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APPROXIMATION METHODS FOR TRIPLE HIERARCHICAL VARIATIONAL INEQUALITIES (I)

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Abstract. In this work, we consider two types of triple hierarchical variational inequalities (in short, THVI), one with a single nonexpansive mapping and another one with a finite family of nonexpansive mappings. In this paper, by combining the viscosity approximation method, hybrid steepest-descent method and Mann's iteration method, we propose the hybrid steepest-descent viscosity approximation method for solving the THVI. The strong convergence of this method to a unique solution of the THVI is studied under some appropriate assumptions. Another iterative algorithm for solving THVI is also presented. Under some mild conditions, we prove that the sequence generated by the proposed algorithm converges strongly to a unique solution of THVI. The case of a finite family of nonexpansive mappings will ve presented in the second part of this work.

Key Words and Phrases: Triple hierarchical variational inequalities, hybrid steepest-descent viscosity approximation method, monotone operators, nonexpansive mappings, fixed points, strong convergence theorems.
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1. Introduction and formulations

Let H be a real Hilbert space with its inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The set of all fixed points of a mapping $T : H \to H$ is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. The mapping $T : H \to H$ is called L-Lipschitzian if there exists a constant $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in H$. In particular, if $L \in [0, 1)$, T is called a contraction mapping, while if $L = 1$, then T is called a nonexpansive mapping.

Let K be a nonempty convex subset of a Hilbert space H and $F : K \to H$ be a monotone mapping, that is, $\langle Fx - Fy, x - y\rangle \geq 0$, $\forall x, y \in K$. The monotone variational inequality problem [19] is to find $x^* \in K$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in K.$$

The set of solutions of this problem is denoted by $\mathrm{VI}(K, F)$. The following variational inequality problem defined over the set $\mathrm{Fix}(T)$ of fixed points of a mapping $T : H \to H$ is called hierarchical variational inequality problem (in short, HVIP).

Problem 1.1. *Given a monotone, continuous operator $A : H \to H$ and a nonexpansive mapping $T : H \to H$,*

$$\textit{find } x^* \in \mathrm{VI}(\mathrm{Fix}(T), A) := \{x^* \in \mathrm{Fix}(T) : \langle Ax^*, v - x^*\rangle \geq 0,\ \forall v \in \mathrm{Fix}(T)\}.$$

Recently, it has been considered and studied by several authors; See, for example, [1, 2, 7, 12, 14, 16, 17, 18, 23, 24, 30] and the references therein. Several iterative methods for computing the approximate solutions of Problem 1.1 are proposed and analyzed in these references. In 2001, Yamada [30] (see also [28]) introduced a hybrid steepest-descent method for finding an element of $\mathrm{VI}(\mathrm{Fix}(T), A)$ so as to reduce the complexity probably caused by the projection $P_{\mathrm{Fix}(T)}$. Zeng et al. [32] introduced and analyzed a modified hybrid steepest-descent algorithm with variable parameters which produces a sequence that converges strongly to a unique element of $\mathrm{VI}(\mathrm{Fix}(T), A)$, where A is η-strongly monotone and κ-Lipschitzian with constants $\eta, \kappa > 0$. They also considered the case where $\Omega = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ and $T_i : H \to H$, $i = 1, 2, \ldots, N$, is a nonexpansive mapping. They proposed another modified hybrid steepest-descent algorithm with variable parameters which produces a sequence that converges strongly to a unique element of $\mathrm{VI}(\Omega, A)$. A hierarchical fixed point problem (in short, HFPP), equivalent to a HVIP, has been discussed in [21, 23]. Some iterative algorithms for solving HFPP are proposed. The solution presented in [21, 23] is not always unique, so that there may be many solutions for this problem. In that case, a solution, that results in practical systems and networks being more stable and reliable, must be found from among candidate solutions. Hence, it wound be reasonable to identify the unique minimizer of an appropriate objective function over the hierarchical fixed point constraint. Such problem would be a three-stage problem. Very recently, Iiduka [13, 15] introduced three-stage variational inequality problem, that is, the monotone variational inequality problem over the solution set of HVIP.

Problem 1.2. *Assume that*

- (A1) $A_1 : H \to H$ *is α-inverse-strongly monotone;*
- (A2) $A_2 : H \to H$ *is η-strongly monotone and κ-Lipschitzian;*
- (A3) $T : H \to H$ *is a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*;*
- (A4) $\mathrm{VI}\,(\mathrm{Fix}(T), A_1) \neq \emptyset$.

Then the objective is to find $x^* \in \mathrm{VI}\,(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2)$, *where*

$$VI\,(VI(\Omega, A_1), A_2) := \Big\{x^* \in VI\,(\Omega, A_1) : \langle A_2x^*, v - x^*\rangle \geq 0, \forall v \in VI\,(\Omega, A_1)\Big\}.$$

We remark that this problem has a triple structure in contrast with bilevel programming problems [20] or hierarchical constrained optimization problems or hierarchical fixed point problem. Thus it is referred as triple-hierarchical variational inequality problem (THVIP). Ceng et al. [3] considered the THVIP and presented its example. They proposed two iterative methods, one is implicit and another one is explicit, to compute the approximate solutions of THVIP. The convergence analysis of the sequences generated by the proposed algorithms is also studied. The THVIP is further considered and studied by Ceng et al. [4]. They proposed relaxed hybrid steepest-descent algorithm with variable parameters for computing the approximate solutions of Problem 1.2.

On the other hand, Ceng et al. [4] also considered the following monotone variational inequality problem over the solution set of the variational inequality which is defined over the set of common fixed points of N nonexpansive mappings $T_i : H \to H$, with $N \geq 1$ an integer.

Problem 1.3. *Assume that*

(B1) $A_1 : H \to H$ *is* α*-inverse-strongly monotone;*
(B2) $A_2 : H \to H$ *is* η*-strongly monotone and* κ*-Lipschitzian;*
(B3) *for* $i = 1, 2, \ldots, N$, $T_i : H \to H$ *is a nonexpansive mapping with* $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$*;*
(B4) $\mathrm{VI}\left(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A_1\right) \neq \emptyset$.

Then the objective is to find $x^* \in \mathrm{VI}\left(\mathrm{VI}\left(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A_1\right), A_2\right)$.

In [4], the authors also proposed another relaxed hybrid steepest-descent algorithm with variable parameters for computing the approximate solutions of Problem 1.3.

We remark that $T_{[k]} := T_{k \bmod N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, \ldots, N\}$, that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $0 < q < N$.

In this paper, by combining the viscosity approximation method [5, 22, 27], hybrid steepest-descent method [6, 30] and Mann's iteration method [25], we introduce two hybrid steepest-descent viscosity approximation algorithms for computing the appropriate solutions of Problems 1.2 and, in the second part of this work, for Problem 1.3, respectively. The strong convergence of the sequences generated by these algorithms is derived under some appropriate conditions. Obviously, whenever $\beta_n = \gamma_n = 0$, $\forall n \geq 0$, these two algorithms reduce to Algorithms 1 and 2, respectively, in [4]. Therefore, our proposed method is quite general and flexible and includes as special cases some other iterative methods in the literature.

2. Preliminaries

Let H be a real Hilbert space. We denote by $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x.

Definition 2.1. An operator $A : H \to H$ is called

(a) *strongly monotone* (or more precisely, α-*strongly monotone*) if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H;$$

(b) *inverse-strongly monotone* (or more precisely, β-*inverse-strongly monotone*) (also called *co-coercive*) if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(c) *hemicontinuous* if for all $x, y \in H$, the mapping $g : [0, 1] \to H$, defined by $g(t) := A(tx + (1 - t)y)$, is continuous.

It is clear that every β-inverse strongly monotone mapping is $\frac{1}{\beta}$-Lipschitzian.

Definition 2.2. Let C be a nonempty convex subset of a real Hilbert space H. A function $\varphi : C \to \mathbb{R}$ is said to be

(a) *convex* if for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y);$$

(b) *strongly convex* if there exists $\alpha > 0$ such that for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y) - \frac{1}{2}\alpha\lambda(1 - \lambda)\|x - y\|^2.$$

Let $\varphi : H \to \mathbb{R}$ be a Fréchet differentiable function. Then, it is well known that φ is convex (respectively, strongly convex) if and only if $\nabla\varphi : H \to H$ is monotone (respectively, strongly monotone) [31, Proposition 25.10], [11, Sect. IV, Theorem 4.1.4]. If $\varphi : H \to \mathbb{R}$ is convex and $\nabla\varphi : H \to H$ is $1/L$-Lipschitzian, then $\nabla\varphi$ is L-inverse-strongly monotone.

The metric projection $P_C : H \to C$ onto the nonempty, closed and convex subset C of H is defined by $P_C x \in C$ and $\|x - P_C x\| = \inf_{x \in C} \|x - y\|, \quad \forall x \in H$. The metric projection P_C onto a given nonempty, closed and convex subset C of H is nonexpansive with $\mathrm{Fix}(P_C) = C$ [25, Theorem 3.1.4 (i)].

Related to the set of all fixed points of a nonexpansive mapping, we have the following result.

Proposition 2.1. *Let C be a nonempty, closed and convex subset C of a real Hilbert space H, and $T : C \to C$ be a nonexpansive mapping. Then,*

(a) [10, Proposition 5.3] $\mathrm{Fix}(T)$ *is closed and convex;*
(b) [10, Theorem 5.1] $\mathrm{Fix}(T) \neq \emptyset$, *provided C is bounded.*

Some properties of the solution set of a monotone variational inequality are mentioned in the following result.

