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# STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR MAXIMAL MONOTONE OPERATORS AND GENERALIZED HYBRID MAPPINGS

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**Abstract.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $T$ be a supper hybrid mapping of $C$ into $H$, let $A$ be an inverse strongly monotone mapping of $C$ into $H$ and let $B$ be a maximal monotone operator on $H$ such that the domain of $B$ is included in $C$. In this paper, we introduce two iterative sequences by hybrid methods of finding a point of $F(T)\cap(A+B)^{-1}0$, where $F(T)$ is the set of fixed points of $T$ and $(A+B)^{-1}0$ is the set of zero points of $A+B$. Then, we prove two strong convergence theorems in a Hilbert space. Using these results, we give some applications.
**Key Words and Phrases**: Hilbert space, nonexpansive mapping, nonspreading mapping, supper hybrid mapping, fixed point, strong convergence, hybrid method.
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## 1. Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a mapping of $C$ into $H$. Then, we denote by $F(T)$ the set of fixed points of $T$. For a constant $\alpha > 0$, the mapping $A : C \to H$ is said to be $\alpha$-inverse strongly monotone if

$$\langle x-y, Ax-Ay\rangle \geq \alpha \|Ax-Ay\|^2$$

for all $x, y \in C$. An $\alpha$-inverse strongly monotone mapping is also Lipschitz continuous with a Lipschitz constant $\frac{1}{\alpha}$. A mapping $S$ of $C$ into $H$ is nonexpansive if $\|Su-Sv\| \leq \|u-v\|$ for all $u, v \in C$. A mapping $T : C \to H$ is said to be a strict pseudo-contraction [6] if there exists a real number $k$ with $0 \leq k < 1$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2 \tag{1.1}$$

for all $x, y \in C$. We also call such a mapping $T$ a $k$-strict pseudo-contraction. A $k$-strict pseudo-contraction $T : C \to H$ is nonexpansive if $k = 0$. A mapping $T : C \to H$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tu-v\| \leq \|u-v\|$ for all $u \in C$ and

$v \in F(T)$. If $S : C \to H$ is a nonexpansive mapping, then $I - S$ is $\frac{1}{2}$-inverse strongly monotone, where $I$ is the identity mapping on $H$. A nonexpansive mapping $S : C \to H$ with $F(S) \neq \emptyset$ is quasi-nonexpansive; see, for instance, [27]. We also know that if $T : C \to H$ is a $k$-strict pseudo-contraction with $0 \leq k < 1$, then $A = I - T$ is a $\frac{1-k}{2}$-strict pseudo-contraction; see, for instance, Marino and Xu [18]. A mapping $S$ of $C$ into $H$ is nonspreading if

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2$$

for all $u, v \in C$; see [15, 16]. A mapping $S$ of $C$ into $H$ is hybrid if

$$3\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2 + \|u - v\|^2$$

for all $u, v \in C$; see [28]. Recently, Kocourek, Takahashi and Yao [14] introduced a broad class of nonlinear mappings which contains nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. They called such mappings generalized hybrid mappings; see Section 2. Furthermore, they defined a class of nonlinear mappings called super hybrid containing generalized hybrid mappings. We know that a super bybrid mapping is not quasi-nonexpansive generally. A multi-valued mapping $B \subset H \times H$ is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in H$, $u \in Bx$ and $v \in By$. A monotone operator $B$ on $H$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $H$.

In this paper, we introduce two iterative sequences by hybrid methods of finding a point of $F(T) \cap (A + B)^{-1}0$, where $T$ is a supper hybrid mapping, $A$ is an inverse strongly monotone mapping and $B$ is a maximal monotone operator in a Hilbert space. Then, we prove two strong convergence theorems in a Hilbert space. Using these results, we obtain well-known, or new results.

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [27], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [27] for more details.

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space $H$, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to $x$ and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to $y$ and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $C_0$ satisfies

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.3}$$

it is said that $\{C_n\}$ converges to $C_0$ in the sense of Mosco [19] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [19]. We know the following theorem [35].

**Theorem 2.1.** *Let $H$ be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $H$. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where $P_{C_n}$ and $P_{C_0}$ are the mertic projections of $H$ onto $C_n$ and $C_0$, respectively.*

