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COUPLED FIXED POINTS OF MONOTONE MAPPINGS IN A METRIC SPACE WITH A GRAPH

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Abstract. In this work, we define the concept of mixed G-monotone mappings defined on a metric space endowed with a graph. Then we obtain sufficient conditions for the existence of coupled fixed points for such mappings when a weak contractivity type condition is satisfied.
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1. Introduction

Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [14] who proved the following result:

Theorem 1.1. [14] *Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

(1) *There exists $k \in [0, 1)$ with*
$$d(f(x), f(y)) \leq k\ d(x, y), \text{ for all } x, y \in X \text{ such that } x \preceq y.$$
(2) *There exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$.*

Then f is a Picard Operator (PO), that is f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered metric spaces; see [2, 4, 7, 11] and references cited therein. Nieto, Pouso and Rodriguez-Lopez in [11] extended the ideas of [14] to prove the existence of solutions to some differential equations.

Generalizing the Banach contraction principle for multivalued mappings, Nadler [10] obtained the following result:

Theorem 1.2. *Let (X, d) be a complete metric space. Denote by $CB(X)$ the set of all nonempty closed bounded subsets of X. Let $F : X \to CB(X)$ be a multivalued mapping. If there exists $k \in [0, 1)$ such that*

$$H(F(x), F(y)) \leq k\ d(x, y)$$

for all $x, y \in X$, where H is the Pompeiu-Hausdorff metric on $CB(X)$, then F has a fixed point in X, i.e., there exists $x \in X$ such that $x \in F(x)$.

Recently, two results have appeared, giving sufficient conditions for f to be a PO, if (X, d) is endowed with a graph. The first result in this direction was given by Jachymski and Lukawska [8, 9] which generalized the results of [4, 11, 12, 13] to single-valued mapping in metric spaces with a graph instead of partial ordering. The extension of Jachymaski's result to multivalued mappings is done in [1].

It is well known that mixed monotone operators were initially considered by Guo and Lakshmikantham [6]. Thereafter, different authors considered the problem of existence of a fixed point for such mappings in Banach spaces and then in partially ordered metric spaces, see for instance [5, 15]. The mixed monotone operator equation is important for applications due to the existence of particular classes of integro-differential equations and boundary value problems that are solved by such equations [7].

The aim of this paper is two folds: first define the mixed G-monotone for both single and multivalued mappings, second extend the conclusion of Theorem 1.1 to both cases in metric spaces endowed with a graph.

2. Preliminaries

Let G be a directed graph (digraph) with set of vertices $V(G)$ and set of edges $E(G)$ contains all the loops, i.e. $(x, x) \in E(G)$ for any $x \in V(G)$. Such digraphs are called reflexive. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=0}^{i=N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected.

In the sequel, we assume that (X, d) is a metric space, and G is a reflexive digraph (digraph) with set of vertices $V(G) = X$ and set of edges $E(G)$.

Definition 2.1. Let (X, d, G) be as described above.

(i) We say that a mapping $F : X \times X \to X$ has the mixed G-monotone property if
$$(x_1, x_2) \in E(G) \Longrightarrow (F(x_1, y), F(x_2, y)) \in E(G),$$
for all $x_1, x_2, y \in X$, and
$$(y_1, y_2) \in E(G) \Longrightarrow (F(x, y_2), F(x, y_1)) \in E(G),$$
for all $x, y_1, y_2 \in X$.

(ii) The pair $(x, y) \in X \times X$ is called a coupled fixed point of $F : X \times X \to X$ if
$$F(x, y) = x, \ \ \textit{and} \ F(y, x) = y.$$

3. Main results

We begin with the extension of the main results of [3] to the case of metric spaces endowed with a graph. Note that if G is a directed graph defined on X as described before, one can construct another graph on $X \times X$, still denoted by G, by
$$\Big((x, y), (u, v)\Big) \in E(G) \Longleftrightarrow (x, u) \in E(G) \ \textit{and} \ (v, y) \in E(G),$$
for any $(x, y), (u, v) \in X \times X$.

