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SEMILINEAR EVOLUTION SYSTEMS WITH NONLINEAR CONSTRAINTS

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Abstract. The purpose of the present paper is to study the existence of solutions to semilinear evolution systems with nonlinear constraints. We establish new existence results using the fixed point principles of Perov and Schauder, combined with the technique that uses matrices with the spectral radius less than one and vector-valued norms. This vectorial approach is fruitful for the treating of systems in general and allows the system nonlinearities to behave independently as much as possible. Moreover, the constants from the Lipschitz or growth conditions are put into connection with the support of the nonlinear operators expressing the constraints. The paper extends and complements previous results from the literature.
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1. Introduction and preliminaries

We are concerned with the existence and the uniqueness of solutions to the semilinear system of abstract evolution equations with constraints

$$\begin{cases} x'(t) + A_1 x(t) = f_1(t, x(t), y(t)) \\ y'(t) + A_2 y(t) = f_2(t, x(t), y(t)) \\ g_1(x,y) = 0 \\ g_2(x,y) = 0. \end{cases} \quad (0 < t < T) \tag{1.1}$$

Here, for each $i = 1, 2$, the linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a strongly continuous semigroup of contractions $\{S_i(t), t \geq 0\}$ on the Banach space $\left(X_i, |.|_{X_i}\right)$, $f_i : [0,T] \times X_1 \times X_2 \to X_i$ is a given function and $g_i : C([0,T], X_1) \times C([0,T], X_2) \to X_i$ is a nonlinear operator.

It is convenient that the constraints are written equivalently under the form of nonlocal conditions

$$x(0) = \alpha_1[x,y], \quad y(0) = \alpha_2[x,y], \tag{1.2}$$

where

$$\alpha_1[x,y] = g_1(x,y) + x(0), \quad \alpha_2[x,y] = g_2(x,y) + y(0).$$

In the literature, it has been given a particular attention and interest to different classes of problems with nonlocal conditions. In this matter, we mention the papers

[6]-[8], [15], [16]-[22], [28], [35], [38] and references therein. This type of problems arise in the study of mathematical modeling of real processes, such as heat, fluid, chemical or biological flow, where the nonlocal conditions can be seen as feedback controls. Other discussions about the importance of nonlocal conditions in different areas of applications, examples of evolution equations and systems with or without delay, subjected to nonlocal initial conditions and references to other works dealing with nonlocal problems can be found in [9], [21], [23], [26], [31], [33], [34].

The constraints (1.2) can be either of discrete type (or multi-point conditions), or of continuous type expressed by continuous operators. Also, the nonlocal conditions can be linear or nonlinear. For the case of nonlinear nonlocal conditions we mention the recent paper [5], as well as [11], [13], [14], [24] and references therein.

In all cases, it is important to take into consideration the *support of the nonlocal conditions*, that is the smallest interval $[0,a] \subset [0,T]$ on which those conditions act in the sense that whenever $x, \overline{x} \in C([0,T], X_1)$ and $y, \overline{y} \in C([0,T], X_2)$,

$$x|_{[0,a]} = \overline{x}|_{[0,a]} \text{ and } y|_{[0,a]} = \overline{y}|_{[0,a]} \text{ imply } \alpha_i[x,y] = \alpha_i[\overline{x},\overline{y}], \quad i = 1,2.$$

The notion of support plays an essential role in the existence results for the nonlocal problems, as shown in the paper [8]. More exactly, it was shown that it is necessary to impose stronger conditions on the nonlinearities on the subinterval $[0,a]$, compared to those required on the rest of the interval $[0,T]$. As shown in the previous works [2]-[5], [8], [18], [19], the integral system equivalent to the nonlocal problem is of Fredholm type on the support $[0,a]$ and of Volterra type on $[a,T]$. From a physical point of view, on the time interval $[0,a]$, the evolution process is subjected to some constraints, and after "the moment a" it becomes free of any constraints.

In connection with the support of the nonlocal conditions, we shall consider a special norm on $C([0,T], X_i)$ $(i = 1,2)$, namely

$$|x|_* = \max\left\{ |x|_{C([0,a],X_i)},\ |x|_{C_\theta([a,T],X_i)} \right\},$$

where $|.|_{C([0,a],X_i)}$ is the usual max norm on $C([0,a], X_i)$,

$$|x|_{C([0,a],X_i)} = \max_{t\in[0,a]} |x(t)|_{X_i},$$

while for any $\theta > 0$, $|x|_{C_\theta([a,T],X_i)}$ is the Bielecki norm on $C([a,T], X_i)$,

$$|x|_{C_\theta([a,T],X_i)} = \max_{t\in[a,T]} |x(t)|_{X_i} e^{-\theta(t-a)}.$$

In what follows, we look for global mild solutions on the interval $[0,T]$, i.e. a pair $(x,y) \in C([0,T], X_1)\times C([0,T], X_2)$ satisfying the integral system

$$\begin{cases} x(t) = S_1(t)\alpha_1[x,y] + \displaystyle\int_0^t S_1(t-s) f_1(s, x(s), y(s))ds \\ y(t) = S_2(t)\alpha_2[x,y] + \displaystyle\int_0^t S_2(t-s) f_2(s, x(s), y(s))ds. \end{cases} \tag{1.3}$$

