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FIXED POINT THEOREMS FOR NONSELF OPERATORS IN b-METRIC SPACES

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Abstract. In this paper we prove some fixed point theorems for different type of contractions in the setting of a b-metric space. The starting point was a recent result of Rus and Şerban [16]. The presented theorems extend, generalize and unify several recent results in the literature.
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1. Introduction

The aim of this paper is to prove some fixed point theorems for nonself singlevalued operators in the context of a b-metric space. The main idea came from some recent results (see [16]) where the authors give another proof of the main result in Reich and Zaslavski [13]. We generalize the reults in the sense that we consider the case of a $b-$metric space. We prove fixed point theorems where the operators are φ-contractions, Kannan contractions, Hardy-Rogers contractions. We also give some data dependence results.

2. Preliminaries

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. We recollect some essential definitions and fundamental results. We begin with the definition of a b-metric space.

Definition 2.1. (Bakhtin [2], Czerwik [10]) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric with constant $s \geq 1$ if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$,

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

For more details and examples on b-metric spaces, see e.g. [1, 2, 3, 6, 7, 9, 10].

We consider next the following families of subsets of a b-metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\},\ P_b(X) := \{Y \in P(X) |\ \delta(Y) < \infty\},$$

$$P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\},\ P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$$

Let us consider the following functionals.

First, we will denote by $\delta(A) = \sup\{d(a, b),\ a, b \in A\}$, the diameter functional.

The maximal displacement functional is given as follows.

Let (X, d) be a b-metric space, $Y \in P_{cl}(X)$, $f : Y \to X$ continuous, $E_f : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}$.

$$E_f(A) = sup\{d(x, f(x))\ x \in A\}$$

We have the following properties:

(i) $A, B \in P(Y),\ A \subset B$ imply $E_f(A) \leq E_f(B)$;
(ii) $E_f(A) = E_f(\bar{A})$ for all $A \in P(Y)$.

Let (X, d) be a b-metric space and $Y \subset X$ and let $f : Y \to X$. The set $Fix(f) := \{x \in X | x = f(x)\}$ is called the fixed point set of f. In the case when f has a unique fixed point $x^* \in X$, we write $Fix(f) = \{x^*\}$.

Let us consider the following definitions and lemmas, which are useful in the proofs of our main theorems.

Definition 2.2. Let (X, d) a metric space, $Y \in P_{cl}(X)$. An operator $f : Y \to X$ is an α-graphic contraction if $0 \leq \alpha < 1$ and $x \in Y,\ f(x) \in Y$ imply

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)).$$

If $f : Y \to X$ is an $\alpha-$Kannan operator, i.e. $0 \leq \alpha < \frac{1}{2}$ and

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))],\ \forall x, y \in Y,$$

then f is $\frac{\alpha}{1-\alpha}-$graphic contraction.

Lemma 2.1. *Let (X, d) be a b-metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous α-graphic contraction. Then:*

(i) *$E_f(f(A)) \leq \alpha \cdot E_f(A)$, for all $A \subset Y$ with $f(A) \subset Y$;*
(ii) *$E_f(f(A) \cap Y) \leq \alpha \cdot E_f(A))$, for all $A \subset Y$ with $f(A) \cap Y \neq \emptyset$.*

Proof. The proof follows from the definition of E_f. □

Definition 2.3. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Then:

(i) φ is called a comparison function if φ is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for all $t \to 0$;

(ii) φ is called a strong comparison function if φ is a comparison function and is monotone increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$;

(iii) φ is called a strict comparison function if φ is a comparison function and $t - \varphi(t) \to \infty$, as $t \to \infty$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. We define the function $\theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ as follows

$$\theta_\varphi(t) := \sup\{r \in \mathbb{R}_+ | r - s \cdot \varphi(r) \leq s \cdot t\}$$

We need the above function when we study the data dependence of the fixed points.

Definition 2.4. Let (X, d) be a b-metric space. $f : X \to X$ is a $\varphi-$contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)).$$

Lemma 2.2. *(Czerwik [10]) Let (X, d) be a b-metric space with constant $s \geq 1$ and let $\{x_k\}_{k=0}^n \subset X$. Then $d(x_n, x_0) \leq sd(x_0, x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n)$.*

For more considerations on the above notions see: [4, 5, 8, 11, 14, 15]. For the multivalued case see [12].

3. Main results

In the following we state and prove our main results.

Theorem 3.1. *Let (X, d) be a complete b-metric space with $s > 1$ with $Y \subset X$ nonempty and closed. Let $f : Y \to X$ be a $\varphi-$ contraction. Suppose there exist a bounded sequence (x_n) such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}$.*

Then $Fix(f) = \{x^*\}$, $f^n(x_n) \to x^*$ and $f^{n-1}(x_n) \to x^*$.

