{I_X}{N}\right)(x) = \frac{x}{N(x)}$,

and $K : C \to X$ be an $M-$contraction map such that

$$N(x)G(y) + K(x) \in \mathcal{P}_{cl,cv}(C), \text{ for all } x, y \in C.$$

Then there exists $x \in C$ such that

$$x \in N(x)G(x) + K(x).$$

Proof. Since $(I - K)^{-1}$ exists and, by i), we deduce that $\frac{I-K}{N}$ has inverse. Then, the multivalued operator $\left(\frac{I-K}{N}\right)^{-1} \circ G : C \to \mathcal{P}_{cp,cv}(C)$ is *u.s.c.* and compact. From Theorem 2.15, the operator $\left(\frac{I-K}{N}\right)^{-1} \circ G$ has at least one fixed point. So $NG + K$ has the above-mentioned property.

6. Integral equations of fractional order

In this section, we will work in the Banach algebra $C([0, 1], \mathbb{R})$. The object of our study is the following system of integral equations:

$$\begin{cases} x(t) = f_1(t, x(t), y(t)) \left(x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V_1(s, x(s), y(s)) ds\right) \\ \qquad +p_1(t, x(t), y(t)), \qquad t \in J, \\ y(t) = f_2(t, x(t), y(t)) \left(y_0(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} V_2(s, x(s), y(s)) ds\right) \\ \qquad +p_2(t, x(t), y(t)), \qquad t \in J, \end{cases} \tag{6.1}$$

where $J := [0,1]$, $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $p_1, p_2, V_1, V_2 \colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\alpha, \beta, \in (0,1)$ and $\Gamma(\cdot)$ denotes the Gamma function. Observe that the above equations can be written in the form

$$(x,y) = N(x,y)G(x,y) + K(x,y),$$

where

$$N(x,y) = (N_1(x,y), N_2(x,y)),$$

and N_1, N_2 are the superposition operators defined by

$$[N_1(x,y)](t) = f_1(t,x(t),y(t)),\ [N_2(x,y)](t) = f_2(t,x(t),y(t))),\ t \in J,$$

G is the Volterra integral operator of fractional order having the form

$$G(x,y) = (G_1(x,y), G_2(x,y)),$$

where

$$[G_1(x,y)](t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{V_1(s,x(s),y(s))}{(t-s)^{1-\alpha}} ds,\ t \in J,$$

and

$$[G_2(x,y)](t) = y_0(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{V_2(s,x(s),y(s))}{(t-s)^{1-\beta}} ds,\ t \in J,$$

and

$$K(x,y) = (K_1(x,y), K_2(x,y)),$$

where

$$[K_1(x,y)](t) = p_1(t,x(t),y(t)),\ [K_2(x,y)](t) = p_2(t,x(t),y(t))),\ t \in J.$$

In what follows, we assume that the following conditions are satisfied:

(H_1) There exist a_1, a_2, a_3, a_4 four nonnegative real numbers such that

$$|f_1(t,u,v) - f_1(t,\bar{u},\bar{v})| \leq \min\{a_1|u-\bar{u}| + a_2|v-\bar{v}|, \sqrt{a_1|u-\bar{u}| + a_2|v-\bar{v}|}\},$$

and

$$|f_2(t,u,v) - f_2(t,\bar{u},\bar{v})| \leq \min\{a_3|u-\bar{u}| + a_4|v-\bar{v}|, \sqrt{a_3|u-\bar{u}| + a_4|v-\bar{v}|}\},$$

for each $t \in J$ and $\forall (u,v), (\bar{u},\bar{v}) \in \mathbb{R} \times \mathbb{R}$.

