Fixed Point Theory, 16(2015), No. 1, 143-162
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

W-GPH-KKM THEOREMS IN PSEUDO H-SPACES AND THEIR APPLICATIONS

HAISHU LU* AND JIHUI ZHANG**

*School of Business, Jiangsu University of Technology
Changzhou 213001, Jiangsu, China
E-mail: luhaishu@126.com

**Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences
Nanjing Normal University, Nanjing 210097, Jiangsu, China
E-mail: zhangjihui@njnu.edu.cn

Abstract. In this paper, some new W-GPH-KKM theorems are established in pseudo H-spaces without any linear and convex structure under much weaker assumptions, and next as their applications, some new coincidence theorems, maximal element theorems, and existence theorems of solutions to generalized equilibrium problems are proved in pseudo H-spaces. The results represented in this paper unify and extend some corresponding known results in the literature.

Key Words and Phrases: Pseudo H-space, W-GPH-KKM mapping, coincidence theorem, maximal element, generalized equilibrium problem, fixed point.
2010 Mathematics Subject Classification: 47H04, 47H10, 49J35, 49J53, 54H25.

1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz [1] first established the classical KKM theorem in the setting of finite dimensional spaces. In 1961, Fan [2] generalized the classical KKM theorem from finite dimensional spaces to infinite dimensional Hausdorff topological vector spaces and established the following FKKM theorem.

Theorem A. *Let X be a arbitrary subset of a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *the convex hull of any finite subset $\{x_0, ..., x_n\}$ of X is contained in $\bigcup_{i=0}^{n} F(x_i)$;*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Since then, a lot of generalized KKM type theorems and their applications have been studied by many authors (see, for example, [3-7] and references therein). Lin and Wan [8] proved some KKM type theorems in the framework of convex spaces. By using these results, they derived some existence theorems of solutions to generalized vector equilibrium problems under suitable assumptions. Pathak and Khan [9] introduced a new class of set-valued mappings called D-KKM mappings and proved a general D-KKM theorem in convex spaces. At the same time, they applied this theorem to getting some existence results for maximal elements, generalized variational inequalities, and price equilibria. Recently, by applying a generalized KKM theorem, Balaj and Lin [10] established some existence theorems of solutions to variational relation problems in convex spaces with applications to fixed point theorems, generalized maximal element theorems, a generalized coincidence theorem, and a section theorem. They also showed that these existence theorems of solutions to variational relation problems are equivalent to the generalized KKM theorem.

It is well known that the linear and convex assumptions play a crucial role in most known KKM theorems and applications, which strictly restricts applications of the KKM principle. In 1983, Horvath [11] gave a purely topological version of the generalized KKM theorem in which the convex assumptions were replaced by contractibility and introduced the concept of H-space. Motivated by Horvath's work, Bardaro and Ceppitelli [12] proved a generalized H-KKM theorem under much weaker conditions and applied it to establishing minimax inequalities without compact and convex hypotheses for functions taking values in ordered topological vector spaces. Later on, Verma [13] generalized H-space by introducing the concept of G-H-space and obtained an intersection theorem involving I-G-H-KKM mapping in G-H-spaces. He also applied this theorem to the theory of a new class of generalized minimax inequalities in the setting of G-H-spaces. Kalmoun and Riahi [14] proved some topological KKM theorems in the framework of G-convex spaces which do not have any linear structure and gave applications to greatest element, fixed point, and vector saddle point problems. Recently, Lin and Yao [15] introduced the concept of pseudo H-space which concludes the spaces mentioned above as special cases and obtained a general Peleg KKM theorem in pseudo H-spaces. As applications of this Peleg KKM theorem, they derived some new results for fixed point theorem and the system of variational inequalities.

Motivated and inspired by these recent works on KKM theorems, in this paper, we prove some new W-GPH-KKM theorems in pseudo H-spaces without any linear and convex structure. As applications, some new coincidence theorems, maximal element theorems, and existence theorems of solutions to generalized equilibrium problems are obtained in pseudo H-spaces.

2. Preliminaries

Let X be a set. We shall denote by 2^X the family of all subsets of X, by $\langle X\rangle$ the family of nonempty finite subsets of X. For any $A \in \langle X\rangle$, we shall denote by $|A|$ the cardinality of A. If A is a subset of a topological space X, then int A and $\overline{A}$ stand for the interior and the closure of A, respectively. If A is a subset of a vector space, we shall denote by co A the convex hull of A. Let Δ_n denote the standard n-dimensional simplex with vertices $\{e_0, e_1, ..., e_n\}$. For a nonempty subset $J \subseteq \{0, 1, ..., n\}$, let $\triangle_{|J|-1}$ denote the convex hull of the vertices $\{e_j : j \in J\}$.