Let $H$ be a Hilbert space and let $C$ be a nonempty closed and convex subset of $H$. Then, a mapping $T : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta)$-generalized hybrid mapping. Notice that the mapping above covers several well-known mappings. For example, an $(\alpha, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|x - y\|^2 + (1 - \beta)\|x - y\|^2$$

and hence $\|x - Ty\| \le \|x - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. We also know a more general class of mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping $S : C \to H$ is called super hybrid [14] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \le \\ \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2\end{aligned} \tag{2.5}$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma)$-super hybrid mapping. We notice that an $(\alpha, \beta, 0)$-super hybrid mapping is $(\alpha, \beta)$-generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. A mapping $U : C \to H$ is called extended hybrid [9] if there exist $\alpha, \beta, r \in \mathbb{R}$ such that

$$\begin{aligned}\alpha(1 + r)\|Ux - Uy\|^2 + (1 - \alpha(1 + r))\|x - Uy\|^2 \\ \le (\beta + \alpha r)\|Ux - y\|^2 + (1 - (\beta + \alpha r))\|x - y\|^2 \\ - (\alpha - \beta)r\|x - Ux\|^2 - r\|y - Uy\|^2\end{aligned} \tag{2.6}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta,\ r)$-extended hybrid mapping. Putting $\gamma = \frac{-r}{1+r}$ in (2.5) with $1 + r > 0$, we get that for all $x, y \in C$,

$$\begin{aligned}\alpha\|Sx - Sy\|^2 + (1 - \alpha + \frac{-r}{1+r})\|x - Sy\|^2 \leq \\ \big(\beta + (\beta - \alpha)\frac{-r}{1+r}\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\frac{-r}{1+r}\big)\|x - y\|^2 \\ + (\alpha - \beta)\frac{-r}{1+r}\|x - Sx\|^2 + \frac{-r}{1+r}\|y - Sy\|^2.\end{aligned}$$

Since $1 + r > 0$, we have

$$\begin{aligned}\alpha(1+r)\|Sx - Sy\|^2 + (1 + r - \alpha(1+r) - r)\|x - Sy\|^2 \leq \\ \big(\beta(1+r) - (\beta - \alpha)r\big)\|Sx - y\|^2 + \big(1 + r - \beta(1+r) + (\beta - \alpha - 1)r\big)\|x - y\|^2 \\ - (\alpha - \beta)r\|x - Sx\|^2 - r\|y - Sy\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha(1+r)\|Sx - Sy\|^2 + (1 - \alpha(1+r))\|x - Sy\|^2 \leq \\ \big(\beta + \alpha r\big)\|Sx - y\|^2 + \big(1 - (\beta + \alpha r)\big)\|x - y\|^2 \\ - (\alpha - \beta)r\|x - Sx\|^2 - r\|y - Sy\|^2.\end{aligned}$$

This implies that $S$ is $(\alpha, \beta, r)$-extended hybrid. Similarly, if $S$ is an $(\alpha, \beta, r)$-extended hybrid mapping with $1 + r > 0$, then $S$ is an $(\alpha, \beta, \frac{-r}{1+r})$-super hybrid mapping. We know the following important lemma from [32].

**Lemma 2.2.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to H$ be a $k$-strict pseudo-contraction with $0 \leq k < 1$. Then, $T$ is a $(1, 0, -k)$-extended hybrid mapping.*

If $T : C \to H$ is a $k$-strict pseudo-contraction with $0 \leq k < 1$, we have $1 - k > 0$. So, we have from Lemma 2.2 that $T$ is a $(1, 0, \frac{k}{1-k})$-supper hybrid mapping.

We know the following theorem from [34].

**Theorem 2.3.** *Let $C$ be a nonempty subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq 1$. Let $S$ and $T$ be mappings of $C$ into $H$ such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, $S$ is $(\alpha,\ \beta,\ \gamma)$-super hybrid if and only if $T$ is $(\alpha,\ \beta)$-generalized hybrid. In this case, $F(S) = F(T)$.*

From [14], we know the following theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 2.4.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T : C \to C$ be a generalized hybrid mapping. Then $T$ has a fixed point in $C$ if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

As a direct consequence of Theorem 2.4, we have the following result.

**Theorem 2.5.** *Let $C$ be nonempty bounded closed convex subset of a Hilbert space $H$ and let $T$ be a generalized hybrid mapping from $C$ to itself. Then $T$ has a fixed point.*

Using Theorems 2.3 and 2.5, we have the following fixed point theorem [14] for super hybrid mappings in a Hilbert space.

**Theorem 2.6.** *Let $C$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an ($\alpha$, $\beta$, $\gamma$)-super hybrid mapping. Then, $S$ has a fixed point in $C$.*

The following lemma for generalized hybrid mappings in a Hilbert space is essencial for proving our main theorems; see Takahashi, Yao and Kocourek [34].

**Lemma 2.7.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to H$ be a generalized hybrid mapping. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.*

## 3. Strong convergence theorems

In this section, using the hybrid method by Nakajo and Takahashi [20], we first prove a strong convergence theorem for maximal monotone operators and super hybrid mappings in a Hilbert space.

