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LEFSCHETZ-TYPE FIXED POINT THEOREMS FOR SPHERIC MAPPINGS

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th birthday

Abstract. Deterministic as well as random Lefschetz-type fixed point theorems are formulated for multivalued spheric maps on various sorts of special retracts in a Euclidean space.
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1. Introduction

In 1957, B. O'Neill [14] constructed an example of a fixed point free multivalued map $\varphi\colon B^2 \multimap B^2$ which is continuous with values homotopically equivalent to S^1 (even homeomorphic to S^1). This example was a motivation for introducing the notion of spheric mappings (see e.g. [10]). Later on, this topic was studied in [5, 6, 9, 11, 12, 13], where some generalizations of the Brouwer fixed point theorem for spheric mappings were presented. In the present paper, the (deterministic) Lefschetz-type fixed point theorems are given for spheric mappings on approximative retracts, weak approximative retracts, and multiretracts. Based on our principle developed in [2], we will also present randomized versions of these Lefschetz-type results. In concluding remarks and comments, we still indicate some further generalizations and possibilities. Finally, we formulate three open problems.

2. Preliminaries

In this paper, all topological spaces are assumed to be metric. We shall use the Čech theory of homology with coefficients in the field of rationals $\mathbb{Q}$ (for more details, see [8]). A nonempty, compact space X is called *acyclic*, provided $H_0(X) = \mathbb{Q}$ and

$H_n(X) = 0$, for $n = 1, 2, \ldots$, where $H_k(X)$ denotes the k-th homology group. A space X is said to be *of a finite type*, provided $H_n(X) = 0$, for almost all n and $\dim H_n(X) < +\infty$, for every n, where $\dim H_n(X)$ stands for the dimension of a linear space $H_n(X)$.

Let X and Y be two compact spaces. A continuous map $p\colon Y \to X$ is called a *Vietoris map*, provided $p^{-1}(x)$ is an acyclic set, for every $x \in X$.

Proposition 2.1 (Vietoris)**.** *If $p\colon Y \to X$ is a Vietoris map, then the induced linear map $p_*\colon H(Y) \xrightarrow{\sim} H(X)$ is an isomorphism.*

For given two compact spaces X and Y, by $\varphi\colon X \multimap Y$, we shall denote (up to Section 4) a multivalued map which is upper semicontinuous with nonempty, compact values $\varphi(x)$, for every $x \in X$. By *upper semicontinuous* (u.s.c) map, we mean the one such that $\{x \in X \mid \varphi(x) \subset A\}$ is open in X, for every open $A \subset Y$, or equivalently, that $\{x \in X \mid \varphi(x) \cap A \neq \emptyset\}$ is closed in X, for every closed $A \subset Y$. A (u.s.c) mapping φ is called *acyclic*, provided $\varphi(x)$ is an acyclic set, for every $x \in X$. For more details, see [1, 8].

Let X be a compact and connected subset of the n-dimensional Euclidean space $\mathbb{R}^n$. In what follows, we shall consider the following classes of compact sets:

(i) *Approximative retracts* (cf. [4, 8])
X is called an *approximative retract* (written, $X \in APR$), provided there exists an open neighbourhood $\mathcal{U}$ of X in $\mathbb{R}^n$ such that, for every $\varepsilon > 0$, there is an ε-retraction $r_\varepsilon\colon \mathcal{U} \to X$, i.e. a continuous mapping such that $|x - r_\varepsilon(x)| < \varepsilon$, for every $x \in X$;

(ii) *Weak approximative retracts* (cf. [7])
X is called a *weak approximative retract* (written, $X \in WAPR$) if, for every $\varepsilon > 0$ there exists an open neighbourhood $\mathcal{U}_\varepsilon$ of X in $\mathbb{R}^n$ and an ε-retraction $r_\varepsilon\colon \mathcal{U}_\varepsilon \to X$;

(iii) *Multiretracts* (cf. [15])
X is called a *multiretract* (written, $X \in MR$), provided there exist an open neighbourhood of a finite type $\mathcal{U}$ of X in $\mathbb{R}^n$ and two mappings $r\colon \mathcal{U} \to X$ and $\varphi\colon X \multimap \mathcal{U}$, such that $r \circ \varphi = \mathrm{Id}_X$, where r is a continuous map and φ is an acyclic map.