(H_2) There exist nonnegative real constants $\bar{a}_1$, $\bar{a}_2, \bar{a}_3, \bar{a}_4 \geq 0$ such that

$$|p_1(t,u,v) - p_1(t,\bar{u},\bar{v})| \leq \min\{\bar{a}_1|u-\bar{u}| + \bar{a}_2|v-\bar{v}|, \sqrt{\bar{a}_1|u-\bar{u}| + \bar{a}_2|v-\bar{v}|}\},$$

and

$$|p_2(t,u,v) - p_2(t,\bar{u},\bar{v})| \leq \min\{\bar{a}_3|u-\bar{u}| + \bar{a}_4|v-\bar{v}|, \sqrt{\bar{a}_3|u-\bar{u}| + \bar{a}_4|v-\bar{v}|}\},$$

for each $t \in J$ and $\forall (u,v), (\bar{u},\bar{v}) \in \mathbb{R} \times \mathbb{R}$.

(H_3) There exist c_1, c_2, c_3, c_4 four nonnegative real numbers such that

$$|V_1(t,u,v)| \leq c_1\sqrt{|u| + |v|} + c_2,$$

and

$$|V_2(t,u,v)| \leq c_3\sqrt{|u| + |v|} + c_4,$$

for each $t \in J$ and $\forall (u,v), (\bar{u},\bar{v}) \in \mathbb{R} \times \mathbb{R}$.

Let $\mathcal{N} : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$ be the operator defined by

$$\mathcal{N}(x,y) = N(x,y)G(x,y) + K(x,y), \quad x,y \in C(J,\mathbb{R}),$$

where N, G and K are defined above. We recall Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 of [19].

Lemma 6.1. *Let $v : [0,b] \to [0,\infty)$ be a real function, and $w(\cdot)$ be a nonnegative, locally integrable function on $[0,b]$. Assume that there are constants $a > 0$ and $0 < \gamma < 1$ such that*

$$v(t) \leq w(t) + a\int_0^t \frac{v(s)}{(t-s)^\gamma} ds,$$

then, there exists a constant $K = K(\gamma) > 0$ such that

$$v(t) \leq w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^\gamma} ds,$$

for every $t \in [0,b]$.

Before giving the main result of this section, we prove some lemmas useful in the sequel.

Lemma 6.2. *Suppose the assumptions $(H_1) - (H_2)$, and assume that*

$$M_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ M_2 = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{a}_3 & \bar{a}_4 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+) \tag{6.2}$$

are two matrices converging to zero. Then N and K are contraction operators and G is completely continuous.

Proof. The proof will be given in several steps. We first show that N and K are contraction operators and, then, we check that G is completely continuous:

- **Step 1.** $N(\cdot,\cdot) = (N_1(\cdot,\cdot), N_2(\cdot,\cdot))$ and $K(\cdot,\cdot) = (K_1(\cdot,\cdot), K_2(\cdot,\cdot))$ are contractive.
  Indeed, let $(x,y), (\widetilde{x},\widetilde{y}) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$, then

  $$\begin{aligned} |[N_1(x,y)](t) - [N_1(\widetilde{x},\widetilde{y})](t)| &= |f_1(t,x(t),y(t)) - f_1(t,\widetilde{x}(t),\widetilde{y}(t))| \\ &\leq a_1\|x-\widetilde{x}\|_\infty + a_2\|y-\widetilde{y}\|_\infty. \end{aligned}$$

  Then

  $$\|N_1(x,y) - N_1(\widetilde{x},\widetilde{y})\|_\infty \leq a_1\|x-\widetilde{x}\|_\infty + a_2\|y-\widetilde{y}\|_\infty.$$

  Similarly, we obtain

  $$\|N_2(x,y) - N_2(\widetilde{x},\widetilde{y})\|_\infty \leq a_3\|x-\widetilde{x}\|_\infty + a_4\|y-\widetilde{y}\|_\infty.$$

  Hence,

  $$d(N(x,y), N(\widetilde{x},\widetilde{y})) \leq M_1 d((x,y),(\widetilde{x},\widetilde{y})),$$

  where

  $$d((x,y),(\widetilde{x},\widetilde{y})) = \begin{pmatrix} \|x-\widetilde{x}\|_\infty \\ \|y-\widetilde{y}\|_\infty \end{pmatrix},$$

  and

  $$M_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Besides,

$$\|K_1(x,y)-K_1(\widetilde{x},\widetilde{y})\|_\infty \le \bar{a}_1\|x-\widetilde{x}\|_\infty + \bar{a}_2\|y-\widetilde{y}\|_\infty.$$

