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AN APPLICATION OF THE COMMON FIXED POINT THEOREMS TO THE THEORY OF STABILITY OF FUNCTIONAL EQUATIONS

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Abstract. We present an application of the common fixed point theorems, i.e., Markov-Kakutani fixed point theorem and DeMarr common fixed point theorem, to the stability of the functional equation of the form

$$f(sx) = F(s, f(x)), \quad s \in G, x \in X.$$

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1. Introduction

The most popular approaches to prove Hyers-Ulam stability of functional equations are: the direct method (see the paper [7] from which the theory originates and the comments in [5]), the method of invariant means (see for example [25] or [26]) and the fixed point method (see, for example, [3] where a recent survey on this topic is provided). In the last method, various already known fixed point theorems are used, as well as some new have been proven and used in a particular situation. However, it seems that the usefulness of the common fixed point theorems to the theory of stability of functional equations have not yet been appreciated. In the paper [21] a new proof of the classical Hyers theorem is given, via the Markov-Kakutani common fixed point theorem. Here we show another example of an application of common fixed point theorems. The functional equation, which stability will be studied, is quite general, it includes, for example, the homogenity equation widely examined in literature, the equation of periodic and microperiodic function, exponential function and others.

For the monographs and survey papers devoted to stability theory see [2, 6, 8, 9, 14, 15, 22, 23, 27] and more references therein.

2. Functional equation $f(sx) = F(s, f(x))$.

Let us start with reminding one of the first examined equations in the monograph [1], i.e.,

$$f(s+t) = F(s, f(t)), \quad s, t \in S, \tag{2.1}$$

with unknown function f defined on a semigroup S with a neutral element. It has been shown there that

• if (2.1) has a solution then $F(s, F(t, x)) = F(s+t, x)$ for every $s, t \in S$ and $x \in f(S)$;

• if $F(s, F(t, x_0)) = F(s+t, x_0)$ for $s, t \in S$ and an x_0, then $f(s) := F(s, x_0)$ is a solution of (2.1).

Moreover, a few particular examples of (2.1) have been given:

(i) $f(s+t) = a^s f(t)$; here $F(s, x) = a^s x$ and $S = (\mathbb{R}, +)$,

(ii) $f(st) = s^a f(t)$; here $F(s, x) = s^a x$ and $S = (\mathbb{R}_+, \cdot)$,

(iii) $f(s+t) = f(t)$; here $F(s, x) = x$ and $(S, +)$ is a group,

(iv) $f(s+t) = s + f(t)$; here $F(s, x) = s + x$, $f \colon S \to S$, where $(S, +)$ is a group.

In what follows assume that X is a set, G is an abelian group with neutral element $\mathbb{1}$, acting on X, i.e., there is a map $S \times X \ni (s, x) \to s \diamond x \in X$ given, such that $s \diamond (t \diamond x) = (st) \diamond x$ and $\mathbb{1} \diamond x = x$, for $s, t \in G$, $x \in X$. Let Y be a set and $F \colon G \times Y \to Y$. Let us consider the following functional equation with unknown function $f \colon X \to Y$

$$f(s \diamond x) = F(s, f(x)), \quad s \in G, x \in X, \tag{2.2}$$

which is a generalization of (2.1). Below we list some other examples of (2.2):

(v) $f(sx) = s^p f(x)$, $s \in \mathbb{K}_0 < \mathbb{K}^*$, $x \in X$; here X and Y are linear spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$;

(vi) $f(x + kp) = f(x)$, $x \in X$, $k \in \mathbb{Z}$; here X is a group, $p \in X$;

(vii) $f(x + qp) = f(x)$, $x \in X$, $q \in \mathbb{Q}$; here X is a linear space, $p \in X$;

(viii) $f(x^y) = y f(x)$, $x \in (0, \infty)$, $y \in \mathbb{R}^*$;

(ix) $f(s+t) = s + f(t)$, $s \in G_0$, $t \in G$, where G_0 is a subgroup of a group G;

(x) $f(x^p) = f(x)^p$, $x \in (0, \infty)$, $p \in G$; here G is a subgroup of $\mathbb{R}_+^*$.

3. An application of the Markov-Kakutani fixed point theorem

For convenience of a reader we remind the well-known Markov-Kakutani fixed point theorem ([17, 20, 24]).

Theorem 3.1. *Let Z be a linear topological space, $\mathcal{F}$ a commuting family of continuous, affine selfmappings of Z, $\mathcal{C}$ a nonempty compact convex subset of Z which is $\mathcal{F}$-invariant. Then there is, in the set $\mathcal{C}$, a common fixed point of all mappings from $\mathcal{F}$.*

The first of the two main results of this paper is the following theorem which we are going to prove via the Markov-Kakutani common fixed point theorem.