Proposition 2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $A : C \to H$ be a monotone and hemicontinuous operator and $\varphi : C \to \mathbb{R}$ be a convex and Fréchet differentiable functional. Then,*

(a) [19] $\mathrm{VI}(C, A)$ *is equivalent to* $\mathrm{MVI}(C, A) := \{x^* \in C : \langle Ay, y - x^*\rangle \geq 0, \ \forall y \in C\}$.
(b) [19] $\mathrm{VI}(C, A) \neq \emptyset$ *when C is bounded.*
(c) [29, Lemma 2.24] $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ *for all $\lambda > 0$, where I is the identity mapping on H.*

(d) [29, Theorem 2.31] VI(C, A) *consists of only one point, if A is strongly monotone and Lipschitzian.*
(e) [8, Chap. II, Proposition 2.1]

$$\mathrm{VI}(C, \nabla\varphi) = \mathrm{Argmin}_{x \in C}\varphi(x) := \left\{ x^* \in C : \varphi(x^*) = \min_{x \in C} \varphi(x) \right\}.$$

The following proposition provides an example of a nonexpansive mapping in which the set of all fixed points of a nonexpansive mapping is equal to the solution set of the monotone variational inequality.

Proposition 2.3. [15, Proposition 2.3] (see also [13]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and $A : C \to H$ be an α-inverse-strongly monotone operator. Then, for any given $\lambda \in [0, 2\alpha]$, the mapping $S_\lambda : H \to H$ defined by $S_\lambda x := P_C(I - \lambda A)x$ is nonexpansive and* $\mathrm{Fix}(S_\lambda) = \mathrm{VI}(C, A)$.

We need the following proposition to prove the main results of this paper.

Proposition 2.4. [30, Lemma 3.1] *Let $A : H \to H$ be η-strongly monotone and κ-Lipschitzian and let $\mu \in (0, 2\eta/\kappa^2)$. For $\lambda \in [0, 1]$, define $T^{(\lambda,\mu)} : H \to H$ by $T^{(\lambda,\mu)}x := x - \lambda\mu Ax$, for all $x \in H$. Then,*

$$\left\| T^{(\lambda,\mu)}x - T^{(\lambda,\mu)}y \right\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H,$$

where $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Recall that a Banach space X is said to satisfy *Opial's condition* if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \ne x.$$

It is well known that every Hilbert space H satisfies Opial's condition (see [9]).

The following lemmas will be used in the proof of the main results of this paper.

Lemma 2.1. [26, Lemma 2.5] *Let $\{a_n\}$ be a sequence of nonnegative numbers such that*

$$a_{n+1} \le (1 - s_n)a_n + s_n t_n + \nu_n, \quad \forall n \ge 0,$$

where $\{s_n\}$, $\{t_n\}$ and $\{\nu_n\}$ are the sequences such that the following conditions hold:

(i) $\{s_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} s_n = \infty$, *or* $\prod_{n=0}^{\infty}(1 - s_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - s_k) = 0$;
(ii) $\limsup_{n\to\infty} t_n \le 0$;
(iii) $\nu_n \ge 0$ *and* $\sum_{n=0}^{\infty} \nu_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

The following lemma is an immediate consequence of the inner product properties,

Lemma 2.2. *In a real Hilbert space H we have that $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, for all $x, y \in H$.*

The following lemma can be easily proved.

Lemma 2.3. [4] *Let* $\{a_n\}$ *be a bounded sequence of nonnegative numbers and* $\{b_n\}$ *be a sequence of real numbers such that* $\limsup_{n\to\infty} b_n \le 0$. *Then,* $\limsup_{n\to\infty} a_n b_n \le 0$.

Lemma 2.4. [9, Demiclosedness Principle] *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and* $T : C \to C$ *be a nonexpansive mapping. If* T *has a fixed point, then* $I - T$ *is demiclosed, that is, whenever* $\{x_n\}$ *is a sequence in* C *and weakly converges to some point* $x \in C$ *and the sequence* $\{(I-T)x_n\}$ *strongly converges to some point* y, *then* $(I-T)x = y$, *where* I *is the identity operator on* H. *In particular, whenever* $y = 0$, *we have* $x \in \mathrm{Fix}(T)$.

3. Iterative methods involving a nonexpansive mapping

In this section, we first consider a hybrid steepest-descent viscosity iterative algorithm for solving Problem 1.2 involving a nonexpansive mapping defined on a real Hilbert space H. Suppose that the assumptions (A1)–(A4) in Problem 1.2 are satisfied.

Algorithm 3.1.
Step 0. Take $\{\lambda_n\} \subset (0, 2\alpha]$, $\{\mu_n\} \subset \left(0, 2\eta/\kappa^2\right)$, $\{\alpha_n\} \subset (0,1]$ and $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\beta_n + \gamma_n \le 1$, $\forall n \ge 0$. Choose $x_0 \in H$ arbitrarily, and let $n := 0$.
Step 1. Given $x_n \in H$, compute $x_{n+1} \in H$ as

$$\begin{aligned} y_n &:= \beta_n x_n + \gamma_n f(x_n) + (1 - \beta_n - \gamma_n) T_n x_n, \\ x_{n+1} &:= y_n - \alpha_n \mu_n A_2 y_n, \quad n \ge 0, \end{aligned}$$

where $T_n := T(I - \lambda_n A_1)$, $\forall n \ge 0$.
Update $n := n+1$ and go to Step 1.

In Algorithm 3.1, we introduce a sequence $\{\mu_n\}$ of positive parameters so as to take into account possible inexact computation. Taking $\mu \in \left(0, 2\eta/\kappa^2\right)$ and putting $\mu_n \equiv \mu$ and $\beta_n = \gamma_n = 0$ for all $n \ge 0$, then Algorithm 3.1 reduces to [13, Algorithm 3.1] (that is, [15, Algorithm 4.1]). Thus, Algorithm 3.1 is more general and more flexible than [13, Algorithm 3.1] (that is, [15, Algorithm 4.1]).

We present the convergence analysis of the sequences generated by Algorithm 3.1.

Theorem 3.1. *Assume that the sequence* $\{y_n\}$ *generated by Algorithm 3.1 is bounded. Let* $\{\mu_n\} \subset \left(0, \eta/\kappa^2\right]$, $\{\alpha_n\} \subset (0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\gamma_n\} \subset (0,1]$ *and* $\{\lambda_n\} \subset (0, 2\alpha]$ *be such that the following conditions hold:*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} (1/\gamma_n)\,|1/\lambda_n - 1/\lambda_{n+1}| = 0$,

(iii) $\lim_{n\to\infty} (1/\lambda_{n+1})\,|1 - \gamma_{n+1}/\gamma_n| = 0$,

(iv) $\lim_{n\to\infty} \lambda_n = 0$,

(v) $\lim_{n\to\infty} \alpha_n \mu_n / \lambda_n = 0$,

(vi) $\lim_{n\to\infty} \left(\lambda_n \beta_n + \gamma_n + \lambda_n^2\right) / \alpha_n \mu_n = 0$,

(vii) $\sum_{n=1}^{\infty} \left(|\beta_n - \beta_{n-1}| + |\alpha_n\mu_n - \alpha_{n-1}\mu_{n-1}| \right) / \lambda_n < \infty.$

Then, the sequence $\{x_n\}$ *generated by Algorithm 3.1 satisfies the following assertions:*

(a) *The sequences* $\{x_n\}$, $\{A_1x_n\}$ *and* $\{A_2y_n\}$ *are bounded;*

(b) $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\lambda_n = 0$, $\lim_{n\to\infty} \|x_n - y_n\|/\lambda_n = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$*;*

(c) *The sequence* $\{x_n\}$ *converges strongly to a unique solution of Problem 1.2 provided that there exists* $r > 0$ *such that* $\|x - Tx\| \geq r \inf_{y\in\mathrm{Fix(T)}} \|x - y\|$ *for all* $x \in H$.

Proof. (a) Suppose that $\mathrm{VI}\left(\mathrm{VI}\left(\mathrm{Fix}(T), A_1\right), A_2\right)$ has a unique solution $\{x^*\}$. Assumption (A2) guarantees that $\|A_2y_n - A_2x^*\| \leq \kappa\|y_n - x^*\|$ for all $n \geq 0$. Hence, the boundedness of $\{y_n\}$ implies that $\{A_2y_n\}$ is bounded. From the definition of $\{x_n\}$ and the boundedness of $\{y_n\}$ and $\{A_2y_n\}$, we deduce that $\{x_n\}$ is also bounded. Since A_1 is α-inverse-strongly monotone, it is $\frac{1}{\alpha}$-Lipschitzian, and hence, $\|A_1x_n - A_1x^*\| \leq (1/\alpha)\|x_n - x^*\|$ for all $n \geq 0$. Therefore, the boundedness of $\{x_n\}$ ensures the boundedness of $\{A_1x_n\}$.

(b) We prove $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\lambda_n = 0$.

