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SOME FIXED POINT THEOREMS FOR NONSELF GENERALIZED CONTRACTION IN GAUGE SPACES

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Abstract. In this paper we extend the result from [18] to the case of for nonself operators on gauge spaces. Some new fixed point theorems for nonself generalized contractions are also given.
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1. Preliminaries

There are many technique in the fixed point theory of nonself operators (see [7], [3], [5], [9], [15], [16], ...). An interesting result is given in [11] (see also, [8] and [12]). This result read as follows:

Theorem 1.1. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be a φ-contraction, where φ is a comparison function. We suppose that there exists a bounded sequence $(x_n)_{n\in\mathbb{N}^*}$ such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$. Then f has a unique fixed point x^* and $f^n(x_n) \to x^*$.*

A new technique of proof for this result was given in [18] and extended with respect to the data dependence of the fixed point, with more conclusions and other new results for nonself generalized contractions.

In case of Theorem 1.1, we have the following result:

Theorem 1.2. *Let $f : Y \to X$ be as in Theorem 1.1, where φ is a strict comparison function. Then:*

(a) $d(f^n(x_n), x^*) \le \varphi(d(x_n, x^*))$, $\forall n \in \mathbb{N}^*$;
(b) $d(x, x^*) \le \theta_\varphi(d(x, f(x)))$, $\forall x \in Y$;
(c) *Let $g : Y \to X$ be such that:*
   (1) *there exists $\eta > 0$ such that* $d(f(x), g(x)) \le \eta$, $\forall x \in Y$;
   (2) $F_g \neq \varnothing$.

Then

$$d(x^*, y^*) \le \theta_\varphi(\eta), \forall y^* \in F_g.$$

The purpose of this paper is to extend the technique and the results given in [18] to the case of nonself generalized contractions on gauge spaces.

1.1. **Notations.** We use the following symbols $\mathbb{N}=\{0,1,2,...\}$, $\mathbb{N}^* = \{1,2,3,...\}$, $\mathbb{R}_+ = \{x\in\mathbb{R} \mid x\geq 0\}$, $\mathbb{R}_+^* = \{x\in\mathbb{R}\mid x>0\}$.

A gauge space is a set endowed with a gauge structure induced by a family $\{d_i : i \in I\}$ of pseudo-metrics, where I is a directed set. The basic definitions and properties of gauge spaces may be found, for example, in [2].

Let $\left(X,(d_i)_{i\in I}\right)$ be a metric a gauge space. We will use the following symbols: $\mathcal{P}(X) = \{Y \mid Y \subset X\}$, $P(X) = \{Y \subset X \mid Y \neq \emptyset\}$, $P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$, $P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$, $P_{b,cl}(X) := P_b(X)\cap P_{cl}(X)$.
If $f : X \to X$ is an operator then $F_f := \{x \in X\mid x = f(x)\}$ denotes the fixed point set of the operator f. In the case when the operator f has an unique fixed point $x^* \in X$ then we write $F_f = \{x^*\}$.

The diameter functional $\delta_i : P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\delta_i(A) := \sup\{d_i(a,b) \mid a,b\in A\},\ i\in I.$$

1.2. **Comparison functions.** Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. We consider the following conditions relative to φ:

(i_φ) φ is increasing;
(ii_φ) $\varphi(t) < t$, $\forall t > 0$;
(iii_φ) $\varphi(0) = 0$;
(iv_φ) $\varphi^n(t) \to 0$ as $n\to\infty$, $\forall t \in \mathbb{R}_+$;
(v_φ) $t-\varphi(t)\to\infty$ as $t\to\infty$;
(vi_φ) $\sum\limits_{n=0}^{\infty}\varphi^n(t) < +\infty$, $\forall t\in\mathbb{R}_+$.