**Theorem 3.1.** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex closed subset of $H$. Let $\alpha > 0$. Let $A : C \to H$ be $\alpha$-inverse strongly monotone, let $B : D(B) \subset C \to 2^H$ be maximal monotone and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an ($\alpha$, $\beta$, $\gamma$)-super hybrid mapping such that $F(S) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\frac{1}{1+\gamma}Sz_n + \frac{\gamma}{1+\gamma}z_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $P_{C_n \cap Q_n}$ is the metric projection of $H$ onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy*

$$0 \leq \alpha_n \leq a < 1 \quad \textit{and} \quad 0 < b \leq \lambda_n \leq c < 2\alpha$$

*for some $a, b, c \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap(A+B)^{-1}(0)}x$, where $P_{F(S)\cap(A+B)^{-1}(0)}$ is the metric projection of $H$ onto $F(S) \cap (A+B)^{-1}(0)$.*

*Proof.* Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have from Theorem 2.3 that $T$ is an ($\alpha$, $\beta$)-generalized hybrid mapping of $C$ into $H$ and $F(S) = F(T)$. Since $F(T)$ is closed and convex, $F(S)$ is closed and convex. We know that $(A+B)^{-1}(0)$ is closed and convex [24]. Then, $F(S) \cap (A+B)^{-1}(0)$ is closed and convex. So, there exists the mertic projection of $H$ onto $F(S) \cap (A+B)^{-1}(0)$. Furthermore, we have

$$y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$$

for all $n \in \mathbb{N}$. Since

$$\|y_n - z\|^2 \le \|x_n - z\|^2$$
$$\Longleftrightarrow \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \le 0,$$

we have that $C_n$, $Q_n$ and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(T) \cap (A+B)^{-1}(0)$. Put $z_n = J_{\lambda_n}(I - \lambda_n A)x_n$. From $z = J_{\lambda_n}(I - \lambda_n A)z$, we have from $0 < b \le \lambda_n \le c < 2\alpha$ that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)z\|^2 \\ &\le \|x_n - \lambda_n A x_n - z + \lambda_n A z\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, Ax_n - Az\rangle + \lambda_n^2 \|Ax_n - Az\|^2 \\ &\le \|x_n - z\|^2 - 2\lambda_n \alpha \|Ax_n - Az\|^2 + \lambda_n^2 \|Ax_n - Az\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2 \\ &\le \|x_n - z\|^2. \end{aligned} \tag{3.1}$$

Since $T$ is quasi-nonexpansive, we have from (3.1) that

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tz_n - z\|^2 \\ &\le \alpha_n \|x_n\|^2 + (1-\alpha_n)\|Tz_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2. \end{aligned}$$

Thus we have $z \in C_n$ and hence $F(T) \cap (A+B)^{-1}(0) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(T) \cap (A+B)^{-1}(0) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(T) \cap (A+B)^{-1}(0) \subset Q_1$, it follows that $F(T) \cap (A+B)^{-1}(0) \subset C_1 \cap Q_1$. Suppose that $F(T) \cap (A+B)^{-1}(0) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k \cap Q_k} x$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(T) \cap (A+B)^{-1}(0) \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in F(T) \cap (A+B)^{-1}(0).$$

This implies $F(T) \cap (A+B)^{-1}(0) \subset Q_{k+1}$. So, we have $F(T) \cap (A+B)^{-1}(0) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(T) \cap (A+B)^{-1}(0) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ and $\{z_n\}$ are well-defined.

Since $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \subset Q_n$, we have from (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, x_n - x_{n+1}\rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned} \tag{3.2}$$

So, we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{3.3}$$

Furthermore, since $x_n = P_{Q_n}x$ and $z \in F(T) \cap (A+B)^{-1}(0) \subset Q_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{3.4}$$

We have from (3.3) and (3.4) that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$ are also bounded. From (3.2), we have

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{3.5}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. From (3.5), we have $\|y_n - x_{n+1}\| \to 0$. So, we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.6}$$

From $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|$, we also have from $0 \le \alpha_n \le a < 1$ that

$$\|Tz_n - x_n\| \to 0. \tag{3.7}$$

Using (3.7), we show $\|Tz_n - z_n\| \to 0$. We have from (3.1) that for any $z \in F(T) \cap (A+B)^{-1}(0)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tz_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2\} \\
&= \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-a)b(2\alpha - c)\|Ax_n - Az\|^2 &\le (1-\alpha_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Az\|^2 \\
&\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|Ax_n - Az\| = 0. \tag{3.8}$$

Since $J_{\lambda_n}$ is firmly nonexpansive, we have that

$$\begin{aligned}\|z_n - z\|^2 &= \|J_{\lambda_n}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)z\|^2 \\ &\leq \langle z_n - z, (I-\lambda_n A)x_n - (I-\lambda_n A)z\rangle \\ &= \frac{1}{2}\{\|z_n - z\|^2 + \|(I-\lambda_n A)x_n - (I-\lambda_n A)z\|^2 \\ &\quad - \|z_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\ &\leq \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 \\ &\quad - \|z_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\ &= \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 \\ &\quad - \|z_n - x_n + \lambda_n(Ax_n - Az)\|^2\} \\ &= \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|z_n - z\|^2 \leq \|x_n - z\|^2 &- \|z_n - x_n\|^2 \\ &- 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2.\end{aligned}$$