From assumption (A3), Proposition 2.3, and the condition $\lambda_n \leq 2\alpha$ $(\forall n \geq 0)$, we obtain, for all $n \geq 0$, that

$$\begin{aligned}
\|T_{n+1}x_{n+1} - T_nx_n\| &= \|T(I - \lambda_{n+1}A_1)x_{n+1} - T(I - \lambda_nA_1)x_n\| \\
&\leq \|(I - \lambda_{n+1}A_1)x_{n+1} - (I - \lambda_nA_1)x_n\| \\
&= \|(I - \lambda_{n+1}A_1)x_{n+1} - (I - \lambda_{n+1}A_1)x_n + (\lambda_n - \lambda_{n+1})A_1x_n\| \\
&\leq \|(I - \lambda_{n+1}A_1)x_{n+1} - (I - \lambda_{n+1}A_1)x_n\| + |\lambda_n - \lambda_{n+1}|\|A_1x_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|A_1x_n\|.
\end{aligned}$$

If we notice that $M_1 := \sup_{n\geq 0}\{\|x_n\| + \|f(x_n)\| + \|T_nx_n\| + \|A_1x_n\|\} < \infty$, then we have that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\beta_{n+1}x_{n+1} + \gamma_{n+1}f(x_{n+1}) + (1 - \beta_{n+1} - \gamma_{n+1})T_{n+1}x_{n+1} \\
&\quad - \beta_nx_n - \gamma_nf(x_n) - (1 - \beta_n - \gamma_n)T_nx_n\| \\
&\leq \|\beta_{n+1}x_{n+1} - \beta_nx_n\| + \|\gamma_{n+1}f(x_{n+1}) - \gamma_nf(x_n)\| \\
&\quad + \|(1 - \beta_{n+1} - \gamma_{n+1})T_{n+1}x_{n+1} - (1 - \beta_n - \gamma_n)T_nx_n\| \\
&\leq |\beta_{n+1} - \beta_n|\|x_{n+1}\| + \beta_n\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|f(x_{n+1})\| \\
&\quad + \gamma_n\|f(x_{n+1}) - f(x_n)\| + |(1 - \beta_{n+1} - \gamma_{n+1}) - (1 - \beta_n - \gamma_n)|\|T_{n+1}x_{n+1}\| \\
&\quad + (1 - \beta_n - \gamma_n)\|T_{n+1}x_{n+1} - T_nx_n\| \\
&\leq |\beta_{n+1} - \beta_n|\|x_{n+1}\| + \beta_n\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|f(x_{n+1})\| \\
&\quad + \gamma_n\rho\|x_{n+1} - x_n\| + (|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n|)\|T_{n+1}x_{n+1}\| \\
&\quad + (1 - \beta_n - \gamma_n)[\|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|A_1x_n\|] \\
&\leq (|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n|)\|x_{n+1}\| + \beta_n\|x_{n+1} - x_n\| \\
&\quad + (|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n|)\|f(x_{n+1})\| + \gamma_n\rho\|x_{n+1} - x_n\| \\
&\quad + (|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n|)\|T_{n+1}x_{n+1}\|
\end{aligned}$$

$$
\begin{aligned}
&+(1-\beta_n-\gamma_n)[\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|A_1x_n\|]\\
&=[1-\gamma_n(1-\rho)]\|x_{n+1}-x_n\|+(1-\beta_n-\gamma_n)|\lambda_n-\lambda_{n+1}|\|A_1x_n\|\\
&+(|\beta_{n+1}-\beta_n|+|\gamma_{n+1}-\gamma_n|)(\|x_{n+1}\|+\|f(x_{n+1})\|+\|T_{n+1}x_{n+1}\|)\\
&\le[1-\gamma_n(1-\rho)]\|x_{n+1}-x_n\|\\
&+(|\beta_{n+1}-\beta_n|+|\gamma_{n+1}-\gamma_n|+|\lambda_{n+1}-\lambda_n|)M_1,
\end{aligned}
$$

From Proposition 2.4 and the above evaluation, we get, for every $n\ge 1$, that

$$
\begin{aligned}
&\|x_{n+1}-x_n\|=\left\|T^{(\alpha_n,\mu_n)}y_n-T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\right\|\\
&\le\left\|T^{(\alpha_n,\mu_n)}y_n-T^{(\alpha_n,\mu_n)}y_{n-1}\right\|+\left\|T^{(\alpha_n,\mu_n)}y_{n-1}-T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\right\|\\
&\le(1-\alpha_n\tau_n)\|y_n-y_{n-1}\|+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|\|A_2y_{n-1}\|\\
&\le(1-\alpha_n\tau_n)\{[1-\gamma_{n-1}(1-\rho)]\|x_n-x_{n-1}\|+(|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|\\
&+|\lambda_n-\lambda_{n-1}|)M_1\}+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|\|A_2y_{n-1}\|\\
&\le[1-\gamma_{n-1}(1-\rho)]\|x_n-x_{n-1}\|+(|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|\\
&+|\lambda_n-\lambda_{n-1}|)M_1+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|M_2,
\end{aligned}
$$

where $\tau_n:=1-\sqrt{1-\mu_n(2\eta-\mu_n\kappa^2)}\in(0,1]$ as in Proposition 2.4 and

$$M_2:=\sup_{n\ge0}\|A_2y_n\|<\infty.$$

So, for all $n\ge1$, we obtain

$$
\begin{aligned}
&\frac{\|x_{n+1}-x_n\|}{\lambda_n}\le[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_n}\\
&+\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|}{\lambda_n}M_1+\frac{|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_2\\
&=[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}+[1-\gamma_{n-1}(1-\rho)]\left\{\frac{\|x_n-x_{n-1}\|}{\lambda_n}-\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}\right\}\\
&+\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|}{\lambda_n}M_1+\frac{|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_2\\
&\le[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}+M_3\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right|\\
&+\frac{|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|}{\lambda_n}M_3+\frac{|\beta_n-\beta_{n-1}|+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3\\
&=[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right|\\
&+\frac{2\alpha M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\frac{|\lambda_n-\lambda_{n-1}|}{2\alpha\lambda_n}+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\frac{|\gamma_n-\gamma_{n-1}|}{\lambda_n}\\
&+\frac{|\beta_n-\beta_{n-1}|+\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3\\
&\le[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right|
\end{aligned}
$$

$$+\frac{2\alpha M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right|+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\lambda_n}\left|1-\frac{\gamma_n}{\gamma_{n-1}}\right|$$
$$+\frac{|\beta_n-\beta_{n-1}|+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3=[1-\gamma_{n-1}(1-\rho)]\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}$$
$$+\gamma_{n-1}(1-\rho)\cdot\frac{M_3}{1-\rho}\Big\{(1+2\alpha)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right|+\frac{1}{\lambda_n}\left|1-\frac{\gamma_n}{\gamma_{n-1}}\right|\Big\}$$
$$+\frac{|\beta_n-\beta_{n-1}|+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3,$$

where $M_3:=\sup_{n\geq 0}\|x_{n+1}-x_n\|+M_1+M_2<\infty$. Therefore, by Lemma 2.1 and the conditions (i), (ii), (iii) and (vii) we have

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\lambda_n}=0. \tag{3.1}$$

Since

$$\|x_n-y_n\|\leq\|x_{n+1}-x_n\|+\|x_{n+1}-y_n\|\leq\quad\|x_{n+1}-x_n\|+\alpha_n\mu_nM_2,$$

we have $\frac{\|x_n-y_n\|}{\lambda_n}\leq\frac{\|x_{n+1}-x_n\|}{\lambda_n}+\frac{\alpha_n\mu_n}{\lambda_n}M_2$. Therefore, by (3.1) and the condition (v), we get

$$\lim_{n\to\infty}\frac{\|x_n-y_n\|}{\lambda_n}=0. \tag{3.2}$$

By the condition (vi), (3.2) and $(\lambda_n/\alpha_n\mu_n)\|x_n-y_n\|=(\lambda_n^2/\alpha_n\mu_n)(\|x_n-y_n\|/\lambda_n)$, $\forall n\geq 0$, we get that $\lim_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\|x_n-y_n\|=0$. Put $z_n:=x_n-\lambda_nA_1x_n$, $\forall n\geq 0$. Then, we have $\|z_n-x_n\|=\lambda_n\|A_1x_n\|\leq\lambda_nM_1$, and hence,

$$(\lambda_n/\alpha_n\mu_n)\|z_n-x_n\|\leq\left(\lambda_n^2/\alpha_n\mu_n\right)M_1,\ \forall n\geq 0.$$

From condition (vi), we have $\lim_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\|z_n-x_n\|=0$. Consequently, we get

$$\lim_{n\to\infty}\frac{\lambda_n\|z_n-y_n\|}{\alpha_n\mu_n}=0. \tag{3.3}$$

Moreover, from assumption (A3), we obtain

$$\begin{aligned}\|y_n-Tx_n\|&=\|\beta_n(x_n-Tx_n)+\gamma_n(f(x_n)-Tx_n)+(1-\beta_n-\gamma_n)(T_nx_n-Tx_n)\|\\&\leq\beta_n\|x_n-Tx_n\|+\gamma_n\|f(x_n)-Tx_n\|+(1-\beta_n-\gamma_n)\|T_nx_n-Tx_n\|\\&\leq\beta_n\|x_n-Tx_n\|+\gamma_n\|f(x_n)-Tx_n\|+\|T(x_n-\lambda_nA_1x_n)-Tx_n\|\\&\leq\beta_n\|x_n-Tx_n\|+\gamma_n\|f(x_n)-Tx_n\|+\lambda_n\|A_1x_n\|.\end{aligned}$$

Also, from conditions (v) and (vi), we have

$$\lim_{n\to\infty}\frac{\gamma_n}{\lambda_n}=\lim_{n\to\infty}\frac{\gamma_n}{\alpha_n\mu_n}\cdot\frac{\alpha_n\mu_n}{\lambda_n}=0\quad\text{and}\quad\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\frac{\lambda_n\beta_n}{\alpha_n\mu_n}\cdot\frac{\alpha_n\mu_n}{\lambda_n}=0.$$

Hence, by the condition (iv), we get $\lim_{n\to\infty}\|y_n-Tx_n\|=0$. Therefore, utilizing (3.2), we obtain $\lim_{n\to\infty}\|x_n-y_n\|=0$, Thus,

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.4}$$

(c) We divide the proof into the following three steps:

(I) We prove $\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2x^*\rangle \leq 0$.