Proof. Let $A \in P_{b,cl}(Y)$ be such that $x_n \in A$ for all $n \in \mathbb{N}^*$. We consider the following construction $A_1 := \overline{f(A)}$, $A_2 := \overline{f(A_1 \cap A)}$, $\cdots$, $A_{n+1} := \overline{f(A_n \cap A)}$, $n \in \mathbb{N}^*$.

We have:

(a) $A_{n+1} \subset A_n, \ \forall n \in \mathbb{N}^*$

(b) $f^n(x_n) \in A_n, \ \forall n \in \mathbb{N}^*$ so $A_n \neq \emptyset, \ \forall n \in \mathbb{N}^*$.

We also have that:

- $\overline{f(A_1 \cap A)} \subset \overline{f(A)}$
- $\overline{f(\overline{f(A)} \cap A)} \subset \overline{f(A)}$

$x_n \in A$, $f^n(x_n) \in A_n$, $A_n = \overline{f(A_{n-1} \cap A)}$

Since f is a φ contraction and:

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in .Y$$

It follows that:

$$\delta(f(B)) \leq \varphi(\delta(B)), \text{ for all } B \in P_b(Y).$$

Using the properties of φ and δ we obtain:

$$\delta(A_{n+1}) = \delta(\overline{f(A_n \cap A)}) = \delta(f(A_n \cap A)) \leq \delta(f(A_n))$$
$$\leq \varphi(\delta(A_n)) \leq \cdots \leq \varphi^{n+1}(\delta(A)) \to 0,$$

as $n \to \infty$.

From Cantor's intersection theorem we have:

$$A_\infty := \cap_{n \in \mathbb{N}} A_n \neq \emptyset, \ \delta(A_\infty) = 0, \ f(A_\infty \cap A) \subset A_\infty.$$

From $A_\infty \neq \emptyset$ and $\delta(A_\infty) = 0$ we have that $A_\infty = \{x^*\}$.

On the other hand $f^n(x_n) \in A_n$, $f^{n-1} \in A_{n-1} \cap Y$. So $\{f^n(x_n)\}_{n\in\mathbb{N}}$, $\{f^{n-1}(x_n)\}_{n\in\mathbb{N}}$ are fundamental sequences. Since $A_n, n \in \mathbb{N}^*$ are closed, we have that

$$f^n(x_n) \to x^*, f^{n-1}(x_n) \to x^*, \text{ as } n \to \infty.$$

Since f is continuous, it follows that $f^n(x_n) \to f(x^*)$ as $n \to \infty$, so $f(x^*) = x^*$. □

Theorem 3.2. *Let $f : Y \to X$ be as in Theorem 3.1, where φ is a strict comparison function. Then:*

(i) $d(f^n(x_n), x^*) \leq \varphi^n(d(x_n, x^*))$;

(ii) $d(x, x^*) \leq \theta_\varphi(d(x, f(x)))$, *for all* $x \in Y$, *where* $\theta_\varphi(t) := \sup\{r \in \mathbb{R}_+ | r - s \cdot \varphi(r) \leq s \cdot t\}$;

(iii) *Let* $g : Y \to X$ *such that there exists* $\eta > 0$ *such that* $d(f(x), g(x)) \leq \eta$, *for all* $x \in Y$ *and* $Fix(g) \neq \emptyset$. *Then* $d(x^*, y^*) \leq \theta_\varphi(\eta)$, *for all* $y^* \in Fix(g)$.

Proof. (i) $d(f^n(x_n), x^*) = d(f^n(x_n), f(x^*)) \leq \varphi(d(x_n, x^*))$.

(ii) Estimating $d(x, x^*)$ we obtain:

$$d(x, x^*) \leq s \cdot [d(x, f(x)) + d(f(x), x^*)] \leq s \cdot d(x, f(x)) + s \cdot \varphi(d(x, x^*)).$$

We obtain:

$$d(x, x^*) - s \cdot \varphi(d(x, x^*)) \leq s \cdot d(x, f(x))$$

Hence:

$$d(x, x^*) \leq \theta_\varphi(d(x, f(x))), \ \forall x \in Y$$

(iii) Choosing in (ii) $x = y^*$ we obtain that

$$d(x^*, y^*) \leq \theta_\varphi(d(y^*, f(y^*))) = \theta_\varphi(d(g(y^*), f(y^*))) \leq \theta_\varphi(\eta)$$ □

The next main result is a fixed point theorem for a nonself Kannan operator.