This system can be viewed as a fixed point problem in $C([0,T], X_1)\times C([0,T], X_2)$ for the nonlinear operator $N = (N_1, N_2) : C([0,T], X_1)\times C([0,T], X_2) \to C([0,T], X_1)$

$\times\, C([0,T],X_2)$ defined by

$$\begin{aligned} N_1(x,y)(t) &= S_1(t)\alpha_1[x,y] + \int_0^t S_1(t-s)f_1(s,x(s),y(s))ds, \\ N_2(x,y)(t) &= S_2(t)\alpha_2[x,y] + \int_0^t S_2(t-s)f_2(s,x(s),y(s))ds. \end{aligned} \tag{1.4}$$

Basic notions and results from the semigroup theory that are frequently used in our work can be found, for example, in [10], [12] and [32]. Next, we recall some basic notions that are used in our vectorial approach. Details can be found in [1], [25]-[27], [29] and [30].

By a *vector-valued metric* on a set X we mean a mapping $d : X \times X \to \mathbb{R}^n_+$ such that (i) $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$ for all $x, y \in X$ and (iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$. Here by $\leq$ we mean the natural componentwise order relation of $\mathbb{R}^n$, more exactly, if $r,s \in \mathbb{R}^n$, $r = (r_1, r_2, ..., r_n)$, $s = (s_1, s_2, ..., s_n)$, then by $r \leq s$ one means that $r_i \leq s_i$ for $i = 1, 2, ..., n$.

A set X together with a vector-valued metric d is called a *generalized metric space*. For such a space, the notions of Cauchy sequence, convergence, completeness, open and closed set are similar to those in usual metric spaces.

Similarly, we speak about a *vector-valued norm* on a linear space X, as being a mapping $\|.\| : X \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda| \, \|x\|$ for $x \in X$, $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$. To any vector-valued norm $\|.\|$ one can associate the vector-valued metric $d\,(x,y) := \|x - y\|$, and one says that $(X, \|.\|)$ is a *generalized Banach space* if X is complete with respect to d.

If (X,d) is a generalized metric space and $N : X \to X$ is any mapping, we say that N is a *generalized contraction* (in Perov's sense) provided that a (Lipschitz) matrix $M \in M_{n\times n}(\mathbb{R}_+)$ exists such that its powers M^k tend to the zero matrix 0 as $k \to \infty$, and

$$d(N(x), N(y)) \leq M d(x,y) \;\text{ for all }\; x, y \in X.$$

For such kind of mappings the following generalization of Banach's contraction principle holds:

Theorem 1.1 (Perov)**.** *If (X,d) is a complete generalized metric space, then any generalized contraction $N : X \to X$ with the Lipschitz matrix M has a unique fixed point x^*, and*

$$d(N^k(x), x^*) \leq M^k(I - M)^{-1} d(x, N(x)),$$

for all $x \in X$ and $k \in \mathbb{N}$.

There are known several characterizations of matrices like in Perov's theorem (see [25] and [30, pp 12, 88]). More exactly, for a matrix $M \in M_{n\times n}(\mathbb{R}_+)$, the following statements are equivalent:

(a) $M^k \to 0$ as $k \to \infty$;
(b) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);
(c) the eigenvalues of M are located inside the unit disc of the complex plane, i.e. $\rho\,(M) < 1$, where $\rho\,(M)$ is the *spectral radius* of M;

(d) $I-M$ is nonsingular and inverse-positive, i.e. $(I-M)^{-1}$ has nonnegative entries.

Remark 1.2. *Recal that in case $n=2$, a matrix $M\in M_{2\times 2}(\mathbb{R}_+)$ satisfies $\rho(M)<1$ if and only if*

$$tr\ M<\min\{2,\ 1+\det M\}.$$

The following almost obvious lemma will be used in the sequel.

Lemma 1.3. *If $A\in M_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(A)<1$, then $\rho(A+B)<1$ for every matrix $B\in M_{n\times n}(\mathbb{R}_+)$ whose elements are small enough.*

The role of matrices with spectral radius less than one in the study of semilinear operator systems was pointed out in [27], also in connection with other abstract principles from nonlinear functional analysis.