Proposition 2.2. *If $X \in APR$ or $X \in MR$, then X is of a finite type.*

Proof. For the proof, in the case that $X \in APR$, see [8, p. 36], and the references therein.

Assume that $X \in MR$. Let $\mathcal{U}$, $\varphi\colon X \multimap \mathcal{U}$ and $r\colon \mathcal{U} \to X$ be chosen according to (iii). Since $r \circ \varphi = \mathrm{Id}_X$, we get that

$$r_* \circ \varphi_* = \mathrm{Id}_{H(X)},$$

and consequently φ_* is a monomorphism.

Thus, our claim for $X \in MR$ follows from the assumption that $\mathcal{U}$ is of a finite type. □

Let us note that if $X \in WAPR$, then X is not necessarily of a finite type. For example, the space called "eye of peacock" (see [4]) is a weak approximative retract, but it is not of a finite type.

It will be convenient to recall the following Lefschetz-type fixed point theorems for acyclic mappings.

Theorem 2.3. *Let $\varphi\colon X \multimap X$ be an acyclic mapping. If $X \in APR$ or $X \in WAPR$ and X is of a finite type or $X \in MR$, then the nontrivial Lefschetz number $\lambda(\varphi) \neq 0$ implies that $\operatorname{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}$ is nonempty set, i.e. $\operatorname{Fix}(\varphi) \neq \emptyset$.*

Proof. For the proof of Theorem 2.3, in the case when $X \in APR$ or $X \in WAPR$ and X is of a finite type, see [8]. □

The case $X \in MR$ was proved in [15].

3. Spheric mappings

Let X be a nonempty, compact subset of $\mathbb{R}^n$, $n \geq 2$. Then $\mathbb{R}^n \setminus X$ consists of two components, namely:

(i) an unbounded component $\mathcal{D}X$ of $\mathbb{R}^n \setminus X$, i.e. $x \in \mathcal{D}X$ if and only if, for every $r > 0$, there exists a continuous function $f\colon [0,1] \to \mathbb{R}^n \setminus X$ such that $f(0) = x$ and $\|f(1)\| > r$. Since X is assumed to be compact, $\mathcal{D}X$ is the unique unbounded component of $\mathbb{R}^n \setminus X$.

(ii) a union of all bounded components $\mathcal{B}X$ of $(\mathbb{R}^n \setminus \mathcal{D}X) \setminus X$.

Moreover, we let

$$\tilde{X} = \mathcal{B}X \cup X = \mathbb{R}^n \setminus \mathcal{D}X \quad \text{and} \quad \tilde{A} = \mathcal{B}A \cup A = \mathbb{R}^n \setminus \mathcal{D}A.$$

The following proposition is evident.

Proposition 3.1. *Let A and X be compact subsets of $\mathbb{R}^n$ such that $A \subset X$ and $\mathbb{R}^n \setminus X$ is connected. Then $\tilde{A} \subset X$.*

Observe also that if X is an acyclic set, then $\tilde{X} = X$.

Definition 3.2. Let X be a compact, nonempty subset of $\mathbb{R}^n$ and let $\varphi\colon X \multimap X$ be a u.s.c. mapping with nonempty, compact, connected values. We say that φ is a *spheric map*, provided:

(i) $\{x \in X \mid x \in \mathcal{B}\varphi(x)\}$ is an open subset of X,

(ii) the map $\tilde{\varphi}\colon X \multimap X$, where $\tilde{\varphi}(x) = \varphi(x) \cup \mathcal{B}\varphi(x)$, is a well defined acyclic mapping.

Evidently, any acyclic map is spheric. Note that the map $\tilde{\varphi}$ is automatically u.s.c. Moreover, if X is an acyclic set or $\mathbb{R}^n \setminus X$ is connected, then $\tilde{\varphi}$ is well defined.

Let $\varphi\colon X \multimap X$ be a spheric map. We let, for the induced linear maps

$$\varphi_*\colon H(X) \multimap H(X),$$

to be equal $\tilde{\varphi}_*$. If X is, additionally, of a finite type, then we define the Lefschetz number $\lambda(\varphi)$ of φ by putting

$$\lambda(\varphi) = \lambda(\tilde{\varphi}_*).$$

4. Lefschetz fixed point theorem for spheric mappings

The main aim of this note is to state the following theorem.