Theorem 3.1. *Let (X, d, G) be as above. Assume that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed G-monotone property on X. Assume there exists $k < 1$ such that*

$$\text{(BL)} \qquad d(F(x, y), F(u, v)) \leq \frac{k}{2}\Big[d(x, u) + d(y, v)\Big],$$

for any $(x, y), (u, v) \in X \times X$ such that $\Big((x, y), (u, v)\Big) \in E(G)$. If there exist $x_0, y_0 \in X$ such that $\Big((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))\Big) \in E(G)$, then there exists (x, y) a coupled fixed point of F , i.e. $F(x, y) = x$ and $F(y, x) = y$.

Proof. By assumption, there exist $x_0, y_0 \in X$ such that
$$(x_0, F(x_0, y_0)) \in E(G) \ \ \textit{and} \ (F(y_0, x_0), y_0) \in E(G).$$
Set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $(x_0, x_1) \in E(G)$ and $(y_1, y_0) \in E(G)$, which implies
$$d\Big(F(x_0, y_0), F(x_1, y_1)\Big) \leq \frac{k}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$
and
$$d\Big(F(y_1, x_1), F(y_0, x_0)\Big) \leq \frac{k}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big].$$
By induction, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $x_{n+1} = F(x_n, y_n)$, and $y_{n+1} = F(y_n, x_n)$;

(ii) $d(x_n, x_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,

(iii) $d(y_n, y_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,

for any $n \geq 1$. From (ii) and (iii), we get

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq k \ \Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big],$$

for any $n \geq 1$. Therefore, we must have

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq k^n \ \Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

for any $n \geq 0$. Hence from (ii), we get

$$d(x_n, x_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big] \leq \frac{k}{2} \ k^{n-1} \ \Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

i.e., $d(x_n, x_{n+1}) \leq \frac{k^n}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big]$, for any $n \geq 0$. Similarly, we will get

$$d(y_n, y_{n+1}) \leq \frac{k^n}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

for any $n \geq 0$. Since $k < 1$, we conclude that $\sum d(x_n, x_{n+1})$ and $\sum d(y_n, y_{n+1})$ are convergent which imply that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since (X, d) is complete, there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} x_n = x \ and \ \lim_{n\to+\infty} y_n = y.$$

Since F is continuous, we get from (i) above

$$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} F(x_n, y_n) = F(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = F(x, y)$$

and similarly $y = F(y, x)$, i.e., (x, y) is a coupled fixed point of F. □

Example 3.1. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ and $F : X \times X \to X$ be defined by

$$F(x, y) = \frac{x + y}{5}, \ (x, y) \in X \times X.$$

Let G be the reflexive digraph defined on X with $((x, y), (u, v)) \in E(G)$ if and only if $x \leq u$ and $v \leq y$. Then F is mixed G-monotone and satisfies condition (BL). Indeed, let $k = \frac{2}{3}$ then

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left|\frac{(x + y)}{5} - \frac{(u + v)}{5}\right| = \left|\frac{(x - u)}{5} + \frac{(y - v)}{5}\right| \\ &\leq \frac{1}{5}(|x - u| + |y - v|) \leq \frac{1}{3}(|x - u| + |y - v|) \\ &= \frac{2/3}{2}\Big[d(x, u) + d(y, v)\Big], \end{aligned}$$

for any $(x, y), (u, v) \in X \times X$ such that $\Big((x, y), (u, v)\Big) \in E(G)$.
Notice that $((0, 0), (0, 0)) \in E(G)$. So by Theorem 3.1 we have that F has a coupled

fixed point $(0,0)$. To illustrate the proof of Theorem 3.1, let us consider

$$(x_0,y_0)=(0,1),\ F(0,1)=F(1,0)=\frac{1}{5}$$

(notice that $((0,1),(\frac{1}{5},\frac{1}{5}))\in E(G)$). Then $x_n=y_n=\frac{1}{5}(\frac{2}{5})^{n-1}\to 0$ as $n\to\infty$. Thus by Theorem 3.1 $(0,0)$ is a couple fixed point of F.