Besides Perov's fixed point theorem, in this work we shall also use the well-known Schauder's fixed point theorem.

2. Main results

Throughout this section, we assume that X_i and A_i are as in the previous section, and that the interval $[0,a]$, $0\leq a\leq T$ is the support of the nonlocal conditions.

Our first result is an existence and uniqueness theorem for the case where the nonlinearities f_1,f_2 and mappings α_1,α_2 are continuous and satisfy the Lipschitz conditions

$$|f_i(t,x,y)-f_i(t,\overline{x},\overline{y})|_{X_i}\leq\begin{cases} a_{i1}(t)\,|x-\overline{x}|_{X_1}+a_{i2}(t)\,|y-\overline{y}|_{X_2}\,, & \text{if } t\in[0,a]\\ b_{i1}(t)\,|x-\overline{x}|_{X_1}+b_{i2}(t)\,|y-\overline{y}|_{X_2}\,, & \text{if } t\in[a,T]\end{cases} \tag{2.1}$$

for all $(x,y),(\overline{x},\overline{y})\in X_1\times X_2$ and some $a_{ij}\in L^1([0,a],\mathbb{R}_+)$, $b_{ij}\in L^p([a,T],\mathbb{R}_+)$ and $p>1$, and

$$|\alpha_i[x,y]-\alpha_i[\overline{x},\overline{y}]|_{X_i}\leq A_{i1}\,|x-\overline{x}|_{C([0,a],X_1)}+A_{i2}\,|y-\overline{y}|_{C([0,a],X_2)}\,, \tag{2.2}$$

for all $x,\overline{x}\in C([0,T],X_1)$ and $y,\overline{y}\in C([0,T],X_2)$ and some $A_{ij}\in\mathbb{R}_+$ $(i,j=1,2)$.

Denote

$$C_i=\sup\{\|S_i(t)\|:\ t\in[0,T]\}\quad(i=1,2),$$

where $\|\cdot\|$ stands for the norm of a continuous linear operator.

Assuming a vectorial condition involving the coefficients a_{ij} and A_{ij}, but independent on the coefficients b_{ij}, we have the following result:

Theorem 2.1. *Assume that the conditions* (2.1) *and* (2.2) *hold. In addition assume that the spectral radius of the matrix*

$$M_0:=\left[C_i\left(A_{ij}+|a_{ij}|_{L^1[0,a]}\right)\right]_{i,j=1,2} \tag{2.3}$$

is less than one. Then the problem (1.1) *has a unique mild solution on* $[0,T]$.

Proof. We shall apply Perov's theorem in $X = C([0,T], X_1) \times C([0,T], X_2)$ endowed with the vector norm $\|.\|_X$ defined by

$$\|u\|_X = (|x|_*\,,\ |y|_*),$$

where $u = (x, y)$ and for any $z \in C([0,T], X_i)$,

$$|z|_* = \max\left\{ |z|_{C([0,a],X_i)}\,, |z|_{C_\theta([a,T],X_i)} \right\}.$$

Clearly, $(X, \|.\|_X)$ is a complete generalized metric space. Finding a mild solution of problem (1.1) is equivalent to finding a fixed point of the nonlinear operator $N = (N_1, N_2)$ defined by (1.4). We have to prove that N is a generalized contraction, more exactly that

$$\|N(u) - N(\overline{u})\|_X \leq M_\theta \|u - \overline{u}\|_X\,,$$

for all $u = (x, y),\ \overline{u} = (\overline{x}, \overline{y}) \in X$ and some matrix M_θ with $\rho(M_\theta) < 1$. To this end, let $u = (x, y),\ \overline{u} = (\overline{x}, \overline{y})$ be any two elements of X. For $t \in [0, a]$, using (2.1) and (2.2), we have

$$\begin{aligned}
&|N_1(u)(t) - N_1(\overline{u})(t)|_{X_1} \leq \|S_1(t)\|\, |\alpha_1[x, y] - \alpha_1[\overline{x}, \overline{y}]|_{X_1} \\
&+ \int_0^t \|S_1(t-s)\|\, |f_1(s, x(s), y(s)) - f_1(s, \overline{x}(s), \overline{y}(s))|_{X_1}\, ds \\
\leq\ & C_1 \left(A_{11} |x - \overline{x}|_{C([0,a],X_1)} + A_{12} |y - \overline{y}|_{C([0,a],X_2)} \right) + \\
&+ C_1 \int_0^a \left(a_{11}(s) |x(s) - \overline{x}(s)|_{X_1} + a_{12}(s) |y(s) - \overline{y}(s)|_{X_2} \right) ds \\
\leq\ & C_1 \left(A_{11} + |a_{11}|_{L^1[0,a]} \right) |x - \overline{x}|_{C([0,a],X_1)} \\
&+ C_1 \left(A_{12} + |a_{12}|_{L^1[0,a]} \right) |y - \overline{y}|_{C([0,a],X_2)}\,.
\end{aligned}$$