Choose a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that $\limsup_{n\to\infty}\langle x^* - x_n, A_2x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, A_2x^*\rangle$. The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|z_n - x_n\| = 0$, we have $\lim_{j\to\infty}\langle x_{n_{i_j}+1}, w\rangle = \lim_{j\to\infty}\langle z_{n_{i_j}}, w\rangle = \lim_{j\to\infty}\langle x_{n_{i_j}}, w\rangle = \langle \hat{x}, w\rangle, \quad \forall w \in H$. Without loss of generality, we may assume that $\lim_{i\to\infty}\langle x_{n_i}, w\rangle = \langle \hat{x}, w\rangle$, $\forall w \in H$. Assume $\hat{x} \neq T\hat{x}$. By (3.4), assumption (A3) and Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - \hat{x}\| &< \liminf_{i\to\infty}\|x_{n_i} - T\hat{x}\| = \liminf_{i\to\infty}\|x_{n_i} - Tx_{n_i} + Tx_{n_i} - T\hat{x}\| \\ &= \liminf_{i\to\infty}\|Tx_{n_i} - T\hat{x}\| \leq \liminf_{i\to\infty}\|x_{n_i} - \hat{x}\|.\end{aligned} \tag{3.5}$$

This is a contradiction, that is, $\hat{x} \in \mathrm{Fix}(T)$. Let $y \in \mathrm{Fix}(T)$ be a fixed arbitrary point and put $M_4 := \sup_{n\geq 0}\{\|x_n - y\| + \|y_n - y\| + \|f(x_n) - y\|\} < \infty$. Then, from assumption (A3) and Proposition 2.3, we have, for every $n \geq 0$, that

$$\begin{aligned}&\|y_n - y\|^2 = \|\beta_n(x_n - y) + \gamma_n(f(x_n) - y) + (1 - \beta_n - \gamma_n)(T_nx_n - y)\|^2 \\ &\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|T(x_n - \lambda_nA_1x_n) - Ty\|^2 \\ &\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|z_n - y\|^2 \\ &= \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - f(y) + f(y) - y\|^2 \\ &+ (1 - \beta_n - \gamma_n)\|(x_n - \lambda_nA_1x_n) - (y - \lambda_nA_1y) - \lambda_nA_1y\|^2 \\ &\leq \beta_n\|x_n - y\|^2 + \gamma_n[\|f(x_n) - f(y)\|^2 + 2\langle f(y) - y, f(x_n) - y\rangle] \\ &+ (1 - \beta_n - \gamma_n)[\|(x_n - \lambda_nA_1x_n) - (y - \lambda_nA_1y)\|^2 + 2\lambda_n\langle y - z_n, A_1y\rangle] \\ &\leq \beta_n\|x_n - y\|^2 + \gamma_n[\rho^2\|x_n - y\|^2 + 2\|f(y) - y\|\|f(x_n) - y\|] \\ &+ (1 - \beta_n - \gamma_n)[\|x_n - y\|^2 + 2\lambda_n\langle y - z_n, A_1y\rangle] \\ &\leq [1 - \gamma_n(1 - \rho)]\|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| \\ &+ 2(1 - \beta_n - \gamma_n)\lambda_n\langle y - z_n, A_1y\rangle \\ &\leq \|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| + 2(1 - \beta_n - \gamma_n)\lambda_n\langle y - z_n, A_1y\rangle,\end{aligned} \tag{3.6}$$

which implies that

$$\begin{aligned}0 &\leq \frac{1}{\lambda_n}\left(c\|x_n - y\|^2 - \|y_n - y\|^2\right) + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\|f(x_n) - y\| \\ &\quad +2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1y\rangle \\ &\leq (\|x_n - y\| + \|y_n - y\|)\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_4 \\ &\quad +2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1y\rangle \\ &\leq M_4\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_4 + 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1y\rangle\end{aligned}$$

$$\leq M_4 \left(\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\| \right) + 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1 y\rangle,$$

for every $n \geq 0$.

By the weak convergence of $\{z_{n_i}\}$ to $\hat{x} \in \mathrm{Fix}(T)$, the condition (v), and (3.2), we get $\langle y - \hat{x}, A_1 y\rangle \geq 0$ for all $y \in \mathrm{Fix}(T)$. By assumption (A1), we have

$$\langle y - \hat{x}, A_1 \hat{x}\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T),$$

that is, $\hat{x} \in \mathrm{VI}\,(\mathrm{Fix}(T), A_1)$. Since $\{x^*\} = \mathrm{VI}\,(\mathrm{VI}\,(\mathrm{Fix}(T), A_1)\,, A_2)$, we have

$$\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2 x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i+1}, A_2 x^*\rangle = \langle x^* - \hat{x}, A_2 x^*\rangle \leq 0. \tag{3.7}$$

(II) We prove $\limsup_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\langle x^* - z_n, A_1 x^*\rangle \leq 0$.

Since $P_{\mathrm{Fix}(T)} z_n \in \mathrm{Fix}(T)$ and $x^* \in \mathrm{VI}\,(\mathrm{Fix}(T), A_1)$, we have

$$\begin{aligned}\langle x^* - z_n, A_1 x^*\rangle &= \langle P_{\mathrm{Fix}(T)} z_n - z_n, A_1 x^*\rangle + \langle x^* - P_{\mathrm{Fix}(T)} z_n, A_1 x^*\rangle \\ &\leq \langle P_{\mathrm{Fix}(T)} z_n - z_n, A_1 x^*\rangle \leq \|P_{\mathrm{Fix}(T)} z_n - z_n\|\|A_1 x^*\|, \quad \forall n \geq 0.\end{aligned}$$

By hypothesis, there exists $r > 0$ such that $\|x - Tx\| \geq r \inf_{y\in\mathrm{Fix(T)}} \|x - y\|$, for all $x \in H$, and therefore, we have

$$\begin{aligned}\langle x^* - z_n, A_1 x^*\rangle &\leq \|P_{\mathrm{Fix}(T)} z_n - z_n\|\|A_1 x^*\| \leq \frac{1}{r}\|z_n - Tz_n\|\|A_1 x^*\| \\ &\leq \frac{1}{r}[\|z_n - y_n\| + \|y_n - Tz_n\|]\|A_1 x^*\| \\ &\leq \frac{1}{r}[\|z_n - y_n\| + \beta_n\|x_n - Tz_n\| + \gamma_n\|f(x_n) - Tz_n\|]\|A_1 x^*\| \\ &\leq \frac{1}{r}[\|z_n - y_n\| + (\beta_n + \gamma_n)M_5]\|A_1 x^*\|,\end{aligned}$$

for every $n \geq 0$, where $M_5 := \sup_{n\geq 0}\{\|x_n - Tz_n\| + \|f(x_n) - Tz_n\|\} < \infty$. So, we obtain

$$\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle \leq \frac{\|A_1 x^*\|}{r}\left\{\frac{\lambda_n\|z_n - y_n\|}{\alpha_n\mu_n} + \frac{\lambda_n(\beta_n + \gamma_n)}{\alpha_n\mu_n}M_5\right\},\ n \geq 0.$$

This together with the condition (vi) and (3.3) implies that

$$\limsup_{n\to\infty}\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle \leq 0. \tag{3.8}$$

(III) Finally, we prove $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Observe that for all $n \geq 0$,

$$\sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \leq \sqrt{1 - \mu_n\eta} \leq 1 - \frac{1}{2}\mu_n\eta,$$

and hence,

$$\tau_n = 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \geq 1 - (1 - \frac{1}{2}\mu_n\eta) = \frac{1}{2}\mu_n\eta, \tag{3.9}$$

where $0 < \mu_n \leq \eta/\kappa^2$ for all $n \geq 0$.