Theorem 3.3. *Let (X, d) be a complete b metric space with $s > 1$, $Y \subset X$ a non-empty, bounded, closed subset and $f : Y \to X$ a continuous operator. Suppose that:*

(a) *f is an α-Kannan operator;*

(b) *there exist a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*

(c) $E_f(Y) < \infty$

Then:

(i) $Fixf = \{x^*\}$;

(ii) $f^{n-1}(x_n) \to x^*$ *and* $f^n(x_n) \to x^*$ *as* $n \to \infty$;

(iii) $d(x, x^*) \leq s \cdot (1 + \alpha) \cdot d(x, f(x))$, *for all* $x, y \in Y$;

(iv) $d(f^{n-1}(x_n), x^*) \leq \cdots d(x_n, f(x_n))$, *for all* $n \in \mathbb{N}$;

(v) *Let* $g : Y \to X$ *such that there exists* $g : Y \to X$ *such that* $d(f(x), g(x)) \leq \eta$, *for all* $x \in Y$ *and let* $Fix(g) \neq \emptyset$.
Then $d(x^*, y^*) \leq \eta \cdot s \cdot (1 + \alpha)$, *for all* $y^* \in Fix(g)$.

Proof. (i)+(ii) Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$, $f^n(x_n) \in Y_n$. So $Y_n \neq \emptyset$, $n \in \mathbb{N}^*$.

Since f is a Kannan operator it follows that f is $\frac{\alpha}{1-\alpha}$ graph contraction. We apply lema 2.1 and we have:

$$\begin{aligned}\delta(Y_{n+1}) &= \delta(\overline{f(Y_n \cap Y)}) = \delta(f(Y_n \cap Y)) \leq \frac{\alpha}{1-\alpha} E_f(Y_n \cap Y) \leq 2\alpha E_f(Y_n \cap Y)\\ &= 2\alpha E_f(\overline{f(Y_{n-1} \cap Y)} \cap Y) = 2\alpha E_f(f(Y_{n-1} \cap Y) \cap Y)\\ &\leq \frac{2\alpha^2}{1-\alpha} E_f(Y_{n-1} \cap Y) \leq \cdots \leq \frac{2\alpha^{n+1}}{(1-\alpha)^n} E_f(Y) \to 0, \text{ as } n \to \infty.\end{aligned}$$

The rest of the proof is similar with the proof from the previous theorem.

(iii) We have the following:

$$\begin{aligned}d(x, x^*) &\leq s[d(x, f(x)) + d(f(x), x^*)] = s[d(x, f(x)) + d(f(x), f(x^*))]\\ &\leq s[d(x, f(x)) + \alpha d(x, f(x)) + \alpha d(x^*, f(x^*))]\\ &= sd(x, f(x)) + s\alpha d(x, f(x)) = d(x, f(x))s \cdot (1+\alpha).\end{aligned}$$

(iv) The conclusion of (iii) follows from (iii).

(v) Now we choose $x = y^*$ in the above inequality and we have:

$$d(x^*, y^*) \leq d(y^*, f(y^*))s \cdot (1+\alpha).$$

From $y^* = g(y^*)$ it follows that

$$d(x^*, y^*) \leq d(g(y^*), f(y^*))(s + s \cdot \alpha) \leq \eta \cdot s \cdot (1+\alpha).$$ □

We will introduce the concept of the Hardy-Rogers operator in the setting of a b-metric space.

Definition 3.1. Let (X, d) be a b-metric space with $s > 1$, $Y \in P_{cl}(X)$, $f : Y \to X$ an operator. f is Hardy-Rogers operator if there exist $a, b, c \in \mathbb{R}_+$ with $a+2b+2cs < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))] + c[d(x, f(y)) + d(y, f(x))], \ \forall x, y \in Y. \tag{3.1}$$

Regarding the above definition we have the following auxiliary result.

Lemma 3.1. *Let (X, d) be a b-metric space with $s > 1$, $Y \in P_{cl}(X)$, $f : Y \to X$ a nonself Hardy-Rogers operator. Then f is a nonself α-graphic contraction with $\alpha = \frac{a+b+c\cdot s}{1-b-c\cdot s}$.*

Proof. Let $x \in Y$ such that $f(x) \in Y$. Then (by choosing $y := x, x := f(x)$) we have:

$$\begin{aligned}&d(f^2(x), f(x)) \leq\\ &\leq a \cdot d(f(x), x) + b \cdot [d(f(x), f^2(x)) + d(x, f(x))] + c \cdot [d(f(x), f(x)) + d(x, f^2(x))]\\ &= a \cdot d(x, f(x)) + b \cdot d(f(x), f^2(x)) + b \cdot d(x, f(x)) + c \cdot s[d(x, f(x)) + d(f(x), f^2(x))]\\ &= d(x, f(x))[a + b + c \cdot s] + d(f(x), f^2(x))[b + c \cdot s]\end{aligned}$$