Let X and Y be two nonempty sets and let $T : X \to 2^Y$ be a set-valued mapping. Then the set-valued mappings $T^{-1} : Y \to 2^X$ and $T^* : Y \to 2^X$ are respectively defined by $T^{-1}(y) = \{x \in X : y \in T(x)\}$ and $T^*(y) = X \setminus T^{-1}(y)$ for each $y \in Y$. $T^c : X \to 2^Y$ is defined by $T^c(x) = Y \setminus T(x)$ for each $x \in X$. Let $T(X_0) = \cup_{x\in X_0} T(x)$ for any $X_0 \subset X$. Note that the graph of T is the set $\{(x, y) \in X \times Y : y \in T(x)\}$. Given a nonempty set $\mathbb{Z}$ and two set-valued mappings $T : X \to 2^Y$ and $S : Y \to 2^Z$, the composition $S \circ T : X \to 2^Z$ is defined by $(S \circ T)(x) = S(T(x)) = \cup\{S(y) : y \in T(x)\}$ for each $x \in X$. Let X and Y be two topological spaces. A set-valued mapping $S : X \to 2^Y$ is called to be upper semicontinuous (resp., lower semicontinuous) at $x \in X$ if for each open set $V \subseteq Y$ with $S(x) \subseteq V$ (resp., $S(x) \cap V \neq \emptyset$), there exists an open neighborhood $U(x)$ of x such that $S(x^{'}) \subseteq V$ (resp., $S(x^{'}) \cap V \neq \emptyset$) for all $x^{'} \in U(x)$. S is said to be upper semicontinuous (resp., lower semicontinuous) on X if S is upper semicontinuous (resp., lower semicontinuous) at each point of X.

Lemma 2.1. ([16]). *Let X and Y be two topological spaces and $S : X \to 2^Y$ be a set-valued mapping. Then S is lower semicontinuous at $x \in X$ if and only if for any $y \in S(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in S(x_\alpha)$ for every α and $\{y_\alpha\}$ converges to y.*

Lemma 2.2. ([17]). *Let X and Y be two topological spaces and $S : X \to 2^Y$ be a set-valued mapping with compact values. Then S is upper semicontinuous at $x \in X$ if and only if for any net $\{x_\alpha\}$ in X converging to x and any net $\{y_\alpha\}$ with $y_\alpha \in S(x_\alpha)$, there exists $y \in S(x)$ and a subnet $\{y_\beta\} \subseteq \{y_\alpha\}$ such that $\{y_\beta\}$ converges to y.*

A subset A of a topological space X is said to be compactly closed (resp., compactly open) in X if for each nonempty compact subset $C \subseteq X$, $A \cap C$ is closed (resp., open) in C. The compact closure and the compact interior of A (see [18]) are defined by

$$\text{ccl } A = \bigcap\{B : A \subseteq B \text{ and } B \text{ is compactly closed in } X\} \text{ and}$$

$$\text{cint } A = \bigcup\{B : B \subseteq A \text{ and } B \text{ is compactly open in } X\},$$

respectively. It is easy to see that $\text{ccl}(X \setminus A) = X \setminus \text{cint}A$, ccl A (resp., cint A) is compactly closed (resp., compactly open) in X, and A is compactly closed (resp., compactly open) if and only if $A = \text{ccl } A$ (resp., $A = \text{cint } A$). For each nonempty compact subset C of X, we have $(\text{ccl } A) \cap C = \text{cl}_C\ (A \cap C)$ and $(\text{cint } A) \cap C = \text{int}_C\ (A \cap C)$, where $\text{cl}_C\ (A \cap C)$ and $\text{int}_C\ (A \cap C)$ denote the closure and the interior of $A \cap C$ in C, respectively.

Definition 2.1. ([18]). Let X be a topological space, Y be a nonempty set, and let $F : Y \to 2^X$ be a set-valued mapping. F is said to be transfer compactly closed-valued (resp., transfer compactly open-valued) on Y if for each $y \in Y$ and for each nonempty compact subset C of X, $x \notin F(y) \cap C$ (resp., $x \in F(y) \cap C$) implies that there exists $y^{'} \in Y$ such that $x \notin \text{cl}_C(F(y^{'}) \cap C)$ (resp., $x \in \text{int}_C(F(y^{'}) \cap C)$).

From the above definition, we can easily verify that F is transfer compactly closed-valued if and only if F^c is transfer compactly open-valued.

Definition 2.2. ([19]). Let X be a topological space, D be a nonempty set. A triple $(X, D; \varphi_N)$ is said to be a pseudo H-space if for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ where some elements in N may be same, there exists an upper semicontinuous set-valued mapping $\varphi_N : \Delta_n \to 2^X$ with nonempty compact values. When $D \subseteq X$, the space is denoted by $(X \supseteq D; \varphi_N)$. In case $X = D$, let $(X; \varphi_N) := (X, X; \varphi_N)$.