Theorem 4.1. *Assume that $X \in APR$ or $X \in WAPR$ and X is of a finite type or $X \in MR$. Assume, furthermore, that $\varphi\colon X \multimap X$ is a spheric map. Then the Lefschetz number $\lambda(\varphi)$ of φ is well defined and if $\lambda(\varphi) \neq 0$, then the set $\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}$ is nonempty, i.e. $\mathrm{Fix}(\varphi) \neq \emptyset$.*

Firstly, we will prove the following lemma.

Lemma 4.2. *Assume that $\varphi\colon X \multimap X$ is a spheric map on a compact nonempty set $X \subset \mathbb{R}^n$. If $\mathrm{Fix}(\tilde{\varphi}) := \{x \in X \mid x \in \tilde{\varphi}(x)\} \neq \emptyset$, then $\mathrm{Fix}(\varphi) \neq \emptyset$.*

Proof (of Lemma 4.2). There are two possibilities:

(i) the boundary ∂X of X in $\mathbb{R}^n$ is equal to X,
(ii) $\partial X \neq X$.

In the case (i), we have $\tilde{\varphi} = \varphi$. Then, our claim follows immediately (cf. Theorem 2.3).

For the proof of (ii), assume on the contrary that $\mathrm{Fix}(\varphi) = \emptyset$. Then $\mathrm{Fix}(\tilde{\varphi})$ is a closed subset of X, and in view of Definition 3.2(i), we deduce that $\mathrm{Fix}(\tilde{\varphi})$ is also an open subset of X. It is however a contradiction, because X is connected. □

Proof of Theorem 4.1. The Lefschetz number $\lambda(\varphi)$ of φ is well defined, because X under our consideration is of a finite type. Thus, the Lefschetz number $\lambda(\tilde{\varphi})$ is well defined, too. If $\lambda(\varphi) \neq 0$, then $\lambda(\tilde{\varphi}) \neq 0$ and, in view of Theorem 2.3, we get that $\mathrm{Fix}(\tilde{\varphi}) \neq \emptyset$. Our claim then already follows from Lemma 4.2. □

Corollary 4.3. *If we assume in Theorem 4.1 that X is additionally acyclic then, for any spheric map $\varphi\colon X \multimap X$, the set $\mathrm{Fix}(\varphi)$ is nonempty, i.e. $\mathrm{Fix}(\varphi) \neq \emptyset$.*

5. Randomized version of Theorem 4.1

In order to randomize Theorem 4.1 by means of our technique developed in [2] (see Proposition 5.5 below), we need to recall some basic facts from the measure theory for multivalued maps.

By a *measurable space*, we shall mean as usually the pair (Ω, Σ), where the set Ω is equipped with a σ-algebra Σ of subsets. We shall use $\mathbb{B}(X)$ to denote the Borel σ-algebra on X. The symbol $\Sigma \otimes \mathbb{B}(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathbb{B}(X)$.

Denoting, for $\varphi\colon X \multimap Y$, by

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\} \quad \text{and} \quad \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$$

the small and large counter-images of $B \subset Y$, we can define measurable multivalued maps as follows.

Definition 5.1. Let (Ω, Σ) be a measurable space and X be a separable metric space. A map $\varphi\colon \Omega \multimap X$ with closed values is called *measurable* if $\varphi^{-1}(B) \in \Sigma$, for each open $B \subset X$, or equivalently, if $\varphi_+^{-1}(B) \in \Sigma$, for each closed $B \subset X$.

Remark 5.2. Since, in our paper, X is always assumed to be compact and since for compact-valued maps $\varphi\colon \Omega \multimap X$ the notions of measurability and weak measurability coincide, we can alternatively define here measurable maps as those which are *weakly measurable*, i.e. if $\varphi_+^{-1}(B) \in \Sigma$, for each open $B \subset X$, or equivalently, if $\varphi^{-1}(B) \in \Sigma$, for each closed $B \subset X$. For more properties and details, see e.g. [1, 8].

The notions of a random operator and a random fixed point are essential in this section. In the sequel, Ω will be always a complete measure space and X be always a complete separable metric space. In particular, we can simply assume that X is compact as before.

Definition 5.3. (cf. [2]) Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is product-measurable (measurable in the whole), i.e. measurable w.r.t. minimal σ-algebra $\Sigma \otimes \mathcal{B}(X)$, generated by $\Sigma \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel sets of X. If $\varphi(\omega, \cdot)\colon A \multimap X$ is still u.s.c., then φ is called a *random u-operator*.