Therefore, taking the supremum when $t \in [0, a]$, we obtain

$$\begin{aligned}
&|N_1(x, y) - N_1(\overline{x}, \overline{y})|_{C([0,a],X_1)} \\
\leq\ & m_{11} |x - \overline{x}|_{C([0,a],X_1)} + m_{12} |y - \overline{y}|_{C([0,a],X_2)}\,,
\end{aligned} \tag{2.4}$$

where

$$m_{1j} = C_1 \left(A_{1j} + |a_{1j}|_{L^1[0,a]} \right), \quad j = 1, 2. \tag{2.5}$$

Next, for $t \in [a, T]$ and any $\theta > 0$, we have

$$\begin{aligned}
&|N_1(x, y)(t) - N_1(\overline{x}, \overline{y})(t)|_{X_1} \leq \|S_1(t)\|\, |\alpha_1[x, y] - \alpha_1[\overline{x}, \overline{y}]|_{X_1} \\
&+ \int_0^a \|S_1(t-s)\|\, |f_1(s, x(s), y(s)) - f_1(s, \overline{x}(s), \overline{y}(s))|_{X_1}\, ds \\
&+ \int_a^t \|S_1(t-s)\|\, |f_1(s, x(s), y(s)) - f_1(s, \overline{x}(s), \overline{y}(s))|_{X_1}\, ds.
\end{aligned} \tag{2.6}$$

The sum of the first two terms can be estimated according to (2.4). As concerns the last term, we have

$$\begin{aligned}
&\int_a^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s)) - f_1(s,\overline{x}(s),\overline{y}(s))\right|_{X_1} ds \qquad (2.7)\\
\leq\ & C_1 \int_a^t \left(b_{11}(s)\left|x(s)-\overline{x}(s)\right|_{X_1} + b_{12}(s)\left|y(s)-\overline{y}(s)\right|_{X_2}\right) ds\\
=\ & C_1 \left(\int_a^t b_{11}(s)\left|x(s)-\overline{x}(s)\right|_{X_1} e^{-\theta(s-a)}e^{\theta(s-a)}ds\right.\\
&\left.+\int_a^t b_{12}(s)\left|y(s)-\overline{y}(s)\right|_{X_2} e^{-\theta(s-a)}e^{\theta(s-a)}ds\right)\\
\leq\ & C_1 \left(\left|x-\overline{x}\right|_{C_\theta([a,T],X_1)} \int_a^t b_{11}(s)e^{\theta(s-a)}ds\right.\\
&\left.+\left|y-\overline{y}\right|_{C_\theta([a,T],X_2)} \int_a^t b_{12}(s)e^{\theta(s-a)}ds\right).
\end{aligned}$$

Now Hölder's inequality gives

$$\int_a^t b_{1j}(s)e^{\theta(s-a)}ds \leq \frac{\left|b_{1j}\right|_{L^p[a,T]}}{(q\theta)^{1/q}}e^{\theta(t-a)}, \qquad j=1,2.$$

Then (2.7) yields

$$\begin{aligned}
&\int_a^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s)) - f_1(s,\overline{x}(s),\overline{y}(s))\right|_{X_1} ds\\
\leq\ & \frac{C_1}{(q\theta)^{1/q}}\left(\left|b_{11}\right|_{L^p[a,T]}\left|x-\overline{x}\right|_{C_\theta([a,T],X_1)} + \left|b_{12}\right|_{L^p[a,T]}\left|y-\overline{y}\right|_{C_\theta([a,T],X_2)}\right)e^{\theta(t-a)}.
\end{aligned}$$

Next (2.6) implies that

$$\begin{aligned}
&\left|N_1(x,y)(t) - N_1(\overline{x},\overline{y})(t)\right|_{X_1}\\
&\leq m_{11}\left|x-\overline{x}\right|_{C([0,a],X_1)} + m_{12}\left|y-\overline{y}\right|_{C([0,a],X_2)}\\
&+\frac{C_1}{(q\theta)^{1/q}}\left(\left|b_{11}\right|_{L^p[a,T]}\left|x-\overline{x}\right|_{C_\theta([a,T],X_1)} + \left|b_{12}\right|_{L^p[a,T]}\left|y-\overline{y}\right|_{C_\theta([a,T],X_2)}\right)e^{\theta(t-a)}.
\end{aligned}$$