Definition 1.3. *(I.A. Rus [13]) By definition the function φ is a comparison function if it satisfies the conditions (i_φ) and (iv_φ).*

Definition 1.4. *A comparison function is:*

(a) *strict comparison function if it satisfies the condition (v_φ);*
(b) *strong comparison function if it satisfies the condition (vi_φ).*

It is clear that if φ is a comparison function then $\varphi(t) < t$, $\forall t > 0$, and $\varphi(0) = 0$. If φ is a strong comparison function then the functions φ and $\sum\limits_{n=0}^{\infty}\varphi^n$ are continuous in $t = 0$. For example, if $\varphi(t) := at$, $t\in\mathbb{R}_+$, $a\in[0;1[$, then φ is a strict and strong comparison function and $\varphi(t) := \frac{t}{1+t}$, $t\in\mathbb{R}_+$, is a strict comparison function which is not a strong comparison function.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. In this case we define the function $\theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by,

$$\theta_\varphi(t) = \sup\{s\in\mathbb{R}_+ \mid s-\varphi(s)\leq t\}.$$

We remark that θ_φ is increasing and $\theta_\varphi(t)\to 0$ as $t\to 0$. The function θ_φ appears when we study the data dependence of the fixed points.

For more considerations on comparison functions see [13], [17] and [4].

2. A fixed point theorem for nonself φ-contractions

Let $\left(X,(d_i)_{i\in I}\right)$ be a gauge space, $Y\in P_{cl}(X)$ and $f:Y\to X$ be an operator.

Definition 2.1. *An operator $f:Y\to X$ is a nonself φ-contractions if for every $i\in I$ there exists a comparison function $\varphi_i:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$d_i\left(f(x),f(y)\right)\le\varphi_i\left(d_i(x,y)\right),\ \forall x,y\in Y.$$

We have:

Theorem 2.2. *Let $\left(X,(d_i)_{i\in I}\right)$ be a complete gauge space, $Y\subset X$ a nonempty bounded closed subset and $f:Y\to X$ a continuous operator. We suppose that:*

(i) *f is a nonself φ-contractions;*
(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n\in\mathbb{N}^*$;*

Then:

(a) *$F_f=\{x^*\}$;*
(b) *$f^{n-1}(x_n)\to x^*$ and $f^n(x_n)\to x^*$ as $n\to+\infty$;*
(c) *if φ_i are strict comparison function for all $i\in I$ then*

$$d_i(x,x^*)\le\theta_{\varphi_i}\left(d_i(x,f(x))\right),\forall x\in Y,\ \forall i\in I.$$

(d) *Let $g:Y\to X$ be such that:*
   (1) *for all $i\in I$ there exists $\eta_i>0$ such that $d_i\left(f(x),g(x)\right)\le\eta_i$, $\forall x\in Y$;*
   (2) *$F_g\ne\varnothing$.*

Then

$$d_i(x^*,y^*)\le\theta_{\varphi_i}(\eta_i),\forall y^*\in F_g,\ \forall i\in I.$$

Proof. $(a)+(b)$. We consider the following standard construction in the fixed point theory for the nonself operators (see for example [6]). Let $Y_1:=\overline{f(Y)}$, $Y_2:=\overline{f(Y_1\cap Y)}$, ..., $Y_{n+1}:=\overline{f(Y_n\cap Y)}$, $n\in\mathbb{N}^*$. We remark that:

(1) $Y_{n+1}\subset Y_n$, $\forall n\in\mathbb{N}^*$;
(2) $f^n(x_n)\in Y_n$, $\forall n\in\mathbb{N}^*$, so $Y_n\ne\varnothing$, $\forall n\in\mathbb{N}^*$.

Since f is a nonself φ-contraction it follows that for every $i\in I$

$$\delta_i\left(f(B)\right)\le\varphi_i\left(\delta(B)\right),\ \forall B\in P_b(Y).$$

From the properties of φ_i and δ_i we have

$$\begin{aligned}\delta_i(Y_{n+1}) &= \delta_i\left(\overline{f(Y_n\cap Y)}\right)=\delta_i\left(f(Y_n\cap Y)\right)\le\delta_i\left(f(Y_n)\right)\le\\ &\le \varphi_i\left(\delta_i(Y_n)\right)\le...\le\varphi_i^{n+1}\left(\delta_i(Y)\right)\to 0,\ \text{as } n\to+\infty.\end{aligned}$$

From Cantor intersection lemma we have

$$Y_\infty:=\bigcap_{n\in\mathbb{N}}Y_n\ne\varnothing,\ \delta_i(Y_\infty)=0\text{ and }f(Y_\infty\cap Y)\subset Y_\infty.$$