Definition 5.4. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a random operator. We say that φ has a *random fixed point* ξ if there exists a measurable mapping $\xi\colon \Omega \to A$ such that:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)), \quad \text{for almost every } \omega \in \Omega.$$

We let $\mathrm{Fix}^{ra}(\varphi) = \{\xi\colon \Omega \to A \mid \xi \text{ is a random fixed point for } \varphi\}$.

The following proposition is crucial in our application. Let us denote by φ_ω the one-parameter family of deterministic maps $\varphi_\omega\colon A \multimap X$ given by $\varphi_\omega(\cdot) := \varphi(\omega, \cdot)$, $\omega \in \Omega$, and by $\mathrm{Fix}(\varphi_\omega) = \{x \in A \mid x \in \varphi_\omega(x)\}$ the set of fixed points of φ_ω, $\omega \in \Omega$.

Proposition 5.5. (cf. [2, Corollary 2.12]) *Let $A \subset X$ be a closed subset of X and $\varphi\colon \Omega \times A \multimap X$ be a random operator. If the set $\mathrm{Fix}(\varphi_\omega)$ of fixed points of $\varphi_\omega\colon A \multimap X$ is nonempty, for almost all $\omega \in \Omega$, i.e. $\mathrm{Fix}(\varphi_\omega) \neq \emptyset$, for a.a. $\omega \in \Omega$, then φ admits a random fixed point, i.e. $\mathrm{Fix}^{ra}(\varphi) \neq \emptyset$.*

We are ready to formulate the main result.

Theorem 5.6. *Assume that $X \in APR$ or $X \in WAPR$ and X is of a finite type or $X \in MR$. Assume, furthermore, that $\varphi\colon \Omega \times X \multimap X$ is a random u-operator such that $\varphi_\omega\colon X \multimap X$ is a spheric map, for almost every $\omega \in \Omega$. If the Lefschetz number $\lambda(\varphi_\omega) \neq 0$, for almost every $\omega \in \Omega$ (observe that, according to Theorem 4.1, it is well defined), then the set $\mathrm{Fix}^{ra}(\varphi)$ of random fixed points is nonempty, i.e. $\mathrm{Fix}^{ra}(\varphi) \neq \emptyset$.*

Proof. In view of Theorem 4.1, $\mathrm{Fix}(\varphi_\omega) \neq \emptyset$, for almost all $\omega \in \Omega$. Applying Proposition 5.5, we get immediately that $\mathrm{Fix}^{ra}(\varphi) \neq \emptyset$. □

Remark 5.7. If we assume in Theorem 5.6 that X is additionally acyclic, then $\lambda(\varphi_\omega) \neq 0$ and, by means of Corollary 4.3, $\mathrm{Fix}(\varphi_\omega) \neq \emptyset$, for almost all $\omega \in \Omega$. Subsequently, $\mathrm{Fix}^{ra}(\varphi) \neq \emptyset$, as claimed.

Remark 5.8. For another randomized Lefschetz-type theorem, but for noncompact multivalued maps, see [3]. In this light (see also Theorem 2.3), Theorem 5.6 has especially meaning for random mappings φ such that φ_ω, $\omega \in \Omega$, is non-acyclic.

6. Some further possibilities

Finally, we shall discuss possible generalizations of results presented in Sections 4 and 5.

Firstly, we recall the notion of admissible mappings introduced in 1976 by the second author (see [7]).

A multivalued map $\varphi\colon X \multimap Y$ is called *admissible*, provided there exists a diagram

$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y$$

such that $\varphi(x) = q(p^{-1}(x))$, where p is a Vietories map and q is a continuous mapping.

Evidently, any acyclic map is admissible. Moreover, composition of admissible mappings is admissible, too; note that the same is not true for acyclic maps.

Now, we shall slightly generalize the notion of multiretracts. Let X be a compact connected subset of $\mathbb{R}^n$. We say that X is an *admissible multiretract* (written $X \in M_aR$, cf. [15]), provided there exists an open subset $\mathcal{U} \subset \mathbb{R}^n$ of a finite type and two maps $r\colon \mathcal{U} \to X$ and $\varphi\colon X \multimap \mathcal{U}$, such that $r \circ \varphi = \mathrm{Id}_X$, where r is a continuous map and φ is an admissible mapping.