Similarly, we obtain

$$\|K_2(x,y)-K_2(\widetilde{x},\widetilde{y})\|_\infty \le \bar{a}_3\|x-\widetilde{x}\|_\infty + \bar{a}_4\|y-\widetilde{y}\|_\infty.$$

Hence

$$d(K(x,y),K(\widetilde{x},\widetilde{y})) \le M_2 d((x,y),(\widetilde{x},\widetilde{y})).$$

- **Step 2.** $G(\cdot,\cdot)=(G_1(\cdot,\cdot),G_2(\cdot,\cdot))$ is continuous.
Let (x_n,y_n) be a sequence such that $(x_n,y_n)\to(x,y)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$ as $n\to\infty$. Then

$$\|G_1(x_n,y_n)-G_1(x,y)\|_\infty \le \frac{1}{\Gamma(\alpha+1)}\|V_1(\cdot,x_n(\cdot),y_n(\cdot))-V_1(\cdot,x(\cdot),y(\cdot))\|_\infty.$$

By the continuity of V_1, it is possible to prove that

$$\|G_1(x_n,y_n)-G_1(x,y)\|_\infty \to 0 \text{ as } n\to\infty.$$

Similarly,

$$\|G_2(x_n,y_n)-G_2(x,y)\|_\infty \le \frac{1}{\Gamma(\beta+1)}\|V_2(\cdot,x_n(\cdot),y_n(\cdot))-V_2(\cdot,x(\cdot),y(\cdot))\|_\infty.$$

Then, by the continuity of V_2,

$$\|G_2(x_n,y_n)-G_2(x,y)\|_\infty \to 0 \text{ as } n\to\infty.$$

Thus, G is continuous.

- **Step 3.** Since V_1, V_2, x_0 and y_0 are continuous functions, then we can easily prove that, for each bounded set $B_1\times B_2$ in $C(J,\mathbb{R})\times C(J,\mathbb{R})$, we have that $G(B_1\times B_2)$ is relatively compact.

We are now in the position to prove our main existence result for problem (6.1).

Lemma 6.3. *Assume that $(H_1)-(H_3)$ hold. Then there exists $K_*>0$ such that, for every $(x,y)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$ solution to the following system*

$$\begin{cases} x=\lambda N_1\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)G_1(x,y)+\lambda K_1\left(\frac{x}{\lambda},\frac{y}{\lambda}\right), \text{ for some } 0<\lambda<1,\\ \\ y=\lambda N_2\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)G_2(x,y)+\lambda K_2\left(\frac{x}{\lambda},\frac{y}{\lambda}\right), \text{ for some } 0<\lambda<1, \end{cases} \tag{6.3}$$

we have $\|x\|_\infty$, $\|y\|_\infty \le K_*$.

Proof. Let $(x,y)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$ be a solution to (6.3), then

$$x=\lambda N_1\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)G_1(x,y)+\lambda K_1\left(\frac{x}{\lambda},\frac{y}{\lambda}\right), \text{ for some } 0<\lambda<1,$$

and

$$y=\lambda N_2\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)G_2(x,y)+\lambda K_2\left(\frac{x}{\lambda},\frac{y}{\lambda}\right), \text{ for some } 0<\lambda<1.$$

We divide by λ both equations and seek for an estimate of $\|\frac{x}{\lambda}\|_\infty$ and $\|\frac{y}{\lambda}\|_\infty$. The estimate obtained would be also valid for $\|x\|_\infty$ and $\|y\|_\infty$. For simplicity, we rename $\frac{x}{\lambda}$ and $\frac{y}{\lambda}$ as x, y, respectively.