By utilizing Lemma 2.2, Proposition 2.4 and relations (3.6) and (3.9), we conclude that for all $n \geq 0$,

$$\|x_{n+1} - x^*\|^2 = \|(y_n - \alpha_n\mu_n A_2 y_n) - (x^* - \alpha_n\mu_n A_2 x^*) - \alpha_n\mu_n A_2 x^*\|^2$$

$$\leq \|(y_n - \alpha_n\mu_n A_2 y_n) - (x^* - \alpha_n\mu_n A_2 x^*)\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$\leq (1 - \alpha_n\tau_n)^2\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$\leq (1 - \alpha_n\tau_n)\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$\leq (1 - \frac{1}{2}\alpha_n\mu_n\eta)\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$\leq (1 - \frac{1}{2}\alpha_n\mu_n\eta)\{\|x_n - x^*\|^2 + 2\gamma_n\|f(x^*) - x^*\|\|f(x_n) - x^*\|$$

$$+2(1 - \beta_n - \gamma_n)\lambda_n\langle x^* - z_n, A_1 x^*\rangle\} + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$= (1 - \frac{1}{2}\alpha_n\mu_n\eta)\{\|x_n - x^*\|^2 + 2\alpha_n\mu_n \cdot \frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\|$$

$$+2\alpha_n\mu_n(1 - \beta_n - \gamma_n) \cdot \frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle\} + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle$$

$$= (1-\frac{1}{2}\alpha_n\mu_n\eta)\|x_n - x^*\|^2 + \frac{1}{2}\alpha_n\mu_n\eta \cdot \frac{2}{\eta}\Big\{2(1-\frac{1}{2}\alpha_n\mu_n\eta)\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|$$

$$+2(1 - \frac{1}{2}\alpha_n\mu_n\eta)(1 - \beta_n - \gamma_n) \cdot \frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle + 2\langle x^* - x_{n+1}, A_2 x^*\rangle\Big\}. \quad (3.10)$$

Since $0 \leq 2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)(1 - \beta_n - \gamma_n) \leq 2$, it follows from Lemma 2.3 and (3.8) that

$$\limsup_{n\to\infty} 2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)(1 - \beta_n - \gamma_n) \cdot \frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle \leq 0.$$

Since $\gamma_n = o(\alpha_n\mu_n)$ (by (vi)) and $\{f(x_n)\}$ is bounded, we have

$$\lim_{n\to\infty}\left\{2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\|\right\} = 0.$$

Now, we put $a_n = \|x_n - x^*\|^2$, $s_n = \frac{1}{2}\alpha_n\mu_n\eta$, $\nu_n = 0$ and

$$\begin{aligned} t_n = \frac{2}{\eta}\Big\{&2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\| \\ &+ 2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)(1 - \beta_n - \gamma_n) \cdot \frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle \\ &+ 2\langle x^* - x_{n+1}, A_2 x^*\rangle\Big\}. \end{aligned} \quad (3.11)$$

Since $\sum_{n=0}^{\infty}\gamma_n=\infty$ and $\gamma_n=o(\alpha_n\mu_n)$ (by (vi)), we have $\sum_{n=0}^{\infty}\alpha_n\mu_n=\infty$, and hence, $\sum_{n=0}^{\infty}s_n=\infty$. It is clear from $\limsup_{n\to\infty}\langle x^*-x_{n+1},A_2x^*\rangle\le 0$, that

$$\begin{aligned}\limsup_{n\to\infty}t_n\le\frac{2}{\eta}\Big\{&\limsup_{n\to\infty}2\left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|\\&+\limsup_{n\to\infty}2\left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)(1-\beta_n-\gamma_n)\cdot\frac{\lambda_n}{\alpha_n\mu_n}\langle x^*-z_n,A_1x^*\rangle\\&+\limsup_{n\to\infty}2\langle x^*-x_{n+1},A_2x^*\rangle\Big\}\le 0.\end{aligned}$$

In terms of (3.10), it can readily be found that $a_{n+1}\le(1-s_n)a_n+s_nt_n+\nu_n,\quad\forall n\ge 0$. By utilizing Lemma 2.1, we obtain

$$\lim_{n\to\infty}\|x_n-x^*\|^2=\lim_{n\to\infty}a_n=0,$$

that is, $x_n\to x^*$. This completes the proof. □

Remark 3.1. In the above proof of Theorem 3.1, we used the similar argument and technique as in the proof of [13, Theorem 3.2], and used Lemma 2.1 to derive $\|x_{n+1}-x_n\|/\lambda_n\to 0$ as $n\to\infty$. If we want only to prove $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, then we may consider only $\sum_{n=0}^{\infty}\gamma_n=\infty$, $\sum_{n=1}^{\infty}(|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|)<\infty$ and $\sum_{n=1}^{\infty}|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|<\infty$. In this case, we have $\|x_{n+1}-x_n\|\le[1-\gamma_{n-1}(1-\rho)]\|x_n-x_{n-1}\|+(|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|)M_1+|\alpha_n\mu_n-\alpha_{n-1}\mu_{n-1}|M_2$. By apply Lemma 2.1 to the last inequality, we obtain $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Remark 3.2. We extended [13, Algorithm 3.1] to develop hybrid steepest-descent viscosity approximation method for solving Problem 1.2. Our hybrid steepest-descent viscosity approximation method is the combination of viscosity approximation method, steepest-descent method and Mann's iteration method. By Remark 3.1, we can readily see that our Algorithm 3.1 is the generalization, improvement, supplement and development of [13, Algorithm 3.1].

Remark 3.3. If for every $n\ge 1$, we take

$$\alpha_n\mu_n=\frac{1}{n^{2/5}},\ \beta_n=\gamma_n=\frac{1}{n^{1/2}},\quad\text{and}\quad\lambda_n=\frac{1}{n^{1/3}},$$

then it is easy to see that the conditions (i)-(vii) are satisfied.

Theorem 3.2. *Assume that the sequence $\{y_n\}$ generated by Algorithm 3.1 is bounded. Suppose that the sequences $\{\mu_n\}\subset(0,\eta/\kappa^2]$, $\{\alpha_n\}\subset(0,1]$, $\{\beta_n\}\subset[0,1]$, $\{\gamma_n\}\subset(0,1]$ and $\{\lambda_n\}\subset(0,2\alpha]$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n\mu_n=\infty$,

(ii) $\lim_{n\to\infty}(1/\alpha_n\mu_n)|1/\lambda_n - 1/\lambda_{n-1}| = 0$,

(iii) $\lim_{n\to\infty}(1/\lambda_n)|1 - \alpha_{n-1}\mu_{n-1}/\alpha_n\mu_n| = 0$,

(iv) $\lim_{n\to\infty}\lambda_n = 0$,

(v) $\lim_{n\to\infty}\alpha_n\mu_n/\lambda_n = 0$,

(vi) $\lim_{n\to\infty}\left(\lambda_n\beta_n + \gamma_n + \lambda_n^2\right)/\alpha_n\mu_n = 0$,

(vii) $\sum_{n=1}^{\infty}\left(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|\right)/\lambda_n < \infty$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 satisfies the following assertions:

(a) *The sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded;*

(b) $\lim_{n\to\infty}\|x_{n+1} - x_n\|/\lambda_n = 0$, $\lim_{n\to\infty}\|x_n - y_n\|/\lambda_n = 0$ *and* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$;

(c) *The sequence $\{x_n\}$ converges strongly to the unique solution of Problem 1.2 provided there exists $r > 0$ such that*

$$\|x_n - Tx_n\| \geq r \inf_{y\in\mathrm{Fix}(T)}\|x_n - y\| \ \forall\ n \geq n_0$$

for some integer $n_0 \geq 1$.

Proof. (a) By using the same argument as in the proof of Theorem 3.1 (a), we see that the sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded.

(b) We prove $\lim_{n\to\infty}\|x_{n+1} - x_n\|/\lambda_n = 0$.

By using the same argument as in the proof of Theorem 3.1 (b), we obtain

$$\|T_{n+1}x_{n+1} - T_nx_n\| \leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|A_1x_n\|, \quad \forall n \geq 0,$$

$$\|y_{n+1}-y_n\| \leq [1-\gamma_n(1-\rho)]\|x_{n+1}-x_n\| + (|\beta_{n+1}-\beta_n| + |\gamma_{n+1}-\gamma_n| + |\lambda_{n+1}-\lambda_n|)M_1,$$

for every $n \geq 0$, where $M_1 := \sup_{n\geq 0}\{\|x_n\| + \|f(x_n)\| + \|T_nx_n\| + \|A_1x_n\|\} < \infty$, and

$$\begin{aligned}\|x_{n+1} - x_n\| \leq (1 - \alpha_n\tau_n)\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\ + |\lambda_n - \lambda_{n-1}|)M_1 + |\alpha_n\mu_n - \alpha_{n-1}\mu_{n-1}|M_2,\end{aligned}$$

for every $n \geq 1$, where $\tau_n := 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \in (0, 1]$ as in Proposition 2.4 and $M_2 := \sup_{n\geq 0}\|A_2y_n\| < \infty$. For all $n \geq 0$, we observe that

$$\sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \leq \sqrt{1 - \mu_n\eta} \leq 1 - \frac{1}{2}\mu_n\eta,$$

and hence,

$$\tau_n = 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \geq 1 - \left(1 - \frac{1}{2}\mu_n\eta\right) = \frac{1}{2}\mu_n\eta, \tag{3.12}$$

where $0 < \mu_n \le \eta/\kappa^2$ for all $n \ge 0$. Therefore, for all $n \ge 1$, we obtain

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\lambda_n} &\le \left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)\frac{\|x_n-x_{n-1}\|}{\lambda_n}\\ &+\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\lambda_n-\lambda_{n-1}|}{\lambda_n}M_1\\ &\le \left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)\frac{\|x_n-x_{n-1}\|}{\lambda_{n-1}}\\ &+\frac{1}{2}\alpha_n\mu_n\eta\cdot\frac{2M_3}{\eta}\Big\{(2\alpha+1)\frac{1}{\alpha_n\mu_n}\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right|+\frac{1}{\lambda_n}|1-\frac{\alpha_{n-1}\mu_{n-1}}{\alpha_n\mu_n}|\Big\}\\ &+\frac{|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|}{\lambda_n}M_3,\end{aligned}$$

where $M_3 = \sup_{n\ge 0}\|x_{n+1}-x_n\| + M_1 + M_2 < \infty$, and also, we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.13}$$

(c) We divide the proof into three steps:

(I) As in the proof of Theorem 3.1 (c) (I), we have

$$\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2x^*\rangle \le 0.$$

(II) We prove $\limsup_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\langle x^* - z_n, A_1x^*\rangle \le 0$.