Evidently, we have $MR \subset M_aR$. In what follows, we shall assume that X is a compact connected subset of $\mathbb{R}^n$.

We let

$$A(X) := \{\varphi\colon X \multimap X \mid \varphi \text{ is admissible}\}.$$

Definition 6.1. Let $\varphi\colon X \multimap X$ be a u.s.c. map with nonempty, compact, connected values. We say that φ is an *a-spheric map*, provided

(i) $\{x \in X \mid x \in \mathcal{B}\varphi(x)\}$ is an open subset of X,
(ii) the map $\tilde{\varphi}\colon X \multimap X$, $\tilde{\varphi}(x) = \varphi(x) \cup \mathcal{B}\varphi(x)$, is well defined and $\tilde{\varphi} \in A(X)$.

Evidently, every spheric map is a-spheric map. For numerous related examples, see [5, 6, 11, 12, 13]. Note that the notion of spheric mappings is defined in a rather geometrical way, while the class of a-spheric mappings is more abstract.

The following Lefschetz-type fixed point theorem for admissible mappings is well known (see e.g. [1, 3, 7, 8, 15]).

Theorem 6.2. *Assume that $X \in APR$ or $X \in WAPR$ and X is of a finite type or $X \in M_aR$. If $\varphi \in A(X)$, then the Lefschetz set $\mathbb{\Lambda}(\varphi)$ is well defined and if $\mathbb{\Lambda}(\varphi) \neq \{0\}$, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Let X be as in Theorem 6.2 and let $\varphi\colon X \multimap X$ be an a-spheric map. Observe that, for the map $\tilde{\varphi}\colon X \multimap X$ defined in Definition 6.1(ii), Theorem 6.2 holds true.

We let

$$\mathbb{\Lambda}(\varphi) = \mathbb{\Lambda}(\tilde{\varphi}). \tag{6.1}$$

In view of (6.1) and Theorem 6.2, we are able to prove the following generalizations of Theorem 4.1 and Theorem 5.6.

Theorem 6.3. *Let X be as in Theorem 6.2 and let $\varphi\colon X \multimap X$ be an a-spheric map. Then the Lefschetz set $\mathbb{\Lambda}(\varphi)$ of φ is well defined and if $\mathbb{\Lambda}(\varphi) \neq \{0\}$, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Remark 6.4. Note that Lemma 4.2 follows from Definition 6.1(i) and, therefore, it is still true for a-spheric mappings. Consequently, the proof of Theorem 6.3 is quite analogous to the one of Theorem 4.1.

We can also formulate the randomized version of Theorem 6.3.

Theorem 6.5. *Assume that X is the same as in Theorem 6.2. Assume, furthermore, that $\varphi\colon \Omega \times X \multimap X$ is a random u-operator such that $\varphi_\omega\colon X \multimap X$ is an a-spheric map, for almost all $\omega \in \Omega$. If the Lefschetz set $\mathbb{\Lambda}(\varphi_\omega) \neq \{0\}$, for almost every $\omega \in \Omega$, then the set* $\mathrm{Fix}^{ra}(\varphi)$ *of random fixed points is nonempty, i.e.* $\mathrm{Fix}^{ra}(\varphi) \neq \emptyset$.

Now, we shall indicate the second (alternative) possibility of generalizations of Theorem 4.1 and Theorem 5.6.

We let

$$A_{1,m}(X,Y) := \{\varphi\colon X \multimap Y \mid \varphi \text{ is continuous, i.e. both u.s.c. and l.s.c., and } \varphi(x) \text{ consists of 1 or } m \text{ acyclic components}\}. \quad (6.2)$$

If $Y = X$, then we shall use the following notations:

$$A_{1,m}(X) := A_{1,m}(X,X),$$

$$A_c(X) := \{\varphi\colon X \multimap X \mid \varphi = \varphi_k \circ \ldots \circ \varphi_1, \varphi_1\colon X \multimap X_1, \ldots, \varphi_k\colon X_{k-1} \multimap X \text{ and } \varphi_i \in A_{1,m}(X_{i-1}, X_i)\}.$$

Observe that if $\varphi \in A_c(X)$, then the induced homomorphism $\varphi_*\colon H(X) \to H(X)$ is well defined (see [8, Sections 50-54]). Consequently, if X is a compact set of a finite type and $\varphi \in A_c(X)$, then the Lefschetz number $\lambda(\varphi)$ of φ is well defined. Moreover, the following theorem is well known (see e.g. [9], and the references therein).