Thus

$$\begin{aligned}
|x(t)| &\leq |[N_1(x,y)](t)| \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |V_1(s, \lambda x(s), \lambda y(s))| \, ds \right) \\
&\quad + |[K_1(x,y)](t)| \\
&\leq |f_1(t, x(t), y(t)) - f_1(t,0,0)| \\
&\quad \times \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(c_1 \sqrt{\lambda |x(s)| + \lambda |y(s)|} + c_2 \right) ds \right) \\
&\quad + |f_1(t,0,0)| \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(c_1 \sqrt{\lambda |x(s)| + \lambda |y(s)|} + c_2 \right) ds \right) \\
&\quad + \sqrt{\bar{a}_1 |x(t)| + \bar{a}_2 |y(t)|} + |p_1(t,0,0)| \\
&\leq \sqrt{a_1 |x(t)| + a_2 |y(t)|} \\
&\quad \times \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(c_1 \sqrt{\lambda |x(s)| + \lambda |y(s)|} + c_2 \right) ds \right) \\
&\quad + |f_1(t,0,0)| \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(c_1 \sqrt{\lambda |x(s)| + \lambda |y(s)|} + c_2 \right) ds \right) \\
&\quad + \sqrt{\bar{a}_1 |x(t)| + \bar{a}_2 |y(t)|} + |p_1(t,0,0)|.
\end{aligned}$$

Hence, taking $c_* := \max\{a_1, a_2, \bar{a}_1, \bar{a}_2, c_1, c_3\}$, we have

$$\begin{aligned}
|x(t)| &\leq \sqrt{c_*(|x(t)| + |y(t)|)} \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_1 \sqrt{\lambda} \sqrt{|x(s)| + |y(s)|} ds \right. \\
&\quad \left. + \frac{c_2}{\Gamma(\alpha+1)} \right) \\
&\quad + |f_1(t,0,0)| \left(\|x_0\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_1 \sqrt{\lambda} \sqrt{|x(s)| + |y(s)|} ds + \frac{c_2}{\Gamma(\alpha+1)} \right) \\
&\quad + \sqrt{c_*(|x(t)| + |y(t)|)} + |p_1(t,0,0)|,
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &\leq \sqrt{c_*(|x(t)| + |y(t)|)} \left(\|y_0\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_3 \sqrt{\lambda} \sqrt{|x(s)| + |y(s)|} ds \right. \\
&\quad \left. + \frac{c_4}{\Gamma(\beta+1)} \right) \\
&\quad + |f_2(t,0,0)| \left(\|y_0\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_3 \sqrt{\lambda} \sqrt{|x(s)| + |y(s)|} ds + \frac{c_4}{\Gamma(\beta+1)} \right) \\
&\quad + \sqrt{c_*(|x(t)| + |y(t)|)} + |p_2(t,0,0)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}|x(t)|+|y(t)| &\leq \sqrt{c_*(|x(t)|+|y(t)|)}\left(c+\sqrt{\lambda}\,\mu\int_0^t(t-s)^{\gamma-1}\sqrt{|x(s)|+|y(s)|}ds\right)\\ &\quad+\widetilde{f}_\infty\left(c+\sqrt{\lambda}\mu\int_0^t(t-s)^{\gamma-1}\sqrt{|x(s)|+|y(s)|}ds\right)\\ &\quad+2\sqrt{c_*(|x(t)|+|y(t)|)}+\widetilde{K}_\infty,\end{aligned}$$

where

$$c:=\|x_0\|_\infty+\|y_0\|_\infty+\frac{c_2}{\Gamma(\alpha+1)}+\frac{c_4}{\Gamma(\beta+1)},\ \gamma:=\min(\alpha,\beta),\ \mu:=\frac{c_*}{\Gamma(\alpha)}+\frac{c_*}{\Gamma(\beta)},$$

and

$$\widetilde{f}_\infty:=\max\{\|f_1(\cdot,0,0)\|_\infty,\|f_2(\cdot,0,0)\|_\infty\},\ \widetilde{K}_\infty:=\|p_1(\cdot,0,0)\|_\infty+\|p_2(\cdot,0,0)\|_\infty.$$

If $\sqrt{|x(t)|+|y(t)|}>1$, then

$$\begin{aligned}\sqrt{|x(t)|+|y(t)|} &\leq \sqrt{c_*}\left(c+\sqrt{\lambda}\,\mu\int_0^t(t-s)^{\gamma-1}\sqrt{|x(s)|+|y(s)|}ds\right)\\ &\quad+\widetilde{f}_\infty\left(c+\sqrt{\lambda}\,\mu\int_0^t(t-s)^{\gamma-1}\sqrt{|x(s)|+|y(s)|}ds\right)\\ &\quad+2\sqrt{c_*}+\widetilde{K}_\infty.\end{aligned}$$