Without loss of generality, we may assume that

$$\|x_n - Tx_n\| \ge r\inf_{y\in \mathrm{Fix}(T)}\|x_n-y\|, \quad \forall n\ge 0 \text{ and for some } r>0.$$

Since $P_{\mathrm{Fix}(T)}z_n \in \mathrm{Fix}(T)$ and $x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1)$, we have

$$\begin{aligned}\langle x^*-z_n, A_1x^*\rangle &= \langle P_{\mathrm{Fix}(T)}z_n - z_n, A_1x^*\rangle + \langle x^* - P_{\mathrm{Fix}(T)}z_n, A_1x^*\rangle\\ &\le \langle P_{\mathrm{Fix}(T)}z_n - z_n, A_1x^*\rangle\\ &\le \|P_{\mathrm{Fix}(T)}z_n - z_n\|\|A_1x^*\|\\ &\le \left[\|P_{\mathrm{Fix}(T)}z_n - P_{\mathrm{Fix}(T)}x_n\| + \|P_{\mathrm{Fix}(T)}x_n - x_n\| + \|x_n - z_n\|\right]\|A_1x^*\|\\ &\le \left[2\|x_n-z_n\| + \|P_{\mathrm{Fix}(T)}x_n - x_n\|\right]\|A_1x^*\|\end{aligned}$$

for every $n \ge 0$. This together with the hypothesis of (c) implies that

$$\begin{aligned}\langle x^*-z_n, A_1x^*\rangle &\le \|P_{\mathrm{Fix}(T)}x_n - x_n\|\|A_1x^*\| + 2\|x_n-z_n\|\|A_1x^*\|\\ &\le \frac{1}{r}\|x_n - Tx_n\|\|A_1x^*\| + 2\|x_n-z_n\|\|A_1x^*\|\\ &\le \frac{1}{r}\left[\|x_n-y_n\| + \|y_n - Tx_n\|\right]\|A_1x^*\| + 2\|x_n-z_n\|\|A_1x^*\|\\ &\le \frac{1}{r}\Big[\|x_n-y_n\| + \beta_n\|x_n-Tx_n\| + \gamma_n\|f(x_n)-Tx_n\|\\ &+(1-\beta_n-\gamma_n)\|Tz_n - Tx_n\|\Big]\|A_1x^*\| + 2\|x_n-z_n\|\|A_1x^*\|\end{aligned}$$

$$\leq \frac{1}{r}\Big[\|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + \gamma_n\|f(x_n) - Tx_n\| + \|z_n - x_n\|\Big]\|A_1x^*\| + 2\|x_n - z_n\|\|A_1x^*\|$$

$$= \frac{1}{r}[\|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + \gamma_n\|f(x_n) - Tx_n\|]\|A_1x^*\| + \left(\frac{1}{r} + 2\right)\|x_n - z_n\|\|A_1x^*\|$$

$$\leq \frac{1}{r}\left[\|x_n - y_n\| + \beta_n M_5 + \gamma_n M_5\right]\|A_1x^*\| + \left(\frac{1}{r} + 2\right)\|x_n - z_n\|\|A_1x^*\|$$

$$= \frac{1}{r}[\|x_n - y_n\| + (\beta_n + \gamma_n)M_5]\|A_1x^*\| + \left(\frac{1}{r} + 2\right)\|x_n - z_n\|\|A_1x^*\|$$

for every $n \geq 0$, where $M_5 := \sup_{n\geq 0}\{\|x_n - Tx_n\| + \|f(x_n) - Tx_n\|\} < \infty$.
So, we obtain

$$\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1x^*\rangle \leq \frac{\|A_1x^*\|}{r}\left\{\frac{\lambda_n\|x_n - y_n\|}{\alpha_n\mu_n} + \frac{\lambda_n(\beta_n + \gamma_n)}{\alpha_n\mu_n}M_5\right\} + \left(\frac{1}{r} + 2\right)\frac{\lambda_n\|x_n - z_n\|}{\alpha_n\mu_n}\|A_1x^*\|$$

$$= \frac{\|A_1x^*\|}{r}\left\{\frac{\lambda_n^2}{\alpha_n\mu_n}\cdot\frac{\|x_n - y_n\|}{\lambda_n} + \frac{\lambda_n(\beta_n + \gamma_n)}{\alpha_n\mu_n}M_5\right\} + \left(\frac{1}{r} + 2\right)\frac{\lambda_n\|x_n - z_n\|}{\alpha_n\mu_n}\|A_1x^*\|$$

for every $n \geq 0$. This together with condition (vi) implies that

$$\limsup_{n\to\infty}\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1x^*\rangle \leq 0. \tag{3.14}$$

(III) Finally, we prove $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

As in the proof of Theorem 3.1 (c) (III), we have for all $n \geq 0$,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1 - \frac{1}{2}\alpha_n\mu_n\eta)\|x_n - x^*\|^2 \\ &+ \frac{1}{2}\alpha_n\mu_n\eta\cdot\frac{2}{\eta}\Big\{2\Big(1 - \frac{1}{2}\alpha_n\mu_n\eta\Big)\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\| \\ &+ 2\left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)(1 - \beta_n - \gamma_n)\cdot\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1x^*\rangle \\ &+ 2\langle x^* - x_{n+1}, A_2x^*\rangle\Big\}.\end{aligned} \tag{3.15}$$

Since $\sum_{n=0}^{\infty}\alpha_n\mu_n = \infty$ implies $\sum_{n=0}^{\infty}s_n = \infty$, it is clear from $\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2x^*\rangle \leq 0$ that $\limsup_{n\to\infty}t_n \leq 0$, where t_n is the same as in (3.11). Following the same argument as in the proof of Theorem 3.1 (c) (III), we obtain $x_n \to x^*$. This completes the proof. □

Remark 3.4. In the above proof of Theorem 3.2, we used the similar argument and technique as in [13, Theorem 3.2] and Lemma 2.1 to derive $\|x_{n+1} - x_n\|/\lambda_n \to 0$

as $n \to \infty$. If we want to prove $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then we may consider $\sum_{n=0}^{\infty} \alpha_n \nu_n = \infty$ in Theorem 3.1 and $\sum_{n=0}^{\infty} \alpha_n \mu_n = \infty$ in Theorem 3.2,

$$\sum_{n=1}^{\infty}(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\lambda_n - \lambda_{n-1}|) < \infty$$

and

$$\sum_{n=1}^{\infty} |\alpha_n \mu_n - \alpha_{n-1}\mu_{n-1}| < \infty.$$

In this case, combining the proven inequality

$$\begin{aligned}\|x_{n+1} - x_n\| \leq (1 - \alpha_n \tau_n)\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\ + |\lambda_n - \lambda_{n-1}|)M_1 + |\alpha_n \mu_n - \alpha_{n-1}\mu_{n-1}|M_2\end{aligned}$$

and (3.12), we have $\|x_{n+1} - x_n\| \leq (1 - \frac{1}{2}\alpha_n\mu_n\eta)\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_1 + |\alpha_n\mu_n - \alpha_{n-1}\mu_{n-1}|M_2$. Then by applying Lemma 2.1 to the last inequality, we immediately obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Remark 3.5. If for every $n \geq 1$, we take $\alpha_n\mu_n = \beta_n = \frac{1}{n^{1/2}}$, $\gamma_n = \frac{1}{n^{3/5}}$, and $\lambda_n = \frac{1}{n^{1/3}}$, then it is easy to see that the conditions (i) - (vii) are satisfied.

The following result is established under some suitable conditions, which are very different from those in Theorems 3.1 and 3.2.

Theorem 3.3. *Assume that the sequence $\{y_n\}$ generated by Algorithm 3.1 is bounded. Let $\{\mu_n\} \subset (0, \eta/\kappa^2]$, $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, 2\alpha]$ be such that the following conditions hold:*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* $\lim_{n\to\infty}(\beta_n + \gamma_n) = 0$;

(ii) $\sum_{n=0}^{\infty} (|\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n| + |\lambda_{n+1} - \lambda_n|) < \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1}\mu_{n+1} - \alpha_n\mu_n| < \infty$ *and* $\lim_{n\to\infty} \alpha_n\mu_n = 0$;

(iv) $\gamma_n = o(\lambda_n)$ *and* $\lambda_n \leq \alpha_n\mu_n$, $\forall n \geq 0$.

Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 satisfies the following assertions:

(a) *The sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded;*

(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$;

(c) *The sequence $\{x_n\}$ converges strongly to a unique solution of Problem 1.2 provided $\|x_n - y_n\| = o(\lambda_n)$.*

Proof. (a) It is similar to the proof of Theorem 3.1 (a).