So, we have

$$\begin{aligned}\|y_n - z\|^2 \leq &\alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Tz_n - z\|^2 \\ \leq &\alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ \leq &\alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 - \|z_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - {\lambda_n}^2\|Ax_n - Az\|^2\} \\ \leq &\|x_n - z\|^2 - (1-a)\|z_n - x_n\|^2 - {\lambda_n}^2(1-\alpha_n)\|Ax_n - Az\|^2 \\ &- 2\lambda_n(1-\alpha_n)\langle z_n - x_n, Ax_n - Az\rangle.\end{aligned}$$

This means that

$$\begin{aligned}(1-a)\|z_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\quad + \|Ax_n - Az\|\{2c\|z_n - x_n\| + c^2\|Ax_n - Az\|\} \\ &\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| \\ &\quad + \|Ax_n - Az\|\{2c\|z_n - x_n\| + c^2\|Ax_n - Az\|\}.\end{aligned}$$

Since $\lim_{n\to\infty}\|Ax_n - Az\| = 0$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, and $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.9}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$, we have $y_n - Tz_n = \alpha_n(x_n - Tz_n)$. So, from (3.7) we have

$$\|y_n - Tz_n\| = \alpha_n\|x_n - Tz_n\| \to 0. \tag{3.10}$$

Since

$$\|z_n - Tz_n\| \le \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Tz_n\|,$$

from (3.6), (3.9) and (3.10) we have

$$\|z_n - Tz_n\| \to 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^*$. We have from (3.9) and $x_{n_i} \rightharpoonup z^*$ that $z_{n_i} \rightharpoonup z^*$. From (3.11) and Lemma 2.7, we have $z^* \in F(T)$. Next, let us show $z \in (A+B)^{-1}(0)$. From the definition of $J_{\lambda_n}$, we have that

$$\begin{aligned} z_n &= J_{\lambda_n}(I - \lambda_n A)x_n \\ &\Leftrightarrow (I - \lambda_n A)x_n \in (I + \lambda_n B)z_n = z_n + \lambda_n B z_n \\ &\Leftrightarrow x_n - z_n - \lambda_n A x_n \in \lambda_n B z_n \\ &\Leftrightarrow \frac{1}{\lambda_n}(x_n - z_n - \lambda_n A x_n) \in B z_n. \end{aligned}$$

Since $B$ is monotone, we have that for $(u, v) \in B$,

$$\langle z_n - u, \frac{1}{\lambda_n}(x_n - z_n - \lambda_n A x_n) - v \rangle \ge 0$$

and hence

$$\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - (Ax_n + v) \rangle \ge 0. \tag{3.12}$$

Furthermore, since $A$ is $\alpha$-inverse strongly monotone,

$$\langle x_{n_i} - z^*, Ax_{n_i} - Az^* \rangle \ge \alpha \left\| Ax_{n_i} - Az^* \right\|^2.$$

From $x_{n_i} \rightharpoonup z^*$ and $Ax_{n_i} \to Az$, we have $\langle x_{n_i} - z^*, Ax_{n_i} - Az^* \rangle \to 0$ and hence $Ax_{n_i} \to Az^*$. We also know from (3.12) that

$$\lim_{i \to \infty} \langle z_{n_i} - u, \frac{x_{n_i} - z_{n_i}}{\lambda_{n_i}} - (Ax_{n_i} + v) \rangle \ge 0.$$

So, we have from $z_{n_i} \rightharpoonup z^*$ that $\langle z^* - u, -Az^* - v \rangle \ge 0$. Since $B$ is maximal monotone, we have $(-Az^*) \in Bz^*$ and hence $z^* \in (A+B)^{-1}(0)$. So, we have $z^* \in F(T) \cap (A+B)^{-1}(0)$.

Put $z_0 = P_{F(T)\cap(A+B)^{-1}(0)}x$. Since $z_0 = P_{F(T)\cap(A+B)^{-1}(0)}x \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n}x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{3.13}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\begin{aligned} \|x - z^*\|^2 &= \|x\|^2 - 2\langle x, z^* \rangle + \|z^*\|^2 \\ &\le \liminf_{i \to \infty} (\|x\|^2 - 2\langle x, x_{n_i} \rangle + \|x_{n_i}\|^2) \\ &= \liminf_{i \to \infty} \|x - x_{n_i}\|^2 \\ &\le \|x - z_0\|^2. \end{aligned}$$

From the definition of $z_0$, we have $z^* = z_0$. So, we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have

$$\|z_0 - x_n\|^2 = \|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

So, we have from (3.13) that

$$\begin{aligned}\limsup_{n\to\infty} \|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &\le \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &= \|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - z_0\rangle \\ &= 0.\end{aligned}$$

So, we obtain $\lim_{n\to\infty} \|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to $z_0$. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [30] for supper hybrid mappings in a Hilbert space.