The continuity assumption of F may be relaxed as it was done by Nieto et al [11]. Indeed, we will say that (X,d,G) has property **(*)** if the following hold:

(i) for any $\{x_n\}$ in X such that $(x_n,x_{n+1})\in E(G)$ and $\lim\limits_{n\to+\infty}x_n=x$, then $(x_n,x)\in E(G)$, and

(ii) for any $\{x_n\}$ in X such that $(x_{n+1},x_n)\in E(G)$ and $\lim\limits_{n\to+\infty}x_n=x$, then $(x,x_n)\in E(G)$.

We have the following result.

Theorem 3.2. *Let (X,d,G) be as above. Assume that (X,d) is a complete metric space and (X,d,G) has property (*). Let $F:X\times X\to X$ be a mapping having the mixed G-monotone property on X. Assume there exists $k<1$ such that*

$$\text{(BL)}\qquad d(F(x,y),F(u,v))\le\frac{k}{2}\Big[d(x,u)+d(y,v)\Big],$$

for any $(x,y),(u,v)\in X\times X$ such that $\Big((x,y),(u,v)\Big)\in E(G)$. If there exist $x_0,y_0\in X$ such that $\Big((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))\Big)\in E(G)$, then there exist (x,y) a coupled fixed point of F.

Proof. As we did in the proof of Theorem 3.1, we construct $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $x_{n+1}=F(x_n,y_n)$, and $y_{n+1}=F(y_n,x_n)$;

(ii) $(x_n,x_{n+1})\in E(G)$ and $(y_{n+1},y_n)\in E(G)$;

(iii) $d(x_n,x_{n+1})\le\frac{k}{2}\Big[d(x_n,x_{n+1})+d(y_n,y_{n+1})\Big]$,

(iv) $d(y_n,y_{n+1})\le\frac{k}{2}\Big[d(x_n,x_{n+1})+d(y_n,y_{n+1})\Big]$,

for any $n\ge 0$. Similar to the proof of Theorem 3.1, we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since (X,d) is complete, then there exist $x,y\in X$ such that

$$\lim_{n\to+\infty}x_n=x\ \textit{and}\ \lim_{n\to+\infty}y_n=y.$$

The property **(*)** implies

$$(x_n,x)\in E(G)\ \textit{and}\ (y,y_n)\in E(G),$$

for any $n\ge 0$. Since F has the mixed G-monotone property on X, we get

$$d\Big(F(x_n,y_n),F(x,y)\Big)\le\frac{k}{2}\Big[d(x_n,x)+d(y_n,y)\Big],$$

and

$$d\Big(F(y_n,x_n),F(y,x)\Big)\le\frac{k}{2}\Big[d(x_n,x)+d(y_n,y)\Big],$$

for any $n \geq 0$. Hence

$$d\Big(x_{n+1}, F(x,y)\Big) \leq \frac{k}{2}\Big[d(x_n,x) + d(y_n,y)\Big],$$

and

$$d\Big(y_{n+1}, F(y,x)\Big) \leq \frac{k}{2}\Big[d(x_n,x) + d(y_n,y)\Big],$$

for any $n \geq 0$. This imply

$$\lim_{n\to+\infty} x_n = F(x,y) \textit{ and } \lim_{n\to+\infty} y_n = F(y,x),$$

i.e., $F(x,y) = x$ and $F(y,x) = y$. □

Under the assumptions of both Theorems 3.1 and 3.2, if assume that $(x_0,y_0) \in E(G)$, then we have $x = y$. Indeed, it is easy to see that for any $u, v \in X$ such that $(u,v) \in E(G)$, then the condition (BL) implies

$$d(F(u,v), F(v,u)) \leq k\ d(u,v).$$

This will imply that $d(x_{n+1}, y_{n+1}) \leq k\ d(x_n,y_n)$, for any $n \geq 0$. In particular, we have $d(x_n,y_n) \leq k^n\ d(x_0,y_0)$, for ay $n \geq 0$. Since $k < 1$, we conclude that

$$d(x,y) = \lim_{n\to+\infty} d(x_n,y_n) = 0, \quad \text{i.e.,} \quad x = y.$$

Remark 3.1. In this remark, we discuss the uniqueness of the coupled fixed point. Under the assumptions of both Theorems 3.1 and 3.2, let (x,y) and (u,v) be two coupled fixed points of F. Assume that $\Big((x,y),(u,v)\Big) \in E(G)$. Since F has the mixed G-monotone property on X, we get