Hence

$$\begin{aligned}\sqrt{|x(t)|+|y(t)|} &\leq (\sqrt{c_*}+\widetilde{f}_\infty)c+2\sqrt{c_*}+\widetilde{K}_\infty\\ &\quad+\sqrt{\lambda}\mu(\sqrt{c_*}+\widetilde{f}_\infty)\int_0^t(t-s)^{\gamma-1}\sqrt{|x(s)|+|y(s)|}ds.\end{aligned}$$

By Lemma 6.1, there exists $K(\gamma)>0$ such that

$$\begin{aligned}\sqrt{|x(t)|+|y(t)|} &\leq (\sqrt{c_*}+\widetilde{f}_\infty)c+2\sqrt{c_*}+\widetilde{K}_\infty+K(\gamma)\sqrt{\lambda}\mu(\sqrt{c_*}+\widetilde{f}_\infty)\int_0^t(t-s)^{\gamma-1}ds\\ &=(\sqrt{c_*}+\widetilde{f}_\infty)c+2\sqrt{c_*}+\widetilde{K}_\infty+\frac{K(\gamma)}{\gamma}\sqrt{\lambda}\mu(\sqrt{c_*}+\widetilde{f}_\infty).\end{aligned}$$

Hence

$$|x(t)|+|y(t)| \leq \left[(\sqrt{c_*}+\widetilde{f}_\infty)c+2\sqrt{c_*}+\widetilde{K}_\infty+\frac{K(\gamma)}{\gamma}\mu(\sqrt{c_*}+\widetilde{f}_\infty)\right]^2=:K_*.$$

Consequently,

$$\|x\|_\infty\leq K_*\text{ and }\|y\|_\infty\leq K_*.$$

We are now in the position to prove our main existence result for problem (6.1).

Theorem 6.4. *Assume that the hypotheses $(H_1)-(H_3)$ hold and consider the matrix*

$$M_*:=\begin{pmatrix}b_1a_1 & b_2a_2\\ b_1a_3 & b_2a_4\end{pmatrix}+\begin{pmatrix}\bar{a}_1 & \bar{a}_2\\ \bar{a}_3 & \bar{a}_4\end{pmatrix}\in\mathcal{M}_{2\times2}(\mathbb{R}_+),$$

with

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \geq \sup\{\|G(x,y)\| : (x,y) \in \overline{U}\},$$

where

$$U = \{(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}) : \|x\|_\infty < K_* + 1,\ \|y\|_\infty < K_* + 1\},$$

$$b_1 = \|x_0\|_\infty + \frac{1}{\Gamma(\alpha+1)}\left(c_1\sqrt{2(K_*+1)} + c_2\right),$$

$$b_2 = \|y_0\|_\infty + \frac{1}{\Gamma(\beta+1)}\left(c_3\sqrt{2(K_*+1)} + c_4\right),$$

and K_* is defined in Lemma 6.3. If the matrix M_* converges to zero, then the problem (6.1) *has a solution in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$.*

Proof. Let $\mathcal{N} : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$ be the operator defined by

$$\mathcal{N}(x,y) = N(x,y)G(x,y) + K(x,y), \quad (x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}),$$

where N, G and K are defined above.

We now show that all the conditions of Theorem 4.4 are satisfied.

At first, let us observe that, from lemma 6.2, the operators N and K are contraction operators and $G : \overline{U} \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is completely continuous.
Let $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ be a solution of

$$(x,y) = \lambda N\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) G(x,y) + \lambda K\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right), \quad \lambda \in (0,1),$$

then by lemma 6.3, we get

$$\|x\|_\infty \leq K_*,\ \|y\|_\infty \leq K_*.$$

As a consequence of Theorem 4.4 we deduce that $\mathcal{N}$ has a fixed point $t \to (x(t), y(t))$, which is a solution to the problem (6.1).

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