**Theorem 3.2.** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex closed subset of $H$. Let $\alpha > 0$. Let $A : C \to H$ be $\alpha$-inverse strongly monotone, let $B : D(B) \subset C \to 2^H$ be maximal monotone and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an $(\alpha, \beta, \gamma)$-super hybrid mapping such that $F(S) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_n = J_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $P_{C_{n+1}}$ is the metric projection of $H$ onto $C_{n+1}$, and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ are sequences such that*

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < b \le \lambda_n \le c < 2\alpha$$

*for some $b, c \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap(A+B)^{-1}(0)} x$, where $P_{F(S)\cap(A+B)^{-1}(0)}$ is the metric projection of $H$ onto $F(S) \cap (A + B)^{-1}(0)$.*

*Proof.* Put $T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I$. Then, we have from Theorem 2.3 that $T$ is an $(\alpha, \beta)$-generalized hybrid mapping of $C$ into $H$ and $F(S) = F(T)$. Since $F(T)$ is closed and convex, so is $F(S)$. We know that $(A + B)^{-1}(0)$ is closed and convex. Then, $F(S) \cap (A + B)^{-1}(0)$ is closed and convex. So, there exists the mertic projection of $H$ onto $F(S) \cap (A + B)^{-1}(0)$. We shall show that $C_n$ are closed and convex, and $F(T) \cap (A + B)^{-1}(0) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = C$ is closed and convex, and $F(T) \cap (A + B)^{-1}(0) \subset C_1$. Suppose that $C_k$ is closed and convex, and $F(T) \cap (A + B)^{-1}(0) \subset C_k$ for some $k \in \mathbb{N}$. From Nakajo and Takahashi

[20], we know that for $z \in C_k$,

$$\begin{aligned} &\|y_k - z\|^2 \le \|x_k - z\|^2 \\ \iff &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \le 0. \end{aligned}$$

So, $C_{k+1}$ is closed and convex. By induction, $C_n$ are closed and convex for all $n \in \mathbb{N}$. Put $z_k = J_{\lambda_k}(I - \lambda_k A)x_k$ and take $z \in F(T) \cap (A+B)^{-1}(0) \subset C_k$. From $z = J_{\lambda_n}(I - \lambda_n A)z$, we have that

$$\begin{aligned} \|z_k - z\|^2 &= \|J_{\lambda_k}(I - \lambda_k A)x_k - J_{\lambda_k}(I - \lambda_k A)z\|^2 \\ &\le \|x_k - \lambda_k Ax_k - z + \lambda_k Az\|^2 \\ &= \|x_k - z\|^2 - 2\lambda_k\langle x_k - z, Ax_k - Az\rangle + \lambda_k^2 \|Ax_k - Az\|^2 \\ &\le \|x_k - z\|^2 - 2\lambda_k\alpha \|Ax_k - Az\|^2 + \lambda_k^2 \|Ax_k - Az\|^2 \\ &= \|x_k - z\|^2 + \lambda_k(\lambda_k - 2\alpha) \|Ax_k - Az\|^2 \\ &\le \|x_k - z\|^2. \end{aligned} \tag{3.14}$$

Since $T$ is quasi-nonexpansive, we have from (3.14) that

$$\begin{aligned} \|y_k - z\|^2 &= \|\alpha_k x_k + (1-\alpha_k)Tz_k - z\|^2 \\ &\le \alpha_k\|x_k\|^2 + (1-\alpha_k)\|Tz_k - z\|^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\|z_k - z\|^2 \\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\|x_k - z\|^2 \\ &= \|x_k - z\|^2. \end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that $F(T) \cap (A+B)^{-1}(0) \subset C_n$ for all $n \in \mathbb{N}$. Since $C_n$ is nonempty, closed and convex, there exists the metric projection $P_{C_n}$ of $H$ onto $C_n$. Thus, $\{x_n\}$ is well-defined. Furthermore, we have

$$y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$$

for all $n \in \mathbb{N}$.

Since $\{C_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of $H$ with respect to inclusion, it follows that

$$\emptyset \ne F(T) \cap (A+B)^{-1}(0) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.15}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.1 we have that $\{P_{C_n}x\}$ converges strongly to $x_0 = P_{C_0}x$, i.e.,

$$x_n = P_{C_n}x \to x_0.$$

To complete the proof, it is sufficient to show that $x_0 = P_{F(T)\cap(A+B)^{-1}(0)}x$.