From $Y_\infty\ne\varnothing$ and $\delta_i(Y_\infty)=0$, $\forall i\in I$, we have that $Y_\infty=\{x^*\}$. On the other hand $f^n(x_n)\in Y_n$ and $f^{n-1}(x_n)\in Y_{n-1}\cap Y$. This implies that $\{f^n(x_n)\}_{n\in\mathbb{N}}$ and

$\left\{f^{n-1}(x_n)\right\}_{n\in\mathbb{N}}$ are fundamental sequences. Since Y_n, $n\in\mathbb{N}^*$, are closed, it follows that

$$f^{n-1}(x_n)\to x^* \text{ and } f^n(x_n)\to x^* \text{ as } n\to+\infty.$$

Since f is continuous then $f^n(x_n)\to f(x^*)$, so $f(x^*)=x^*$.

(c). The conclusion (c) follows from the following estimation:

$$d_i(x,x^*)\le d_i(x,f(x))+d_i(f(x),x^*)\le d_i(x,f(x))+\varphi_i(d_i(x,x^*)),\ \forall x\in Y,\ \forall i\in I.$$

So,

$$d_i(x,x^*)-\varphi_i(d(x,x^*))\le d_i(x,f(x)),\ \forall x\in Y,\quad \forall i\in I.$$

(d). Let $y^*\in F_g$ then from (c) it follows that

$$d_i(x^*,y^*)\le\theta_{\varphi_i}(d_i(y^*,f(y^*)))=\theta_{\varphi_i}(d_i(g(y^*),f(y^*)))\le\theta_{\varphi_i}(\eta_i),\ \forall i\in I. \qquad \square$$

2.1. **Maximal displacement functional.** Let $\left(X,(d_i)_{i\in I}\right)$ be a gauge space, $Y\in P_{cl}(X)$ and $f:Y\to X$ be a continuous nonself operator. By the maximal displacement functional corresponding to f and d_i, $i\in I$, we understand the functional $E_{f,i}:P(Y)\to\mathbb{R}_+\cup\{+\infty\}$ defined by

$$E_{f,i}(A):=\sup\{d_i(x,f(x))\ \mid\ x\in A\},$$

We have that:

(i) $A,B\in P(Y)$, $A\subset B$ imply $E_{f,i}(A)\le E_{f,i}(B)$;

(ii) $E_{f,i}(A)=E_{f,i}\left(\overline{A}\right)$ for all $A\in P(Y)$.

Definition 2.3. *An operator $f:Y\to X$ is a nonself graphic contraction if for every $i\in I$ there exists $\alpha_i\in[0;1[$ such that if $x\in Y$, $f(x)\in Y$ imply*

$$d_i\left(f^2(x),f(x)\right)\le\alpha_i d_i(x,f(x)),\ \forall i\in I.$$

Example 2.4. *If $f:Y\to X$ is contraction, then f is graphic contraction.*

Example 2.5. *If $f:Y\to X$ is Kannan operator, i.e., for every $i\in I$ there exists $\alpha_i\in\left[0;\frac{1}{2}\right[$ such that*

$$d_i(f(x),f(y))\le\alpha_i[d_i(x,f(x))+d_i(y,f(y))],\ \forall x,y\in Y,\ \forall i\in I,$$

then f is graphic contraction.

Proof. Let $i\in I$, $x\in Y$ such that $f(x)\in Y$ then

$$d_i\left(f^2(x),f(x)\right)\le\alpha_i\left[d_i\left(f(x),f^2(x)\right)+d_i(x,f(x))\right],$$

so,

$$d_i\left(f^2(x),f(x)\right)\le\frac{\alpha_i}{1-\alpha_i}d_i(x,f(x)) \qquad \square$$

We also have the following auxiliary result.