Theorem 6.6. *If X is an absolute neighbourhood retract of a finite type and $\varphi \in A_c(X)$ is a compact map, then $\lambda(\varphi) \neq 0$ implies that* $\mathrm{Fix}(\varphi) \neq \emptyset$.

We will prove a modification of Theorem 6.6 in the case when $X \in WAPR$ is of a finite type.

Theorem 6.7. *Assume that $X \in WAPR$ is of a finite type. If $\varphi \in A_c(X)$, then the Lefschetz number $\lambda(\varphi)$ of φ is well defined and $\lambda(\varphi) \neq 0$ implies that* $\mathrm{Fix}(\varphi) \neq \emptyset$.

For the proof of Theorem 6.7, we need the following proposition.

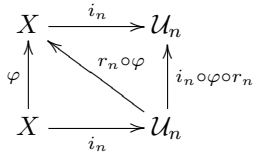
Proposition 6.8. (cf. [7, Theorem I.3.4]) *Let (X,d) be a compact space of a finite type. Then there exists an $\varepsilon > 0$ such that, for every two continuous maps $f, g\colon Y \to X$, where Y is an arbitrary metric space, the condition*

$$d(f(y), g(y)) < \varepsilon, \quad \text{for every } y \in Y,$$

implies $f_* = g_*$.

Proof of Theorem 6.7. Since $X \in WAPR$, for each $n = 1, 2, \ldots$, we have $r_n\colon \mathcal{U}_n \to X$ such that $\|x - r_n(x)\| < \frac{1}{n}$, for all $x \in X$, where $\mathcal{U}_n$ is an open subset of $\mathbb{R}^n$. Let $i_n\colon X \to \mathcal{U}_n$, $i_n(x) = x$, be the inclusion map.

We shall consider the following commutative diagram:



In view of Proposition 6.8, there exists n_0 such that, for every $n \geq n_0$, we get $r_{n_*} \circ i_{n_*} = (\mathrm{Id}_X)_*$. It follows that, for $n \geq n_0$, the above diagram is homologically commutative. Consequently, $\lambda(\varphi) = \lambda(i_n \circ \varphi \circ r_n)$, for all $n \geq n_0$.

Thus, using Theorem 6.6, for every $n \geq n_0$, we get $x_n \in (i_n \circ \varphi \circ r_n)(x)$. Therefore, $x_n \in X$ and $\|x_n - r_n(x_n)\| < \frac{1}{n}$, for every $n \geq n_0$. Hence, from the compactness of X, we deduce that

$$\lim_{m\to\infty} r_{n_m}(x_{n_m}) = \lim_{m\to\infty} x_{n_m} = x_0,$$

and since φ is u.s.c., we get $x_0 \in \varphi(x_0)$, provided $\lambda(\varphi) \neq 0$. □

Remark 6.9. Since every *APR*-set is of a finite type, Theorem 6.7 holds true for *APR*-sets.

Definition 6.10. Let X be a compact subset of $\mathbb{R}^n$ and $\varphi\colon X \multimap X$ be a u.s.c. map with nonempty compact values. We say that φ is a *c-spheric map*, provided

(i) $\{x \in X \mid x \in \mathcal{B}\varphi(x)\}$ is an open subset of X,
(ii) the map $\tilde{\varphi} \in A_c(X)$, and $\tilde{\varphi}\colon X \multimap X$, $\tilde{\varphi}(x) = \varphi(x) \cup \mathcal{B}\varphi(x)$, is well defined.

Putting

$$\lambda(\varphi) = \lambda(\tilde{\varphi}),$$

we can obtain the analogies of Theorem 4.1 and Theorem 5.6 in the same way as Theorem 6.3 and Theorem 6.5, but for $X \in APR$ or $X \in WAPR$ of a finite type.

7. Open problems

Open problem 7.1. Is it possible to prove Theorem 4.1 and Theorem 5.6 for $X \in M_aR$ or $X \in MR$ and $\varphi\colon X \multimap X$ to be a c-spheric map?

Open problem 7.2. Is the Lefschetz fixed point theorem true for $X \in APR$ or $X \in WAPR$ of a finite type and φ to be a finite composition of spheric maps or a-spheric maps or c-spheric maps?

Open problem 7.3. How to define spheric maps on subsets of arbitrary Banach spaces?

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