Since $x_n = P_{C_n}x$ and $x_{n+1} = P_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from (2.2) that

$$\begin{aligned} 0 &\leq 2\langle x - x_n, x_n - x_{n+1}\rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\leq \|x - x_{n+1}\|^2 - \|x - x_n|^2. \end{aligned} \tag{3.16}$$

So, we get that

$$\|x - x_n\|^2 \leq \|x - x_{n+1}\|^2. \tag{3.17}$$

Furthermore, since $x_n = P_{C_n}x$ and $z \in F(T) \cap (A+B)^{-1}(0) \subset C_n$, we have

$$\|x - x_n\|^2 \leq \|x - z\|^2. \tag{3.18}$$

So, we have that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$ are also bounded. From (3.16), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.$$

So, we have that

$$\|x_n - x_{n+1}\|^2 \to 0. \tag{3.19}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. So, we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.20}$$

From $0 \leq \liminf_{n\to\infty} \alpha_n < 1$, we have a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_i} \to \gamma$ and $0 \leq \gamma < 1$. From

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|,$$

we have that

$$\|Tz_{n_i} - x_{n_i}\| \to 0. \tag{3.21}$$

Using (3.21), let us show $\|Tz_{n_i} - z_{n_i}\| \to 0$. As in the proof of Theorem 3.1, we have that for any $z \in F(T) \cap (A+B)^{-1}(0)$,

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tz_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2\} \\ &= \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} (1-\alpha_n)b(2\alpha - c)\|Ax_n - Az\|^2 &\leq (1-\alpha_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\ &\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|. \end{aligned}$$

From $\|y_n - x_n\| \to 0$ and $\alpha_{n_i} \to \gamma$, we have that

$$\lim_{i\to\infty} \|Ax_{n_i} - Az\| = 0. \tag{3.22}$$

Since $J_{\lambda_n}$ is firmly nonexpansive, we have that

$$\begin{aligned}
\|z_n - z\|^2 &= \|J_{\lambda_n}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)z\|^2 \\
&\le \langle z_n - z, (I-\lambda_n A)x_n - (I-\lambda_n A)z\rangle \\
&= \frac{1}{2}\{\|z_n - z\|^2 + \|(I-\lambda_n A)x_n - (I-\lambda_n A)z\|^2 \\
&\quad - \|z_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\
&\le \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 \\
&\quad - \|z_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\
&= \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 \\
&\quad - \|z_n - x_n + \lambda_n(Ax_n - Az)\|^2\} \\
&= \frac{1}{2}\{\|z_n - z\|^2 + \|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|z_n - z\|^2 \le \|x_n - z\|^2 &- \|z_n - x_n\|^2 \\
&- 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|y_n - z\|^2 &\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|Tz_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\{\|x_n - z\|^2 - \|z_n - x_n\|^2 \\
&\qquad - 2\lambda_n\langle z_n - x_n, Ax_n - Az\rangle - {\lambda_n}^2\|Ax_n - Az\|^2\} \\
&\le \|x_n - z\|^2 - (1-\alpha_n)\|z_n - x_n\|^2 - {\lambda_n}^2(1-\alpha_n)\|Ax_n - Az\|^2 \\
&\quad - 2\lambda_n(1-\alpha_n)\langle z_n - x_n, Ax_n - Az\rangle.
\end{aligned}$$

This means that

$$\begin{aligned}
(1-\alpha_n)\|z_n - x_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&\quad + \|Ax_n - Az\|\{2c\|z_n - x_n\| + c^2\|Ax_n - Az\|\} \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| \\
&\quad + \|Ax_n - Az\|\{2c\|z_n - x_n\| + c^2\|Ax_n - Az\|\}.
\end{aligned}$$

Since $\lim_{i\to\infty}\|Ax_{n_i} - Az\| = 0$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\alpha_{n_i} \to \gamma < 1$ and $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|z_{n_i} - x_{n_i}\| = 0. \tag{3.23}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$, we have $y_n - Tz_n = \alpha_n(x_n - Tz_n)$. So, from (3.21) we have

$$\|y_{n_i} - Tz_{n_i}\| = \alpha_{n_i}\|x_{n_i} - Tz_{n_i}\| \to 0. \tag{3.24}$$

Since

$$\|z_{n_i} - Tz_{n_i}\| \leq \|z_{n_i} - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - Tz_{n_i}\|,$$

from (3.20), (3.23) and (3.24) we have

$$\|z_{n_i} - Tz_{n_i}\| \to 0. \tag{3.25}$$

Since $x_{n_i} = P_{C_{n_i}}x \to x_0$, we have from (3.23) that $z_{n_i} \to x_0$. So, $z_{n_i} \rightharpoonup x_0$. From (3.25) and Lemma 2.7 we have $x_0 \in F(T)$. Let us show $x_0 \in (A+B)^{-1}(0)$. As in the proof of Theorem 3.1, we have for $(u,v) \in B$,

$$\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - (Ax_n + v)\rangle \geq 0. \tag{3.26}$$