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}\Big[d(x,u) + d(y,v)\Big],$$

and

$$d(F(v,u), F(y,x)) \leq \frac{k}{2}\Big[d(x,u) + d(y,v)\Big],$$

with $k < 1$. Since (x,y) and (u,v) are coupled fixed points of F, we get

$$d(x,u) \leq \frac{k}{2}\Big[d(x,u) + d(y,v)\Big], \textit{ and } d(y,v) \leq \frac{k}{2}\Big[d(x,u) + d(y,v)\Big],$$

which implies

$$d(x,u) + d(y,v) \leq k\ \Big(d(x,u) + d(y,v)\Big).$$

Hence $d(x,u) + d(y,v) = 0$, which yields $(x,y) = (u,v)$. Moreover assume that there exist $x_0, y_0 \in X$ such that $\Big((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))\Big) \in E(G)$. Let (u,v) be a coupled fixed point of F such that $\Big((x_0,y_0),(u,v)\Big) \in E(G)$, then

$$d(F(x_0,y_0), F(u,v)) = d(F(x_0,y_0), u) \leq \frac{k}{2}\Big[d(x_0,u) + d(y_0,v)\Big],$$

and

$$d(F(v,u), F(y_0,x_0)) = d(v, F(y_0,x_0)) \leq \frac{k}{2}\Big[d(x_0,u) + d(y_0,v)\Big],$$

since F has the mixed G-monotone property. If $\{x_n\}$ and $\{y_n\}$ are the two sequences generated by x_0, y_0, F in the proof of both Theorems 3.1 and 3.2, then we have

$$d(x_{n+1}, u) \leq \frac{k}{2}\Big[d(x_n, u) + d(y_n, v)\Big] \leq \frac{k^n}{2}\Big[d(x_0, u) + d(y_0, v)\Big],$$

and

$$d(v, y_{n+1}) \leq \frac{k}{2}\Big[d(x_n, u) + d(y_n, v)\Big] \leq \frac{k^n}{2}\Big[d(x_0, u) + d(y_0, v)\Big],$$

for any $n \geq 1$. Since $k < 1$, we get

$$\lim_{n\to+\infty} x_n = u \text{ and } \lim_{n\to+\infty} y_n = v.$$

Therefore given $x_0, y_0 \in X$ such that $\Big((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))\Big) \in E(G)$, there exists a unique coupled fixed point (x, y) of F such that $\Big((x_0, y_0), (x, y)\Big) \in E(G)$.

In the next section we discuss the multivalued version of the main results of this section.

4. Coupled fixed points of multivalued monotone mappings

Let (X, d) be a metric space. We denote by $\mathcal{CB}(X)$ the collection of all nonempty closed and bounded subsets of X. The Pompeiu-Hausdorff distance on $\mathcal{CB}(X)$ is defined by

$$H(A, B) := \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

for $A, B \in \mathcal{CB}(X)$, where $d(a, B) := \inf_{b\in B} d(a, b)$. Let $F : X \times X \to \mathcal{CB}(X)$ be a multivalued mapping. We will say that F is continuous if for any sequences $\{x_n\}$ and $\{y_n\}$ which converge respectively to x and y, we have

$$\lim_{n\to\infty} H(F(x_n, y_n), F(x, y)) = 0.$$

The following technical result is useful to explain our definition later on.

Lemma 4.1. *Let (X, d) be a metric space. For any $A, B \in \mathcal{CB}(X)$ and $\varepsilon > 0$, we have:*

(i) *for $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon;$$

(ii) *for $b \in B$, there exists $a \in A$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Note that from Lemma 4.1, whenever one uses multivalued mappings which involves the Pompeiu-Hausdorff distance, then one must assume that the multivalued mappings have bounded values. Otherwise, one has only to assume that the multivalued mappings have nonempty closed values.