Dividing by $e^{\theta(T-a)}$ and taking the supremum when $t\in[a,T]$ we obtain

$$\begin{aligned}
&\left|N_1(x,y) - N_1(\overline{x},\overline{y})\right|_{C_\theta([a,T],X_1)} \leq m_{11}\left|x-\overline{x}\right|_{C([0,a],X_1)} + m_{12}\left|y-\overline{y}\right|_{C([0,a],X_2)}\\
&+\frac{C_1}{(q\theta)^{1/q}}\left(\left|b_{11}\right|_{L^p[a,T]}\left|x-\overline{x}\right|_{C_\theta([a,T],X_1)} + \left|b_{12}\right|_{L^p[a,T]}\left|y-\overline{y}\right|_{C_\theta([a,T],X_2)}\right). \qquad (2.8)
\end{aligned}$$

Thus, from (2.4), (2.8), we have that

$$\left|N_1(x,y) - N_1(\overline{x},\overline{y})\right|_* \leq (m_{11}+n_{11})\left|x-\overline{x}\right|_* + (m_{12}+n_{12})\left|y-\overline{y}\right|_* \qquad (2.9)$$

where

$$n_{1j} = \frac{C_1}{(q\theta)^{1/q}}\left(\left|b_{1j}\right|_{L^p(a,T)}\right), \quad j=1,2. \qquad (2.10)$$

Similarly

$$|N_2(x,y) - N_2(\overline{x},\overline{y})|_* \leq (m_{21} + n_{21})\,|x - \overline{x}|_* + (m_{22} + n_{22})\,|y - \overline{y}|_*\,, \qquad (2.11)$$

where

$$m_{2j} = C_2\left(A_{2j} + |a_{2j}|_{L^1[0,a]}\right), \quad n_{2j} = \frac{C_2}{(q\theta)^{1/q}}\left(|b_{2j}|_{L^p[a,T]}\right), \quad j = 1,2. \qquad (2.12)$$

Next, (2.9) and (2.11) can be put together under the vectorial form

$$\begin{bmatrix} |N_1(x,y) - N_1(\overline{x},\overline{y})|_* \\ |N_2(x,y) - N_2(\overline{x},\overline{y})|_* \end{bmatrix} \leq M_\theta \begin{bmatrix} |x - \overline{x}|_* \\ |y - \overline{y}|_* \end{bmatrix}$$

or equivalently,

$$\|N(u) - N(\overline{u})\|_X \leq M_\theta \|u - \overline{u}\|_X\,,$$

for all $u = (x,y), \overline{u} = (\overline{x},\overline{y}) \in X$. Clearly M_θ can be represented as $M_\theta = M_0 + M_1$, where

$$M_0 = [m_{ij}]_{i,j=1,2}\,, \qquad M_1 = [n_{ij}]_{i,j=1,2}\,.$$

Since $\rho(M_0) < 1$, from Lemma 1.3 we have that $\rho(M_\theta) < 1$ for a large enough $\theta > 0$ making n_{ij} as small as necessary. Therefore, Perov's theorem applies and gives the conclusion. □

If we assume that the operator N is completely continuous, we can weaken conditions (2.1) and (2.2) to at most linear growth conditions. In this case, we can apply Schauder's fixed point theorem that guarantees the existence of the solution for our studied problem, but not also the uniqueness.

Therefore, we give our second result assuming that the nonlinearities f_1, f_2 and the mappings α_1, α_2 satisfy, instead of the Lipschitz conditions, some conditions of at most linear growth, namely

$$|f_i(t,x,y)|_{X_i} \leq \begin{cases} a_{i1}(t)\,|x|_{X_1} + a_{i2}(t)\,|y|_{X_2} + a_{i3}(t), & \text{if } t \in [0,a] \\ b_{i1}(t)\,|x|_{X_1} + b_{i2}(t)\,|y|_{X_2} + b_{i3}(t), & \text{if } t \in [a,T] \end{cases} \qquad (2.13)$$

for all $(x,y) \in X_1 \times X_2$ and some $a_{ij} \in L^1([0,a],\mathbb{R}_+)$, $b_{ij} \in L^p([a,T],\mathbb{R}_+)$, $p > 1$, and

$$|\alpha_i[x,y]|_{X_i} \leq A_{i1}\,|x|_{C([0,a],X_1)} + A_{i2}\,|y|_{C([0,a],X_2)} + A_{i3}, \qquad (2.14)$$

for all $x \in C([0,T],X_1)$, $y \in C([0,T],X_2)$ and some $A_{ij} \in \mathbb{R}_+$ $(i = 1,2, j = 1,2,3)$.