Furthermore, since $A$ is $\alpha$-inverse strongly monotone and $x_{n_i} = P_{C_{n_i}}x \to x_0$, we have that $Ax_{n_i} \to Ax_0$. So, we have from (3.26)

$$\langle x_0 - u, -(Ax_0 + v)\rangle \geq 0. \tag{3.27}$$

Since $B$ is maximal monotone, we have $-Ax_0 \in Bx_0$ and hence $x_0 \in (A+B)^{-1}(0)$. So, we have $x_0 \in F(T) \cap (A+B)^{-1}(0)$. Put $z_0 = P_{F(T)\cap(A+B)^{-1}(0)}x$. Since $z_0 = P_{F(T)\cap(A+B)^{-1}(0)}x \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x$, we have that

$$\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2. \tag{3.28}$$

So, we have that

$$\|x - x_0\|^2 = \lim_{n\to\infty}\|x - x_n\|^2 \leq \|x - z_0\|^2.$$

So, we get $z_0 = x_0$. Hence, $\{x_n\}$ converges strongly to $z_0$. This completes the proof. $\square$

## 4. Applications

In this section, we give some applications. Let $H$ be a Hilbert space and let $f$ be a proper lower semicontinuous convex function of $H$ into $(-\infty, \infty]$. Then, the subdifferential $\partial f$ of $f$ is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ y \in H\}$$

for all $x \in H$. From Rockafellar [22], we know that $\partial f$ is maximal monotone. Let $C$ be a closed convex subset of $H$ and let $i_C$ be the indicator function of $C$, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Since $i_C$ is a proper lower semicontinuous convex function on $H$, the subdifferential $\partial i_C$ of $i_C$ is a maximal monotone operator. So, we can define the resolvent $J_\lambda$ of $\partial i_C$ for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x$$

for all $x \in H$. We have that, for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_\lambda x &\Longleftrightarrow x \in u + \lambda \partial i_C u \\ &\Longleftrightarrow x \in u + \lambda N_C u \\ &\Longleftrightarrow x - u \in \lambda N_C u \\ &\Longleftrightarrow \frac{1}{\lambda}\langle x - u, v - u\rangle \leq 0, \quad \forall v \in C \\ &\Longleftrightarrow \langle x - u, v - u\rangle \leq 0, \quad \forall v \in C \\ &\Longleftrightarrow u = P_C x, \end{aligned}$$

where $N_C u$ is the normal cone to $C$ at $u$, i.e.,

$$N_C u = \{z \in H : \langle z, v - u\rangle \leq 0, \ \forall v \in C\}.$$

Similarly, if $A : C \to H$ is a nonlinear mapping, then we have that for $x \in C$,

$$\begin{aligned} x \in (A + \partial i_C)^{-1}(0) &\Leftrightarrow 0 \in Ax + \partial i_C x \\ &\Leftrightarrow -Ax \in \partial N_C x \\ &\Leftrightarrow \langle -Ax, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow x \in VI(A, C), \end{aligned}$$

where $VI(A, C) = \{x \in C : \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C\}$. This is called the set of solutions of the variational inequality for $A$.

Now, using Theorem 3.1, we can obtain a strong convergence theorem for finding a common element of the set of solutions of the variational inequality for an inverse strongly-monotone mapping and the set of fixed points of a super hybrid mapping in a Hilbert space.

**Theorem 4.1.** *Let $H$ be a Hilbert space and let $C$ be a closed convex subset of $H$. Let $\alpha > 0$ and let $A$ be an $\alpha$-inverse strongly-monotone mapping of $C$ into $H$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an ($\alpha$, $\beta$, $\gamma$)-super hybrid mapping such that $F(S) \cap (A + \partial i_C)^{-1}0 = F(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_n = P_C(I - \lambda_n A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $P_{C_n \cap Q_n}$ is the metric projection of $H$ onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy*

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n \leq c < 2\alpha$$

*for some $a, b, c \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S) \cap VI(A,C)} x$, where $P_{F(S) \cap VI(A,C)}$ is the metric projection of $H$ onto $F(S) \cap VI(A, C)$.*

*Proof.* Setting $B = \partial i_C$ in Theorem 3.1, we know that $J_{\lambda_n} = P_C$ for all $\lambda_n$ with $0 < a \leq \lambda_n \leq b < 2\alpha$. So we obtain the desired result by Theorem 3.1. □

Using Theorem 4.1, we get the following theorem for nonexpansive mappings and strict pseudo-contractions in a Hilbert space.