Lemma 2.6. *Let $\left(X,(d_i)_{i\in I}\right)$ be a gauge space, $Y\in P_{cl}(X)$ and $f:Y\to X$ be a continuous graphic contraction. Then:*

(a) *$E_{f,i}(f(A))\le\alpha_i\cdot E_{f,i}(A)$, for all $A\subset Y$ with $f(A)\subset Y$, $\forall i\in I$;*

(b) *$E_{f,i}(f(A)\cap Y)\le\alpha_i E_{f,i}(A)$, for all $A\subset Y$ with $f(A)\cap Y\neq\varnothing$, $\forall i\in I$.*

Proof. (a) Let $A \subset Y$ with $f(A) \subset Y$. Then

$$\begin{aligned} E_{f,i}(f(A)) &= \sup\{d_i(x, f(x)) \mid x \in f(A)\} \\ &= \sup\left\{d_i\left(f(u), f^2(u)\right) \mid u \in A\right\} \\ &\leq \alpha_i \cdot \sup\{d_i(u, f(u)) \mid u \in A\} = \alpha_i \cdot E_{f,i}(A) \end{aligned}$$

(b) We have

$$\begin{aligned} E_{f,i}(f(A) \cap Y) &= \sup\{d_i(x, f(x)) \mid x \in f(A) \cap Y\} \\ &= \sup\left\{d_i\left(f(u), f^2(u)\right) \mid u \in A,\ f(u) \in Y\right\} \\ &\leq \alpha_i \cdot \sup\{d_i(u, f(u)) \mid u \in A\} = \alpha_i \cdot E_{f,i}(A) \end{aligned}$$

□

3. A fixed point theorem for nonself Kannan operators

The following theorem is the main result of this section.

Theorem 3.1. *Let $\left(X, (d_i)_{i \in I}\right)$ be a complete gauge space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:*

(i) *f is a nonself Kannan operator;*
(ii) *there exists a sequence $(x_n)_{n \in \mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*
(iii) *$E_{f,i}(Y) < +\infty$, $\forall i \in I$.*

Then, the following conclusions hold:

(a) *$F_f = \{x^*\}$;*
(b) *$f^{n-1}(x_n) \to x^*$ and $f^n(x_n) \to x^*$ as $n \to +\infty$;*
(c) *$d_i(x, x^*) \leq (1 + \alpha_i) d_i(x, f(x))$, $\forall x \in Y$, $\forall i \in I$;*
(d) *$d_i\left(f^{n-1}(x_n), x^*\right) \leq \alpha_i^{n-1}(1 - \alpha_i)^{1-n}(1 + \alpha_i) d_i(x_n, f(x_n))$, $\forall n \in \mathbb{N}^*$, $\forall i \in I$;*
(e) *Let $g : Y \to X$ be such that:*
   (1) *for every $i \in I$ there exists $\eta_i > 0$ such that $d_i(f(x), g(x)) \leq \eta_i$, $\forall x \in Y$;*
   (2) *$F_g \neq \varnothing$.*
   Then, we have $d_i(x^*, y^*) \leq (1 + \alpha_i)\eta_i, \forall y^* \in F_g$, $\forall i \in I$.

Proof. $(a) + (b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1 \cap Y)}$, ..., $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$.
Since f is Kannan operator, from Example 2.5 and Lemma 2.6, we have that:

$$\begin{aligned} \delta_i(Y_{n+1}) &= \delta_i\left(\overline{f(Y_n \cap Y)}\right) = \delta_i(f(Y_n \cap Y)) \leq 2\alpha_i \cdot E_{f,i}(Y_n \cap Y) \\ &= 2\alpha_i \cdot E_{f,i}\left(\overline{f(Y_{n-1} \cap Y)} \cap Y\right) = 2\alpha_i \cdot E_{f,i}(f(Y_{n-1} \cap Y) \cap Y) \\ &\leq \frac{2\alpha_i^2}{1 - \alpha_i} E_{f,i}(Y_{n-1} \cap Y) \leq ... \leq \frac{2\alpha_i^{n+1}}{(1 - \alpha_i)^n} E_{f,i}(Y) \to 0,\ n \to +\infty, \forall i \in I. \end{aligned}$$

Now, the rest of the proof is similar with the proof of Theorem 2.2.

(c). Let $x \in Y$. From the definition of the Kannan operator we have:

$$d_i(x, x^*) \leq d_i(x, f(x)) + d_i(f(x), x^*) \leq d_i(x, f(x)) + \alpha_i d_i(x, f(x)),\ \forall x \in Y,\ \forall i \in I.$$

(d) and (e) follow from (c). □

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