Theorem 2.2. *Assume that conditions* (2.13) *and* (2.14) *are satisfied. If the spectral radius of the matrix* (2.3) *is less than one, then problem* (1.1) *has at least one mild solution.*

Proof. In order to apply Schauder's fixed point principle, we need to find a nonempty closed bounded convex set $B \subset X$ such that

$$N(B) \subset B. \qquad (2.15)$$

We shall look for the set B under the form $B := \overline{B}_1(0;R_1) \times \overline{B}_2(0;R_2)$, where $\overline{B}_i(0;R_i)$ is the closed ball of radius R_i centered in the origin of the space $C([0,T],X_i)$,

$i=1,2$. Thus we have to find two positive numbers R_1, R_2 such that (2.15) holds. Let $u=(x,y)\in X$. For $t\in[0,a]$, using (2.13) and (2.14), we have

$$
\begin{aligned}
&|N_1(u)(t)|_{X_1} \leq \|S_1(t)\| \left|\alpha_1[x,y]\right|_{X_1} + \int_0^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s))\right|_{X_1} ds \\
&\leq C_1 \left(A_{11}|x|_{C([0,a],X_1)} + A_{12}|y|_{C([0,a],X_2)} + A_{13}\right) + \\
&+ C_1 \int_0^a \left(a_{11}(s)|x(s)|_{X_1} + a_{12}(s)|y(s)|_{X_2} + a_{13}(s)\right) ds \\
&\leq C_1 \left(A_{11} + |a_{11}|_{L^1(0,a)}\right)|x|_{C([0,a],X_1)} + C_1\left(A_{12} + |a_{12}|_{L^1(0,a)}\right)|y|_{C([0,a],X_2)} + \widehat{C}_1,
\end{aligned}
$$

where $\widehat{C}_1 := C_1\left(A_{13} + |a_{13}|_{L^1(0,a)}\right)$. Taking the supremum when $t\in[0,a]$, we obtain

$$
|N_1(x,y)|_{C([0,a],X_1)} \leq m_{11}|x|_{C([0,a],X_1)} + m_{12}|y|_{C([0,a],X_2)} + \widehat{C}_1, \tag{2.16}
$$

where m_{1j}, $j=1,2$ are given by (2.5). Next, for $t\in[a,T]$ and any $\theta>0$, we have

$$
\begin{aligned}
& |N_1(x,y)(t)|_{X_1} \\
\leq\; & \|S_1(t)\| \left|\alpha_1[x,y]\right|_{X_1} + \int_0^a \|S_1(t-s)\| \left|f_1(s,x(s),y(s))\right|_{X_1} ds \\
& + \int_a^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s))\right|_{X_1} ds.
\end{aligned} \tag{2.17}
$$

Again, the sum of the first two terms can be estimated according to (2.16). For the last term, we obtain

$$
\begin{aligned}
& \int_a^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s))\right|_{X_1} ds \\
\leq\; & C_1 \int_a^t \left(b_{11}(s)|x(s)|_{X_1} + b_{12}(s)|y(s)|_{X_2} + b_{13}(s)\right) ds \\
\leq\; & C_1 \int_a^t b_{11}(s)|x(s)|_{X_1} e^{-\theta(s-a)} e^{\theta(s-a)} ds \\
& + C_1 \int_a^t b_{12}(s)|y(s)|_{X_2} e^{-\theta(s-a)} e^{\theta(s-a)} ds + C_1 \int_a^t b_{13}(s) ds \\
\leq\; & C_1 |x|_{C_\theta([a,T],X_1)} \int_a^t b_{11}(s) e^{\theta(s-a)} ds + C_1 |y|_{C_\theta([a,T],X_2)} \int_a^t b_{12}(s) e^{\theta(s-a)} ds \\
& + C_1 |b_{13}|_{L^1(a,T)}.
\end{aligned}
$$

Now, using Hölder's inequality, we deduce that

$$
\begin{aligned}
& \int_a^t \|S_1(t-s)\| \left|f_1(s,x(s),y(s))\right|_{X_1} ds \\
\leq\; & \frac{C_1}{(q\theta)^{1/q}} \left(|b_{11}|_{L^p(a,T)} |x|_{C_\theta([a,T],X_1)} + |b_{12}|_{L^p(a,T)} |y|_{C_\theta([a,T],X_2)}\right) e^{\theta(t-a)} \\
& + C_1 |b_{13}|_{L^1(a,T)}.
\end{aligned}
$$