Theorem 3.2. *Let G be an abelian group acting on a set X, Y a linear topological space, K compact convex subset of Y containing* 0. *Let $F \colon G \times Y \to Y$ satisfy assumptions* (A)*:*

• $F(s, F(t, y)) = F(st, y), \quad s, t \in G, y \in Y,$

• *$F(t,\cdot)\colon Y \to Y$ is continuous and affine, for every $t \in G$. Suppose that $f_0\colon X \to Y$ satisfies*

$$F(s, f_0(x)) - f_0(s \diamond x) \in K, \quad x \in X,\, s \in G,$$

then there is $\bar{f}\colon X \to Y$ such that

$$\bar{f}(s \diamond x) = F(s, \bar{f}(x)), \quad x \in X,\, s \in G \tag{3.1}$$

and

$$\bar{f}(x) - f_0(x) \in K, \quad x \in X. \tag{3.2}$$

Proof. Let us consider $Z := Y^X$ with the Tychonoff topology. For any $t \in G$ let us define $G_t\colon Z \to Z$ by $G_t f(x) = F(t, f(t^{-1} \diamond x))$. Put $\mathcal{F} = \{G_t : t \in G\}$. It is easy to verify that

$$G_t \circ G_s = G_{ts}, \quad t, s \in G, \tag{3.3}$$

in particular, the family $\mathcal{F}$ is commuting. Moreover, due to the second of assumptions (A), every G_t is continuous and affine. Let us put

$$\begin{aligned}\mathcal{C} := \{f \in Z :\ & f(x) - f_0(x) \in K,\, x \in X,\\ & G_t f(x) - f_0(x) \in K,\, x \in X, t \in G\}.\end{aligned}$$

The set $\mathcal{C}$ is nonempty, as $f_0 \in \mathcal{C}$. Moreover, it is convex, by convexity of K and affinity of G_t. Notice also that $\mathcal{C}$ is compact, as a closed subset of the compact set $\{f \in Z : f(x) - f_0(x) \in K\}$. Moreover, due to (3.3), $G_t(\mathcal{C}) \subset \mathcal{C}$ for every $t \in G$. By Markov-Kakutani theorem we infer that there is $\bar{f}$ in the set $\mathcal{C}$ such that $G_t(\bar{f}) = \bar{f}$, $t \in G$. This means that (3.2) and (3.1) hold. The proof is completed. □

4. An application of DeMarr theorem

Let $(\mathcal{C}, \leq)$ be a partially ordered set. We say that $\mathcal{C}$ is *complete*, if every bounded subset has infimum and supremum. A map $g\colon \mathcal{C} \to \mathcal{C}$ is *isotone* if $x \leq y$ implies $g(x) \leq g(y)$, for every $x, y \in \mathcal{C}$. We remind the corollary from DeMarr theorem ([4], see also [19]):

Theorem 4.1. *If $\mathcal{C}$ is a nonempty complete partially ordered set which has a largest element, and $\mathcal{F}$ is a commuting family of isotone mappings from $\mathcal{C}$ to $\mathcal{C}$, then there is a common fixed point $\bar{f}$ of $\mathcal{F}$.*

Our second main result reads as follows.

Theorem 4.2. *Let G be an abelian group acting on a set X, $(Y, \leq)$ partially ordered set such that every bounded subset of Y has infimum and supremum in Y, $F\colon G \times Y \to Y$ satisfy assumptions* (B)*:*

- *$F(s, F(t, y)) = F(st, y)$, $s, t \in G$, $y \in Y$,*
- *$[y_1 \leq y_2$ implies $F(t, y_1) \leq F(t, y_2)]$, $t \in G$,*
- *if Y_0 is a nonempty bounded subset of Y, $t \in G$, then*

$$[F(t, y) \leq z,\ y \in Y_0] \Rightarrow F(t, \sup Y_0) \leq z,$$

$$[z \leq F(t, y),\ y \in Y_0] \Rightarrow z \leq F(t, \inf Y_0).$$

Let $f_0, a, b\colon X \to Y$ satisfy

$$f_0(x), F(t, f_0(t^{-1} \diamond x)) \in [a(x), b(x)], \quad x \in X,\, t \in G.$$

Then there is $\bar{f}\colon X \to Y$ *such that*

$$\bar{f}(s \diamond x) = F(s, \bar{f}(x)), \quad s \in G, x \in X \tag{4.1}$$

and

$$\bar{f}(x) \in [a(x), b(x)], \quad x \in X. \tag{4.2}$$

Proof. Let $Z = Y^X$. We define partial order in Z by

$$f \le g \iff f(x) \le g(x), x \in X.$$

Let $\mathcal{F} = \{G_t\colon Z \to Z;\, t \in G\}$, where $G_t f(x) = F(t, f(t^{-1} \diamond x))$. Put $\mathcal{C} = \{f \in Z : f(x), G_t f(x) \in [a(x), b(x)],\, x \in X,\, t \in G\}$. Then $\mathcal{C}$ is nonempty complete partially ordered set which has a largest element, and $\mathcal{F}$ is a commuting family of isotone mappings from $\mathcal{C}$ to $\mathcal{C}$. From the theorem of DeMarr we infer that there is a common fixed point $\bar{f}$ of $\mathcal{F}$ in $\mathcal{C}$, hence (4.1) and (4.2) hold. □