(b) We prove $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

As in the proof of Theorem 3.1 (b), we have

$$\|T_{n+1}x_{n+1} - T_n x_n\| \leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|A_1 x_n\|, \quad n \geq 0,$$

and hence,

$$\|y_{n+1}-y_n\| \leq [1-\gamma_n(1-\rho)]\|x_{n+1}-x_n\|+(|\beta_{n+1}-\beta_n|+|\gamma_{n+1}-\gamma_n|+|\lambda_{n+1}-\lambda_n|)M_1,$$

where $M_1 := \sup_{n\geq 0}\{\|x_{n+1}\| + \|f(x_{n+1})\| + \|T_{n+1}x_{n+1}\| + \|A_1 x_n\|\} < \infty$, and

$$\begin{aligned}\|x_{n+1} - x_n\| \leq [1 - \gamma_{n-1}(1-\rho)]\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\ + |\lambda_n - \lambda_{n-1}|)M_1 + |\alpha_n\mu_n - \alpha_{n-1}\mu_{n-1}|M_2,\end{aligned}$$

where $\tau_n := 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \in (0,1]$ as in Proposition 2.4 and

$$M_2 := \sup_{n\geq 0}\|A_2 y_n\| < \infty.$$

Hence, for all $n, m \geq 0$, we get

$$\begin{aligned}
&\|x_{n+m+1} - x_{n+m}\| \leq [1 - \gamma_{n+m-1}(1-\rho)]\|x_{n+m} - x_{n+m-1}\| \\
&+ (|\beta_{n+m} - \beta_{n+m-1}| + |\gamma_{n+m} - \gamma_{n+m-1}| + |\lambda_{n+m} - \lambda_{n+m-1}|)\, M_1 \\
&+ |\alpha_{n+m}\mu_{n+m} - \alpha_{n+m-1}\mu_{n+m-1}|M_2 \\
&\leq [1 - \gamma_{n+m-1}(1-\rho)]\{[1 - \gamma_{n+m-2}(1-\rho)]\|x_{n+m-1} - x_{n+m-2}\| \\
&+ (|\beta_{n+m-1} - \beta_{n+m-2}| + |\gamma_{n+m-1} - \gamma_{n+m-2}| \\
&+ |\lambda_{n+m-1} - \lambda_{n+m-2}|)M_1 + |\alpha_{n+m-1}\mu_{n+m-1} - \alpha_{n+m-2}\mu_{n+m-2}|M_2\} \\
&+ M_1(|\beta_{n+m} - \beta_{n+m-1}| + |\gamma_{n+m} - \gamma_{n+m-1}| + |\lambda_{n+m} - \lambda_{n+m-1}|) \\
&+ M_2|\alpha_{n+m}\mu_{n+m} - \alpha_{n+m-1}\mu_{n+m-1}| \\
&\leq \prod_{k=m}^{n+m-1}[1 - \gamma_k(1-\rho)]\|x_{m+1} - x_m\| \\
&+ M_1\sum_{k=m}^{n+m-1}(|\beta_{k+1} - \beta_k| + |\gamma_{k+1} - \gamma_k| + |\lambda_{k+1} - \lambda_k|) \\
&+ M_2\sum_{k=m}^{n+m-1}|\alpha_{k+1}\mu_{k+1} - \alpha_k\mu_k|.
\end{aligned}$$

By condition (i), we have $\prod_{k=m}^{\infty}[1 - \gamma_k(1-\rho)] = 0, \quad \forall m \geq 0$, and hence, for all $m \geq 0$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\|x_{n+1} - x_n\|^2 &= \limsup_{n\to\infty}\|x_{n+m+1} - x_{n+m}\|^2 \\
&\leq M_1\sum_{k=m}^{\infty}(|\beta_{k+1} - \beta_k| + |\gamma_{k+1} - \gamma_k| + |\lambda_{k+1} - \lambda_k|) \\
&+ M_2\sum_{k=m}^{\infty}|\alpha_{k+1}\mu_{k+1} - \alpha_k\mu_k|.
\end{aligned}$$

This together with the conditions (ii) and (iii) ensures that $\limsup_{n\to\infty} \|x_{n+1} - x_n\| \le 0$, that is,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

From condition (iii) and $\|x_{n+1} - y_n\| = \alpha_n \mu_n \|A_2 y_n\| \le M_2 \alpha_n \mu_n$, we get that $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$.

Since $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, it follows from (3.16) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.17}$$

Moreover, assumption (A3) guarantees that

$$\begin{aligned}
\|y_n - Tx_n\| &= \|\beta_n(x_n - Tx_n) + \gamma_n(f(x_n) - Tx_n) + (1 - \beta_n - \gamma_n)(T_n x_n - Tx_n)\| \\
&\le \beta_n \|x_n - Tx_n\| + \gamma_n \|f(x_n) - Tx_n\| + (1 - \beta_n - \gamma_n)\|T_n x_n - Tx_n\| \\
&\le \beta_n \|x_n - Tx_n\| + \gamma_n \|f(x_n) - Tx_n\| + \|T(I - \lambda_n A_1)x_n - Tx_n\| \\
&\le \beta_n \|x_n - Tx_n\| + \gamma_n \|f(x_n) - Tx_n\| + \lambda_n \|A_1 x_n\|.
\end{aligned}$$

Hence, from conditions (i) and (iii), we have $\lim_{n\to\infty} \|y_n - Tx_n\| = 0$. Therefore, from (3.17), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.18}$$

(c) We first prove $\limsup_{n\to\infty} \langle x^* - x_n, A_1 x^* \rangle \le 0$.

Choose a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x^* - x_n, A_1 x^* \rangle = \lim_{i\to\infty} \langle x^* - x_{n_i}, A_1 x^* \rangle.$$

The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. We may assume without loss of generality that

$$\lim_{i\to\infty} \langle x_{n_i}, w \rangle = \langle \hat{x}, w \rangle, \quad \forall w \in H.$$

If we assume $\hat{x} \notin \mathrm{Fix}(T)$, then (3.18) and (A3) guarantee that $\liminf_{i\to\infty} \|x_{n_i} - \hat{x}\| < \liminf_{i\to\infty} \|x_{n_i} - T\hat{x}\| = \liminf_{i\to\infty} \|x_{n_i} - Tx_{n_i} + Tx_{n_i} - T\hat{x}\| = \liminf_{i\to\infty} \|Tx_{n_i} - T\hat{x}\| \le \liminf_{i\to\infty} \|x_{n_i} - \hat{x}\|$, which is a contradiction. Therefore, $\hat{x} \in \mathrm{Fix}(T)$. Since $x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1)$, we have

$$\limsup_{n\to\infty} \langle x^* - x_n, A_1 x^* \rangle = \lim_{i\to\infty} \langle x^* - x_{n_i}, A_1 x^* \rangle = \langle x^* - \hat{x}, A_1 x^* \rangle \le 0. \tag{3.19}$$

Next, we prove $\limsup_{n\to\infty} \langle x^* - x_{n+1}, A_2 x^* \rangle \le 0$.

The relation (3.16) guarantees the existence of two subsequences $\{x_{n_k}\}$ and $\{x_{n_k+1}\}$ of $\{x_n\}$ and $\bar{x} \in H$ such that one have

$$\limsup_{n\to\infty} \langle x^* - x_{n+1}, A_2 x^* \rangle = \lim_{k\to\infty} \langle x^* - x_{n_k+1}, A_2 x^* \rangle$$

and

$$\lim_{k\to\infty} \langle x_{n_k}, w \rangle = \lim_{k\to\infty} \langle x_{n_k+1}, w \rangle = \langle \bar{x}, w \rangle, \quad \forall w \in H.$$

By the same argument as that in the proof of $\hat{x} \in \mathrm{Fix}(T)$, we have $\bar{x} \in \mathrm{Fix}(T)$. Let $y \in \mathrm{Fix}(T)$ be fixed arbitrarily.

Then, it follows from (A1) and (A3) that for all $n \geq 0$,

$$\begin{aligned}
\|y_n - y\|^2 &= \|\beta_n(x_n - y) + \gamma_n(f(x_n) - y) + (1 - \beta_n - \gamma_n)(T_n x_n - Ty)\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|T(I - \lambda_n A_1)x_n - Ty\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - f(y) + f(y) - y\|^2 \\
&+ (1 - \beta_n - \gamma_n)\|(I - \lambda_n A_1)x_n - y\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n[\|f(x_n) - f(y)\|^2 + 2\langle f(y) - y, f(x_n) - y\rangle] \\
&+ (1 - \beta_n - \gamma_n)[\|x_n - y\|^2 \\
&+ 2\lambda_n\langle y - x_n, A_1 x_n\rangle + \lambda_n^2\|A_1 x_n\|^2] \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n[\rho^2\|x_n - y\|^2 + 2\|f(y) - y\|\|f(x_n) - y\|] \\
&+ (1 - \beta_n - \gamma_n)[\|x_n - y\|^2 \\
&+ 2\lambda_n\langle y - x_n, A_1 y\rangle + \lambda_n^2 M_1^2] \\
&\leq [1 - \gamma_n(1 - \rho)]\|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| \\
&+ (1 - \beta_n - \gamma_n)[2\lambda_n\langle y - x_n, A_1 y\rangle + \lambda_n^2 M_1^2] \\
&\leq \|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| \\
&+ (1 - \beta_n - \gamma_n)[2\lambda_n\langle y - x_n, A_1 y\rangle + \lambda_n^2 M_1^2],
\end{aligned} \tag{3.20}$$

which implies that for all $n \geq 0$,

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_n}(\|x_n - y\|^2 - \|y_n - y\|^2) + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\|f(x_n) - y\| \\
&\quad +(1 - \beta_n - \gamma_n)[2\langle y - x_n, A_1 y\rangle + \lambda_n M_1^2] \\
&= (\|x_n - y\| + \|y_n - y\|)\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\|f(x_n) - y\| \\
&\quad +(1 - \beta_n - \gamma_n)\left[2\langle y - x_n, A_1 y\rangle + \lambda_n M_1^2\right] \\
&\leq M_3\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_3 + (1 - \beta_n - \gamma_n)\left[2\langle y - x_n, A_1 y\rangle + \lambda_n M_1^2\right] \\
&\leq M_3\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_3 + (1 - \beta_n - \gamma_n)\left[2\langle y - x_n, A_1 y\rangle + \lambda_n M_1^2\right],
\end{aligned}$$

where $M_3 := \sup_{n\geq 0}\{\|x_n - y\| + \|y_n - y\| + \|f(x_n) - y\|\} < \infty$. This shows that

$$\frac{M_3}{1 - \beta_n - \gamma_n}\left[\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\right] + 2\langle y - x_n, A_1 y\rangle + \lambda_n M_1^2 \geq 0.$$