Thus, (2.17) becomes

$$\begin{aligned}&|N_1(x,y)(t)|_{X_1} \le m_{11}\,|x|_{C([0,a],X_1)} + m_{12}\,|y|_{C([0,a],X_2)} + \widehat{C}_1\\&+\frac{C_1}{(q\theta)^{1/q}}\left(|b_{11}|_{L^p(a,T)}\,|x|_{C_\theta([a,T],X_1)} + |b_{12}|_{L^p(a,T)}\,|y|_{C_\theta([a,T],X_2)}\right)e^{\theta(t-a)}\\&+C_1\,|b_{13}|_{L^1(a,T)}\,.\end{aligned}$$

Dividing by $e^{\theta(T-a)}$ and taking the supremum when $t\in[a,T]$, we obtain

$$\begin{aligned}&|N_1(x,y)|_{C_\theta([a,T],X_1)} \le m_{11}\,|x|_{C([0,a],X_1)} + m_{12}\,|y|_{C([0,a],X_2)} + \widehat{C}_1\\&+\frac{C_1}{(q\theta)^{1/q}}\left(|b_{11}|_{L^p(a,T)}\,|x|_{C_\theta([a,T],X_1)} + |b_{12}|_{L^p(a,T)}\,|y|_{C_\theta([a,T],X_2)}\right)\\&+C_1\,|b_{13}|_{L^1(a,T)}\,.\end{aligned}\tag{2.18}$$

Therefore, from (2.16) and (2.18) we have

$$|N_1(x,y)|_* \le (m_{11}+n_{11})\,|x|_* + (m_{12}+n_{12})\,|y|_* + \overline{C}_1,\tag{2.19}$$

where we have denoted $\overline{C}_1 := \widehat{C}_1 + C_1\,|b_{13}|_{L^1(a,T)}$ and $n_{1j}, j=1,2$ are given by (2.10).

Similarly,

$$|N_2(x,y)|_* \le (m_{21}+n_{21})\,|x|_* + (m_{22}+n_{22})\,|y|_* + \overline{C}_2,\tag{2.20}$$

where $\overline{C}_2 := C_2\left(A_{23} + |a_{23}|_{L^1(0,a)} + |b_{23}|_{L^1(a,T)}\right)$ and $m_{2j}, n_{2j}, j=1,2$ are given by (2.12).

Now, (2.19) and (2.20) can be put together as

$$\begin{bmatrix}|N_1(x,y)|_*\\|N_2(x,y)|_*\end{bmatrix} \le M_\theta\begin{bmatrix}|x|_*\\|y|_*\end{bmatrix} + \begin{bmatrix}\overline{C}_1\\\overline{C}_2\end{bmatrix},$$

where $M_\theta = M_0 + M_1$ is given by (2.3) and has the spectral radius less than one according to the hypotheses of the theorem.

In what follows, we look for two positive numbers R_1, R_2 such that if $|x|_* \le R_1, |y|_* \le R_2$, then $|N_1(x,y)|_* \le R_1, |N_2(x,y)|_* \le R_2$. To this end, it is sufficient that

$$M_\theta\begin{bmatrix}R_1\\R_2\end{bmatrix} + \begin{bmatrix}\overline{C}_1\\\overline{C}_2\end{bmatrix} \le \begin{bmatrix}R_1\\R_2\end{bmatrix}.$$

Since $\rho(M_0) < 1$, from Lemma 1.3 we have that $\rho(M_\theta) < 1$ for a large enough $\theta > 0$. Then the matrix $I - M_\theta$ is invertible and its inverse $(I-M_\theta)^{-1}$ has nonnegative elements and thus

$$\begin{bmatrix}R_1\\R_2\end{bmatrix} \ge (I-M_\theta)^{-1}\begin{bmatrix}\overline{C}_1\\\overline{C}_2\end{bmatrix}.\tag{2.21}$$

Therefore we can take R_1, R_2 the numbers corresponding to the equality in (2.21). Thus Schauder's fixed point theorem can be applied and gives the conclusion. □

Sufficient conditions for the complete continuity of the operator N, as well as for the mild solutions to be classical solutions can be found in the literature, for example in [10], [32].

We note that Theorem 2.1 and Theorem 2.2 can be easily extended to the general n-dimensional case, when the assumption about the spectral radius of the corresponding matrix of order n can be checked using computer algebra programs such as Maple, Mathematica or Sage.

We conclude this paper by two examples illustrating our main results.

Example 2.1. Let us consider the semilinear transport system

$$\begin{cases} u'(t) + d_1 \cdot \nabla u(t) = a_1(t) \sin u(t) + b_1(t) v(t) \\ v'(t) + d_2 \cdot \nabla v(t) = a_2(t) u(t) + b_2(t) \cos v(t) \\ u(0) = \displaystyle\int_0^{T/2} [m_1 u(s) + n_1 v(s)]\, ds \\ v(0) = \displaystyle\int_0^{T/2} [m_2 u(s) + n_2 v(s)]\, ds, \end{cases} \tag{2.22}$$

where $d_1, d_2 \in \mathbb{R}^n$; $a_i, b_i \in C([0,T], \mathbb{R}_+)$; $m_i, n_i \in \mathbb{R}_+$, for $i = 1, 2$.