Let (X, d, G) be as before. We denote by $\mathcal{C}(X)$ the collection of all nonempty closed subsets of X. Let $F : X \times X \to \mathcal{C}(X)$ be a multivalued mapping. We will say that F has the mixed G-monotone property on X if:

(i) for any $x_1, x_2, y \in X$ such that $(x_1, x_2) \in E(G)$, for any $u \in F(x_1, y)$, there exists $v \in F(x_2, y)$ such that $(u, v) \in E(G)$;
(ii) for any $x, y_1, y_2 \in X$ such that $(y_1, y_2) \in E(G)$, for any $u \in F(x, y_2)$, there exists $v \in F(x, y_1)$ such that $(u, v) \in E(G)$;

The pair $(x, y) \in X \times X$ is called a coupled fixed point of $F : X \times X \to \mathcal{C}(X)$ if

$$x \in F(x, y), \quad and\ y \in F(y, x).$$

The multivalued version of the condition (BL) may be stated as

Definition 4.1. The multivalued mapping $F : X \times X \to \mathcal{C}(X)$ is said to satisfy the condition (MBL) if there exists $k < 1$ such that for any $(x, y), (u, v) \in X \times X$ with $\Big((x, y), (u, v)\Big) \in E(G)$, and for any $a \in F(x, y)$ there exists $b \in F(u, v)$ such that

$$d(a, b) \leq \frac{k}{2}\Big[d(x, u) + d(y, v)\Big]. \tag{MBL}$$

Next we give an analogue result of Theorem 3.1 to the case of mixed G-monotone multivalued mappings in metric spaces.

Theorem 4.1. *Let (X, d, G) be as above. Assume that (X, d) is a complete metric space. Let $F : X \times X \to \mathcal{CB}(X)$ be a continuous multivalued mapping having the mixed G-monotone property on X and satisfying (MBL) condition. If there exist $x_0, y_0 \in X$ and $x_1 \in F(x_0, y_0)$, $y_1 \in F(y_0, x_0)$ such that $\Big((x_0, y_0), (x_1, y_1)\Big) \in E(G)$, then there exists (x, y) a coupled fixed point of F.*

Proof. By assumption, there exist $x_0, y_0 \in X$ and $x_1 \in F(x_0, y_0)$, $y_1 \in F(y_0, x_0)$ such that $\Big((x_0, y_0), (x_1, y_1)\Big) \in E(G)$. Then $(x_0, x_1) \in E(G)$ and $(y_1, y_0) \in E(G)$. Since F satisfies the (MBL) condition, then there exists $x_2 \in F(x_1, y_1)$ and $y_2 \in F(y_1, x_1)$ with

$$d\Big(x_1, x_2\Big) \leq \frac{k}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

and

$$d\Big(y_1, y_2\Big) \leq \frac{k}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big].$$

By induction, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $x_{n+1} \in F(x_n, y_n)$, and $y_{n+1} \in F(y_n, x_n)$;
(ii) $d(x_n, x_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,
(iii) $d(y_n, y_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,

for any $n \geq 1$. As we did in the proof of Theorem 3.1, we have

$$d(x_n, x_{n+1}) \leq \frac{k^n}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

and

$$d(y_n, y_{n+1}) \leq \frac{k^n}{2}\Big[d(x_0, x_1) + d(y_0, y_1)\Big],$$

for any $n \geq 1$. Since $k < 1$ we conclude that $\sum d(x_n, x_{n+1})$ and $\sum d(y_n, y_{n+1})$ are convergent which imply that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since (X, d) is complete, then there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} x_n = x \textit{ and } \lim_{n\to+\infty} y_n = y.$$

Since F is continuous, we get

$$\lim_{n\to\infty} H(F(x_n, y_n), F(x, y)) = 0.$$

Since $x_{n+1} \in F(x_n, y_n)$, Lemma 4.1 implies the existence of $b_n \in F(x, y)$ such that

$$d(x_{n+1}, b_n) \leq H(F(x_n, y_n), F(x, y)) + \frac{1}{n},$$

for any $n \geq 1$. Clearly, we have $\lim\limits_{n\to\infty} b_n = x$. Since $F(x, y)$ is closed, we conclude that $x \in F(x, y)$. Similarly, we will show that $y \in F(y, x)$, i.e., (x, y) is a coupled fixed point of F. □