5. Remarks

Notice that, by Theorem 3.2, we have proven the stability of functional equations from examples (i)-(ix) under suitable condition on domain and target space of f. By Theorem 4.2, the functional equation from example (x) is stable in the sense of Hyers-Ulam. In paper [16] the stability of functional equation from example (ix) has been investigated, in paper [30]- from examples (v) and (viii). Moreover, stability of homogenity (example (v)) has been widely examined in papers [10, 11, 12, 13, 18, 28, 29]. Actually, in the mentioned papers, it has been proven that the functional equations from examples (v) and (viii) are *superstable* (i.e., every approximate solution is an exact solution of the equation). Notice that for some of the investigated equations from the proven stability we can easily derive superstability.

Corollary 5.1. (a) *Let* $\varepsilon \ge 0$ *and* $f\colon \mathbb{R} \to \mathbb{R}$, $a > 0$. *Assume that*

$$|f(s+t) - a^s f(t)| \le \varepsilon, \quad s, t \in \mathbb{R}.$$

Then $f(s+t) = a^s f(t)$ *for all* $s, t \in \mathbb{R}$.

(b) *Let* $\varepsilon \ge 0$ *and* $f\colon (0,\infty) \to (0,\infty)$, $a \ne 0$. *Assume that*

$$|f(st) - s^a f(t)| \le \varepsilon, \quad s, t \in (0,\infty).$$

Then $f(st) = s^a f(t)$ *for all* $s, t \in (0,\infty)$.

(c) *Let* $p \ne 0$ *and* $f\colon X \to Y$, *where* X *and* Y *are linear spaces over* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $\mathbb{K}_0$ *be a subgroup of* $(\mathbb{K} \setminus \{0\}, \cdot)$, *different from* $\{1\}$ *and let* K *be a nonempty convex compact subset of* Y *containing* 0. *Assume that*

$$s^p f(x) - f(sx) \in K, \quad s \in \mathbb{K}_0, x \in X.$$

Then $f(sx) = s^p f(x)$, *for all* $s \in \mathbb{K}_0$ *and* $x \in X$.

(d) *Let* $\varepsilon \ge 0$ *and* $f\colon (0,\infty) \to (0,\infty)$. *Assume that*

$$|f(x^y) - y f(x)| \le \varepsilon, \quad x, y \in (0,\infty).$$

Then $f(x^y) = y f(x)$ *for all* $x, y \in (0,\infty)$.

Proof. We will show only part (a), since the proofs of the other parts are analogous. Let f be as in the assumptions. We already know (from Theorem 3.2 with $G = X =$

$Y = \mathbb{R}$ and $K = [-\varepsilon, \varepsilon]$, $\diamond = +$ and $F(s,x) = a^s x$) that there is $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\bar{f}(s+t) = a^s \bar{f}(t)$ and $|\bar{f}(s) - f(s)| \leq \varepsilon$, $s, t \in \mathbb{R}$. We have

$$|a^s \bar{f}(t) - a^s f(t)| = |\bar{f}(s+t) - f(s+t)| + |f(s+t) - a^s f(t)| \leq 2\varepsilon, \quad s, t \in \mathbb{R},$$

hence

$$|\bar{f}(t) - f(t)| \leq \frac{2\varepsilon}{a^s}, \quad s, t \in \mathbb{R}.$$

Passing with a^s to infinity gives $\bar{f} = f$. □

Next, we are going to discuss the assumption that F satisfies the translation equation. As it has been proven in [1], satisfying the translation equation at least in a single point x_0 (i.e., $F(s, F(t, x_0)) = F(s+t, x_0)$, $s, t \in G$) is equivalent to the existence of solution of equation (2.1). However, assuming that the translation equation is satisfied only in one point is not enough to obtain stability.

Example 5.2. Let $\varepsilon > 0$ and $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$F(s,x) = \begin{cases} s + x + \varepsilon, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Notice that the translation equation is satisfied only for $x = 0$. Hence the only solution of (2.1) is $f = 0$. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(t) = \begin{cases} t, & t \neq 0; \\ \varepsilon, & t = 0. \end{cases}$$

One can check directly that

$$|\phi(s+t) - F(s, \phi(t))| \leq \varepsilon, \quad s, t \in \mathbb{R}.$$

But the difference between ϕ and the only solution of (2.1) is unbounded.

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