Since $\|x_n - y_n\| + \gamma_n = o(\lambda_n)$, from condition (i), for any $\varepsilon > 0$, there exists an integer $m_0 \geq 0$ such that

$$\frac{M_3}{1 - \beta_n - \gamma_n}\left[\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\right] + \lambda_n M_1^2 \leq \varepsilon, \ \forall\ n \geq m_0.$$

Hence,

$$0 \le \varepsilon + 2\langle y - x_n, A_1 y\rangle, \quad \forall n \ge m_0.$$

Putting $n := n_k$, we have $\varepsilon + 2\langle y - \bar{x}, A_1 y\rangle \ge 0$ as $k \to \infty$, from the weak convergence of $\{x_{n_k}\}$ to $\bar{x} \in \mathrm{Fix}(T)$. Since $\varepsilon > 0$ is arbitrary, we get

$$\langle y - \bar{x}, A_1 y\rangle \ge 0, \quad \forall y \in \mathrm{Fix}(T).$$

By assumption (A1) and Proposition 2.2 (i), we have

$$\langle y - \bar{x}, A_1 \bar{x}\rangle \ge 0, \quad \forall y \in \mathrm{Fix}(T);$$

that is, $\bar{x} \in \mathrm{VI}\,(\mathrm{Fix}(T), A_1)$. Since $\{x^*\} = \mathrm{VI}\,(\mathrm{VI}\,(\mathrm{Fix}(T), A_1)\,, A_2)$, we have

$$\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2 x^*\rangle = \lim_{k\to\infty}\langle x^* - x_{n_k+1}, A_2 x^*\rangle = \langle x^* - \bar{x}, A_2 x^*\rangle \le 0. \tag{3.21}$$

Finally, we prove that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Observe that $\sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \le \sqrt{1 - \mu_n\eta} \le 1 - \frac{1}{2}\mu_n\eta$, for all $n \ge 0$ and hence,

$$\tau_n = 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \ge 1 - \left(1 - \frac{1}{2}\mu_n\eta\right) = \frac{1}{2}\mu_n\eta, \tag{3.22}$$

where $0 < \mu_n \le \eta/\kappa^2$ for all $n \ge 0$.

By utilizing Lemma 2.2 and Proposition 2.4, from (3.20), (3.22) and the condition (iv), we conclude that for all $n \ge 0$,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(I - \alpha_n\mu_n A_2)y_n - (I - \alpha_n\mu_n A_2)x^* - \alpha_n\mu_n A_2 x^*\|^2 \\
&\le \|(I - \alpha_n\mu_n A_2)y_n - (I - \alpha_n\mu_n A_2)x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le (1 - \alpha_n\tau_n)^2\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le (1 - \alpha_n\tau_n)\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le (1 - \frac{1}{2}\alpha_n\mu_n\eta)\|y_n - x^*\|^2 + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le (1 - \frac{1}{2}\alpha_n\mu_n\eta)\{\|x_n - x^*\|^2 + 2\gamma_n\|f(x^*) - x^*\|\|f(x_n) - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)[2\lambda_n\langle x^* - x_n, A_1 x^*\rangle + \lambda_n^2 M_1^2]\} + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&= \left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)\Big\{\|x_n - x^*\|^2 + 2\alpha_n\mu_n\cdot\frac{\gamma_n}{\alpha_n\mu_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)\Big[2\alpha_n\mu_n\cdot\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - x_n, A_1 x^*\rangle \\
&\quad + \alpha_n\mu_n\cdot\frac{\lambda_n^2}{\alpha_n\mu_n}M_1^2\Big]\Big\} + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le \left(1 - \frac{1}{2}\alpha_n\mu_n\eta\right)\Big\{\|x_n - x^*\|^2 + 2\alpha_n\mu_n\cdot\frac{\gamma_n}{\lambda_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\| \\
&\quad + (1 - \beta_n - \gamma_n)\Big[2\alpha_n\mu_n\cdot\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - x_n, A_1 x^*\rangle \\
&\quad + \alpha_n\mu_n\cdot\lambda_n M_1^2\Big]\Big\} + 2\alpha_n\mu_n\langle x^* - x_{n+1}, A_2 x^*\rangle \\
&\le (1 - \frac{1}{2}\alpha_n\mu_n\eta)\|x_n - x^*\|^2 + 2\alpha_n\mu_n\cdot\frac{\gamma_n}{\lambda_n}\|f(x^*) - x^*\|\|f(x_n) - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&+(1-\frac{1}{2}\alpha_n\mu_n\eta)(1-\beta_n-\gamma_n)2\alpha_n\mu_n\cdot\frac{\lambda_n}{\alpha_n\mu_n}\langle x^*-x_n,A_1x^*\rangle\\
&+\alpha_n\mu_n\cdot\lambda_nM_1^2+2\alpha_n\mu_n\langle x^*-x_{n+1},A_2x^*\rangle\\
&=(1-\frac{1}{2}\alpha_n\mu_n\eta)\|x_n-x^*\|^2+\frac{1}{2}\alpha_n\mu_n\eta\cdot\frac{2}{\eta}\Big\{\frac{2\gamma_n}{\lambda_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|\\
&+(1-\frac{1}{2}\alpha_n\mu_n\eta)(1-\beta_n-\gamma_n)\frac{2\lambda_n}{\alpha_n\mu_n}\langle x^*-x_n,A_1x^*\rangle\\
&+\lambda_nM_1^2+2\langle x^*-x_{n+1},A_2x^*\rangle\Big\}. \qquad (3.23)
\end{aligned}
$$

Note that $0\le\left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)(1-\beta_n-\gamma_n)\frac{2\lambda_n}{\alpha_n\mu_n}\le 2$ (by the condition (iv)) and $\limsup_{n\to\infty}\langle x^*-x_n,A_1x^*\rangle\le 0$. Hence, it follows from Lemma 2.3 that

$$\limsup_{n\to\infty}(1-\frac{1}{2}\alpha_n\mu_n\eta)(1-\beta_n-\gamma_n)\frac{2\lambda_n}{\alpha_n\mu_n}\langle x^*-x_n,A_1x^*\rangle\le 0.$$

Since $\gamma_n=o(\lambda_n)$ and $\{f(x_n)\}$ is bounded, we have

$$\lim_{n\to\infty}\left(\frac{2\gamma_n}{\lambda_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|+\lambda_nM_1^2\right)=0.$$

Now, we put $a_n=\|x_n-x^*\|^2$, $s_n=\frac{1}{2}\alpha_n\mu_n\eta$ and

$$
\begin{aligned}
t_n=\frac{2}{\eta}\Big\{&\left(\frac{2\gamma_n}{\lambda_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|+\lambda_nM_1^2\right)\\
&+\left(1-\frac{1}{2}\alpha_n\mu_n\eta\right)(1-\beta_n-\gamma_n)\frac{2\lambda_n}{\alpha_n\mu_n}\langle x^*-x_n,A_1x^*\rangle\\
&+2\langle x^*-x_{n+1},A_2x^*\rangle\Big\}.
\end{aligned}
$$

Utilizing the condition (iv), we have

$$\sum_{n=0}^{\infty}s_n=\sum_{n=0}^{\infty}\frac{1}{2}\alpha_n\mu_n\eta\ge\frac{1}{2}\eta\sum_{n=0}^{\infty}\lambda_n=\infty.$$

Since $\limsup_{n\to\infty}\langle x^*-x_{n+1},A_2x^*\rangle\le 0$, we have

$$
\begin{aligned}
\limsup_{n\to\infty}t_n\le\frac{2}{\eta}\Big\{&\limsup_{n\to\infty}\left(\frac{2\gamma_n}{\lambda_n}\|f(x^*)-x^*\|\|f(x_n)-x^*\|+\lambda_nM_1^2\right)\\
&+\limsup_{n\to\infty}(1-\frac{1}{2}\alpha_n\mu_n\eta)(1-\beta_n-\gamma_n)\frac{2\lambda_n}{\alpha_n\mu_n}\langle x^*-x_n,A_1x^*\rangle\\
&+\limsup_{n\to\infty}2\langle x^*-x_{n+1},A_2x^*\rangle\Big\}\le 0.
\end{aligned}
$$

In terms of (3.23), it can be easily seen that $a_{n+1}\le(1-s_n)a_n+s_nt_n,\quad\forall n\ge 0$. Therefore, utilizing Lemma 2.1, we obtain $\lim_{n\to\infty}\|x_n-x^*\|^2=\lim_{n\to\infty}a_n=0$, that is, $x_n\to x^*$. This completes the proof. □

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