**Theorem 4.2.** *Let $H$ be a Hilbert space and let $C$ be a closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into itself and let $S$ be a $k$-strict pseudo-contraction with $0 \leq k < 1$ of $C$ into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} z_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Sz_n + kz_n\}, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $P_{C_n \cap Q_n}$ is the metric projection of $H$ onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy*

$$0 \leq \alpha_n \leq a < 1 \quad \textit{and} \quad 0 < b \leq \lambda_n \leq c < 1$$

*for some $a, b, c \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection of $H$ onto $F(S) \cap F(T)$.*

*Proof.* Put $A = I - T$ in Theorem 4.1. Then, we know that $A$ is a $\frac{1}{2}$-inverse strongly-monotone operator. We also have that for all $x \in C$,

$$\begin{aligned} P_C(x - \lambda_n Ax) &= P_C(x - \lambda_n(I-T)x) \\ &= P_C((1-\lambda_n)x + \lambda_n Tx) \\ &= (1-\lambda_n)x + \lambda_n Tx. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} u \in (A + \partial i_C)^{-1}0 &\Longleftrightarrow 0 \in Au + \partial i_C u \\ &\Longleftrightarrow 0 \in u - Tu + N_C u \\ &\Longleftrightarrow Tu - u \in N_C u \\ &\Longleftrightarrow \langle Tu - u, v - u \rangle \leq 0, \ \forall v \in C \\ &\Longleftrightarrow P_C Tu = u \\ &\Longleftrightarrow Tu = u. \end{aligned}$$

So, we obtain $(A+\partial i_C)^{-1}0 = F(T)$. On the other hand, we know from Lemma 2.2 that if $S$ is a $k$-strict pseudo-contraction with $0 \leq k < 1$, then $S$ is a $(1, 0, -k)$-extended hybrid mapping and hence a $(1, 0, \frac{k}{1-k})$-supper hybrid mapping. Furthermore, we have if $\gamma = \frac{k}{1-k}$, then $\frac{1}{1+\gamma} = 1 - k$ and $\frac{\gamma}{1+\gamma} = k$. Thus, we get the desired result by Theorem 4.1. □

Next, using Theorem 3.1, we consider the problem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $C$ be a nonempty closed convex subset of a Hilbert space and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) $f$ is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Then, the equilibrium problem (with respect to $C$) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{4.1}$$

for all $y \in C$. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. The following lemma appears implicitly in Blum and Oettli [4].

**Lemma 4.3** (Blum and Oettli)**.** *Let $C$ be a nonempty closed convex subset of $H$ and let $f$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [7].

**Lemma 4.4.** *Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1) − (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}.$$

*Then, the following hold:*

(1) *$T_r$ is single-valued;*
(2) *$T_r$ is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such $T_r$ the resolvent of $f$ for $r > 0$. Using Lemmas 4.3 and 4.4, we know the following theorem from Takahashi, Takahashi and Toyoda [24]. See [2] for a more general result.

**Theorem 4.5.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $f : C \times C \to \mathbb{R}$ satisfy (A1) − (A4). Let $A_f$ be a multivalued mapping of $H$ into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, \ x \in C, \\ \emptyset, \ x \notin C. \end{cases}$$

*Then,* $EP(f) = A_f^{-1}0$ *and* $A_f$ *is a maximal monotone operator with* $D(A_f) \subset C$*. Furthermore, for any* $x \in H$ *and* $r > 0$*, the resolvent* $T_r$ *of* $f$ *coincides with the resolvent of* $A_f$*; i.e.,*

$$T_r x = (I + rA_f)^{-1}x.$$

Using Theorem 4.5, we obtain the following result which was proved by Takahashi and Takahashi [23, Theorem 4.1].

**Theorem 4.6.** *Let* $H$ *be a real Hilbert space and let* $C$ *be a nonempty closed convex subset of* $H$*. Let* $f$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(A1)-(A4)$ *and let* $T_\lambda$ *be the resolvent of* $f$ *for* $\lambda > 0$*. Let* $S$ *be a* $k$*-strict pseudo-contraction with* $0 \le k < 1$ *of* $C$ *into itself such that* $F(S) \cap EP(f) \neq \emptyset$*. Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{cases} z_n = T_{\lambda_n} x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Sz_n + kz_n\}, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where* $P_{C_n \cap Q_n}$ *is the metric projection of* $H$ *onto* $C_n \cap Q_n$*, and* $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy*

$$0 \le \alpha_n \le a < 1 \quad \textit{and} \quad 0 < b \le \lambda_n \le c$$

*for some* $a, b, c \in \mathbb{R}$*. Then,* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap EP(f)}x$*, where* $P_{F(S)\cap EP(f)}$ *is the metric projection of* $H$ *onto* $F(S) \cap EP(f)$*.*

*Proof.* Put $A = 0$ in Theorem 3.1. From Theorem 4.5 we also know that $J_{\lambda_n} = T_{\lambda_n}$ for all $n \in \mathbb{N}$. For a $k$-strict pseudo-contraction, we follow the proof of Theorem 4.2. Thus, we obtain the desired result by Theorem 3.1. □

As in the proofs of Theorems 4.1, 4.2 and 4.6, we also get similar results from Theorem 3.2.

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