Hence

$$d(f^2(x), f(x)) \leq \frac{a+b+c \cdot s}{1-(b+c \cdot s)} \cdot d(x, f(x))$$

It follows that f is an α-graphic contraction with $\alpha = \frac{a+b+c\cdot s}{1-(b+c\cdot s)}$. □

Lemma 3.2. *Let (X,d) be a b-metric space with $s>1$, $Y \in P_{cl}(X)$, $f : Y \to X$ a nonself Hardy-Rogers operator. Then:*

(a) $\delta(f(A)\cap Y) \leq (a+2cs)\delta(A) + (2b+2cs)E_f(A)$, *for all* $A \subset Y$*;*
(b) $E_f(f(A)\cap Y) \leq \alpha E_f(A)$, *for all* $A \subset Y$*, where* $\alpha = \frac{a+b+c\cdot s}{a-b-c\cdot s}$

Proof. (a) Let $A \subset Y$. Then, we have:

$$\begin{aligned}
&\delta(f(A)\cap Y) = \sup\{d(x,y)|x,y \in f(A)\cap Y\}\\
&\quad = \sup\{d(f(u),f(v))|u,v \in A, f(u),f(v) \in Y\}\\
&\quad \leq a\cdot \sup\{d(u,v)|u,v\in A\} + 2b\cdot \sup\{d(u,f(u))|u \in A\}\\
&\quad + 2c\cdot \sup\{d(u,f(v))|u \in A, f(v)\in Y\}\\
&\quad \leq a\cdot\delta(A) + 2b\cdot E_f(A) + 2c\cdot [sup\{s\cdot d(u,v)|u,v\in A\}\\
&\quad + s\cdot \sup\{d(v,f(v))|v\in A, f(v)\in Y\}] \leq (a+2cs)\cdot\delta(A) + (2b+2cs)\cdot E_f(A)
\end{aligned}$$

(b) The proof follows from Lemma 2.1 and Lemma 3.1. □

The next result is a fixed point theorem for a nonself Hardy-Rogers operator.

Theorem 3.4. *Let (X,d) be a complete b-metric space with $1 < s < \frac{1-2a-2b}{2c}$, $Y \subset X$ a nonempty, bounded, closed subset and $f : Y \to X$ a continuous operator. We suppose:*

(a) *f is Hardy-Rogers operator;*
(b) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*
(c) $E_f(Y) < \infty$.

Then:

(i) $Fixf = \{x^*\}$*;*
(ii) $d(x,x^*) \leq \frac{s+sb+s^2c}{1-sa-2s^2c}\cdot d(x,f(x))$*, for all* $x \in Y$*, with* $s \in (1, \frac{-a+\sqrt{a^2+8c}}{4c})$*;*
(iii) *Let $g : Y \to X$ such that there exist $\eta > 0$ such that $d(f(x),g(x)) \leq \eta$, for all $x \in Y$ and $Fixg \neq \emptyset$. Then*

$$d(x^*,y^*) \leq \frac{s+sb+s^2c}{1-sa-s^2c-s^2c}\cdot\eta,\ \forall y^* \in Fix(g) \quad \textit{and}\ s \in (1, \frac{-a+\sqrt{a^2+8)}}{4c}).$$

Proof. (i) Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1\cap Y)}$, $\cdots$, $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$.
We remark that $Y_{n+1} \subset Y_n$, $f^n(x_n) \in Y_n$. So $Y_n \neq \emptyset$, $n\in\mathbb{N}^*$.
Since f is a Hardy-Rogers operator, from Lemma 3.2 denoting by $a_1 := a+2cs$ and

$b_1 := 2b + 2cs$ we have:

$$\delta(Y_{n+1}) = \delta(\overline{f(Y_n \cap Y)}) \leq a_1\delta(Y_n) + b_1E_f(Y_n) \leq a_1\delta(Y_n) + b_1E_f(f(Y_{n-1}) \cap Y)$$
$$\leq a_1\delta(Y_n) + b_1\alpha E_f(Y_{n-1}) \leq a_1[a_1\delta(Y_{n-1}) + b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})]$$
$$\leq a_1^2\delta(Y_{n-1}) + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})$$
$$\leq a_1^2[a_1\delta(Y_{n-2}) + b_1\alpha E_f(Y_{n-3}) + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})]$$
$$\leq a_1^3\delta(Y_{n-2}) + a_1^2b_1\alpha E_f(Y_{n-3}) + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})$$
$$\leq a_1^3[a_1\delta(Y_{n-3}) + b_1\alpha E_f(Y_{n-4}) + a_1^2b_1\alpha E_f(Y_{n-3}) + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})]$$
$$\leq a_1^4\delta(Y_{n-3}) + a_1^3b_1\alpha E_f(Y_{n-4}) + a_1^2b_1\alpha E_f(Y_{n-3}) + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})$$
$$\leq \cdots \leq a_1^{n-1}\delta(Y_2) + a_1^{n-2}b_1\alpha E_f(Y_1) + a_1^{n-3}b_1\alpha E_f(Y_2)$$
$$+ \cdots + a_1b_1\alpha E_f(Y_{n-2}) + b_1\alpha E_f(Y_{n-1})$$
$$\leq a_1^{n-1}[a_1\delta(Y_1) + b_1\alpha E_f(Y)] + b_1\alpha\sum_{k=0}^{n-2} a_1^kE_f(Y_{n-k-1})$$
$$\leq a_1^n\delta(Y_1) + a_1^{n-1}b_1\alpha E_f(Y)] + b_1\alpha\sum_{k=0}^{n-2} a_1^kE_f(Y_{n-k-1})$$
$$\leq a_1^n\delta(f(Y)) + a_1^{n-1}b_1\alpha E_f(Y) + b_1\alpha[a_1^{n-2}E_f(Y_1) + a_1^{n-3}E_f(Y_2)$$
$$+ \cdots + a_1E_f(Y_{n-2}) + E_f(Y_{n-1})]$$
$$\leq a_1^n\delta(f(Y)) + a_1^{n-1}b_1\alpha E_f(Y) + b_1\alpha[a_1^{n-2}\alpha E_f(Y) + a_1^{n-3}\alpha^2E_f(Y)$$
$$+ \cdots + a_1\alpha^{n-2}E_f(Y) + \alpha^{n-1}E_f(Y)]$$
$$\leq a_1^n\delta(f(Y)) + a_1^{n-1}b_1\alpha E_f(Y) + b_1\alpha E_f(Y)[a_1^{n-2}\alpha + a_1^{n-3}\alpha^2 + \cdots + a_1^0\alpha^{n-1}]$$
$$\leq a_1^n\delta(f(Y)) + b_1\alpha[a_1^{n-1}\alpha^0 + a_1^{n-2}\alpha + a_1^{n-3}\alpha^2 + \cdots + a_1^0\alpha^{n-1}]$$
$$\leq a_1^n\delta(f(Y)) + b_1\alpha E_f(Y)\sum_{i=0}^{n-1} a_1^i\alpha^{n-i-1}.$$

We have that $a_1^n \to 0$, as $n \to \infty$. From $\alpha < 1$ and applying a Cauchy type Lemma (see [17]) it follows that the sum written above tends to 0. Thus $\delta(Y_{n+1}) \to 0$, as $n \to \infty$. The rest of the proof follows as in the above main theorems.

(ii) Let $x \in Y$. From the definition of Hardy-Rogers operator we have:

$$\begin{aligned}
d(x,x^*) &\leq sd(x,f(x)) + sd(f(x),x^*) \leq sd(x,f(x)) \\
&\quad + s\{ad(x,x^*) + b[d(x,f(x)) + d(x^*,f(x^*))] + c[d(x,f(x^*)) + d(x^*,f(x))]\} \\
&\leq sd(x,f(x)) + sad(x,x^*) + sb[d(x,f(x)) + scd(x,x^*) \\
&\quad + sc[sd(x,x^*) + sd(x,f(x))] \\
&= (s + sb + s^2c)d(x,f(x)) + (sa + sc + s^2c)d(x,x^*).
\end{aligned}$$

From the hypothesis we have that $1 - sa - s^2c - s^2c > 0$. Hence

$$d(x,x^*) \leq \frac{s + sb + s^2c}{1 - sa - s^2c - s^2c}d(x,f(x)).$$

(iii) In (ii) we choose $x = y^*$, where $y^* \in Fix(g)$ and we obtain:

$$\begin{aligned} d(x^*, y^*) &\leq \frac{s + sb + s^2c}{1 - sa - s^2c - s^2c} d(y^*, f(y^*)) \\ &= \frac{s + sb + s^2c}{1 - sa - s^2c - s^2c} d(g(y^*), f(y^*)) \leq \frac{s + sb + s^2c}{1 - sa - s^2c - s^2c}\eta. \end{aligned}$$

□

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