In this case, for each $i = 1, 2$, we may take

$$X_i = L^{r_i}(\mathbb{R}^n), \quad r_i \geq 1,$$

and A_i the directional derivative operator along direction d_i,

$$\begin{aligned} A_i u &= d_i \cdot \nabla u = \sum_{j=1}^{n} d_{ij} \frac{\partial u}{\partial x_j}, \\ D(A_i) &= \{u \in L^{r_i}(\mathbb{R}^n) : d_i \cdot \nabla u \in L^{r_i}(\mathbb{R}^n)\}. \end{aligned}$$

Notice that here ∇u is the gradient of u in the sense of distributions, and in (2.22) by $\sin u(t)$ we mean the function $x \mapsto \sin(u(t)(x))$. The meaning of $\cos v(t)$ is similar. According to [32, p. 88], the operator $-A_i$ generates even a C_0-group of isometries, namely

$$S_i(t) u(x) = u(x - t d_i), \quad u \in L^{r_i}(\mathbb{R}^n),\ t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Hence $C_i = 1$. For this example, we have

$$\begin{aligned} f_1(t, u(t), v(t)) &= a_1(t) \sin u(t) + b_1(t) v(t), \\ f_2(t, u(t), v(t)) &= a_2(t) u(t) + b_2(t) \cos v(t), \\ \alpha_1[u, v] &= \int_0^{T/2} [m_1 u(s) + n_1 v(s)]\, ds, \\ \alpha_2[u, v] &= \int_0^{T/2} [m_2 u(s) + n_2 v(s)]\, ds. \end{aligned}$$

Also, in this case $a = T/2$, while f_1, f_2 satisfy (2.1) with

$$\begin{aligned} a_{i1} &= a_i|_{[0,T/2]}, \quad a_{i2} = b_i|_{[0,T/2]}, \\ b_{i1} &= a_i|_{[T/2,T]}, \quad b_{i2} = b_i|_{[T/2,T]}. \end{aligned}$$

In addition, the mappings α_1, α_2 satisfy (2.2) with

$$A_{11} = Tm_1/2,\ A_{12} = Tn_1/2,\ A_{21} = Tm_2/2,\ A_{22} = Tn_2/2.$$

Then

$$\begin{aligned} M_0 &= \left[C_i\left(A_{ij} + |a_{ij}|_{L^1[0,a]}\right)\right]_{i,j=1,2} \qquad (2.23)\\ &= \begin{bmatrix} Tm_1/2 + |a_1|_{L^1[0,T/2]} & Tn_1/2 + |b_1|_{L^1[0,T/2]} \\ Tm_2/2 + |a_2|_{L^1[0,T/2]} & Tn_2/2 + |b_2|_{L^1[0,T/2]} \end{bmatrix}. \end{aligned}$$

Therefore, according to Theorem 2.1, if the spectral radius of the matrix M_0 is less than one, then the problem (2.22) has a unique mild solution in $C([0,T], L^{r_1}(\mathbb{R}^n))$ $\times C([0,T], L^{r_2}(\mathbb{R}^n))$.

The next example illustrates the existence result given by Theorem 2.2.

Example 2.2. Consider the semilinear transport system

$$\begin{cases} u'(t) + d_1 \cdot \nabla u(t) = a_1(t)u(t)\sin(u(t) + v(t)) + b_1(t)v(t) + c_1(t) \\ v'(t) + d_2 \cdot \nabla v(t) = a_2(t)u(t) + b_2(t)v(t)\cos(u(t) + v(t)) + c_2(t) \\ u(0) = \displaystyle\int_0^{T/2} [m_1u(s) + n_1v(s)]\, ds \\ v(0) = \displaystyle\int_0^{T/2} [m_2u(s) + n_2v(s)]\, ds, \end{cases} \qquad (2.24)$$

where $d_1, d_2 \in \mathbb{R}^n$; $a_i, b_i, c_i \in C([0,T], \mathbb{R}_+)$; $m_i, n_i \in \mathbb{R}_+$, for $i = 1, 2$.

If the spectral radius of the matrix (2.23) is less than one, then the problem (2.24) has at least one mild solution in $C([0,T], L^{r_1}(\mathbb{R}^n)) \times C([0,T], L^{r_2}(\mathbb{R}^n))$.

Notice that the nonlinearities f_1 and f_2 from this example do not satisfy conditions (2.1) and therefore Theorem 2.1 does not apply.

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