Example 4.1. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ and $F : X \times X \to \mathcal{CB}(X)$ be defined by

$$F(x, y) = \left\{-\frac{x + y}{5}, \frac{x + y}{5}\right\},\ (x, y) \in X \times X.$$

Let G be the reflexive digraph defined on X with $((x, y),(u, v)) \in E(G)$ if and only if $x \leq u$ and $v \leq y$. Then F is mixed G-monotone and satisfies condition (MBL). Indeed, let $k = \frac{2}{3}$ and for any $u \in F(x.y)$ take $v = u \in F(y, x)$, then

$$0 = d(u, v) \leq \frac{1}{5}(|u - x| + |v - y|) \leq \frac{1}{3}(|u - x| + |v - y|) = \frac{2/3}{2}\Big[d(u, x) + d(v, y)\Big],$$

for any $(x, y),(u, v) \in X\times X$ with $\Big((x, y),(u, v)\Big) \in E(G)$. Notice that $((0, 0),(0, 0)) \in E(G)$. So by Theorem 4.1 we have that F has a coupled fixed point $(0, 0)$. To illustrate the proof of Theorem 4.1, let us consider $(x_0, y_0) = (0, 1)$, if $u = \frac{-1}{5} \in F(0, 1)$ take $v = \frac{-1}{5}$ (notice that $((0, 1),(\frac{-1}{5}, \frac{-1}{5})) \in E(G)$). Then $x_n = y_n = \frac{-1}{5}(\frac{2}{5})^{n-1} \to 0$ as $n \to \infty$. Thus by Theorem 4.1 $(0, 0)$ is a couple fixed point of F.

As we did in the single valued case, the continuity assumption of F can be relaxed using property **(*)**. We have the following result.

Theorem 4.2. *Let (X, d, G) be as above. Assume that (X, d) is a complete metric space and (X, d, G) has property (*). Let $F : X\times X \to \mathcal{C}(X)$ be a multivalued mapping having the mixed G-monotone property on X and satisfying (MBL) condition. If there exist $x_0, y_0 \in X$ and $x_1 \in F(x_0, y_0)$, $y_1 \in F(y_0, x_0)$ such that $\Big((x_0, y_0),(x_1, y_1)\Big) \in E(G)$, then there exist (x, y) a coupled fixed point of F.*

Proof. As we did in the proof of Theorem 3.1, we construct $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $x_{n+1} \in F(x_n, y_n)$, and $y_{n+1} \in F(y_n, x_n)$;
(ii) $(x_n, x_{n+1}) \in E(G)$ and $(y_{n+1}, y_n) \in E(G)$;
(iii) $d(x_n, x_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,

(iv) $d(y_n, y_{n+1}) \leq \frac{k}{2}\Big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big]$,

for any $n \geq 1$. Clearly both sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since (X, d) is complete, then there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} x_n = x \textit{ and } \lim_{n\to+\infty} y_n = y.$$

The property **(*)** implies

$$(x_n, x) \in E(G) \textit{ and } (y, y_n) \in E(G),$$

for any $n \geq 1$. Since F has the mixed G-monotone property on X, there exist $x_n^* \in F(x, y)$ and $y_n^* \in F(y, x)$ with

$$d(x_{n+1}, x_n^*) \leq \frac{k}{2}\Big[d(x_n, x) + d(y_n, y)\Big],$$

and

$$d(y_{n+1}, y_n^*) \leq \frac{k}{2}\Big[d(x_n, x) + d(y_n, y)\Big],$$

for any $n \geq 1$. This will imply

$$\lim_{n\to+\infty} d(x_{n+1}, x_n^*) = 0 \text{ and } \lim_{n\to+\infty} d(y_{n+1}, y_n^*) = 0.$$

Therefore, we have

$$\lim_{n\to+\infty} x_n^* = x \text{ and } \lim_{n\to+\infty} y_n^* = y.$$

Since $F(x, y)$ and $F(y, x)$ are closed, we conclude that $x \in F(x, y)$ and $y \in F(y, x)$, i.e., (x, y) is a coupled fixed point of F. □

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