Let $A \subseteq D$ and $B \subseteq Y$. B is said to be a pseudo H-subspace of $(X, D; \varphi_N)$ relative to A if for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and for each $\{d_{i_0}, ..., d_{i_k}\} \subseteq A \cap \{d_0, ..., d_n\}$, we have $\varphi_N(\Delta_k) \subseteq B$, where $\Delta_k = \text{co}(\{e_{i_0}, ..., e_{i_k}\})$. We note that if A is nonempty and B is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to A, then B is automatically nonempty. When $A = B$, B is said to be a pseudo H-subspace of $(X \supseteq D; \varphi_N)$.

Remark 2.1. It is worthwhile noticing that convex subsets of topological vector spaces, Lassonde's convex spaces in [20], H-spaces introduced by Horvath [11], G-convex spaces introduced by Park and Kim [21], L-convex spaces introduced by Ben-El-Mechaiekh et al. [22], G-H-spaces introduced by Verma [13], GFC-spaces due to Khanh et al. [23], FC-spaces due to Ding [24], and many other topological spaces with abstract convex structure (see, for example, [25-26] and references therein) are all particular forms of pseudo H-spaces.

Lemma 2.3. *Let I be any index set. For each $i \in I$, let $(X_i, D_i; \varphi_{N_i})$ be a pseudo H-space. Let $X = \prod_{i\in I} X_i$, $D = \prod_{i\in I} D_i$, and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then $(X, D; \varphi_N)$ is also a pseudo H-space.*

Proof. Let X be equipped with the product topology and for each $i \in I$, let $\pi_i : X \to X_i$ be the projection of X onto X_i. For any given $N = \{d_0, ..., d_n\} \in \langle D\rangle$, let $N_i = \pi_i(N) = \{\pi_i(d_0), ..., \pi_i(d_n)\}$. Since each $(X_i, D_i; \varphi_{N_i})$ is a pseudo H-space, it follows that there exists an upper semicontinuous set-valued mapping $\varphi_{N_i} : \Delta_n \to 2^{X_i}$ with nonempty compact values. Define a set-valued mapping $\varphi_N : \Delta_n \to 2^X$ by

$$\varphi_N(z) = \prod_{i\in I} \varphi_{N_i}(z) \text{ for each } z \in \Delta_n.$$

By Lemma 3 of Fan [27], φ_N is an upper semicontinuous set-valued mapping with nonempty compact values. Therefore, $(X, D; \varphi_N)$ is also a pseudo H-space.

Definition 2.3. Let $(X, D; \varphi_N)$ be a pseudo H-space and Y be a topological space. The class $\widetilde{\mathcal{B}}(X, D, Y)$ of better admissible mappings is defined as follows: a set-valued mapping $T : X \to 2^Y$ belongs to $\widetilde{\mathcal{B}}(X, D, Y)$ if and only if T is upper semicontinuous mapping with compact values such that for any $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and for any continuous mapping $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point. When $X = D$, we shall write $\widetilde{\mathcal{B}}(X, Y)$ instead of $\widetilde{\mathcal{B}}(X, D, Y)$.

The class $\widetilde{\mathcal{B}}(X, D, Y)$ unifies and extends many important classes of mappings, for example, the class $\mathcal{U}_C^K(X, Y)$ of Park [21], the class $\mathcal{A}(Y, X)$ of Ben-El-Mechaiekh et al. [22] and the class $\mathcal{B}(Y, X)$ of Ding [24].

Definition 2.4. Let $(X, D; \varphi_N)$ be a pseudo H-space and Y be a nonempty set. Let $S : D \to 2^Y$ and $T : X \to 2^Y$ be two set-valued mappings. S is said to be a W-GPH-KKM mapping with respect to T if for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and each $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$, we have $T(\varphi_N(\Delta_k)) \subseteq \cup_{j=0}^k S(d_{i_j})$.

Remark 2.2. Definition 2.4 generalizes the generalized R-KKM mapping of Verma [13] and the R-KKM mapping of Deng and Xia [28] to pseudo H-spaces.

3. W-GPH-KKM theorems

In this section, we shall establish some new W-GPH-KKM theorems in pseudo H-spaces, which are needed in the sequel.

Theorem 3.1. *Let $(X, D; \varphi_N)$ be a pseudo H-space, Y be a Hausdorff topological space. Let $S : D \to 2^Y$, $R : D \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, D, Y)$ be three set-valued mappings such that*

(i) *for each* $d \in D$*,* $S(d)$ *is compactly closed;*

(ii) S *is a W-GPH-KKM mapping with respect to* T*;*

(iii) *there exists a nonempty compact subset* K *of* Y *such that either*

(iii$_1$) $\bigcap_{d\in N_0} S(d) \subseteq K$ *for some* $N_0 \in \langle D\rangle$*; or*

(iii$_2$) *for each* $N \in \langle D\rangle$*, there exists a subset* L_N *of* D *containing* N *such that* $R(L_N)$ *is a compact pseudo H-subspace of* $(X, D; \varphi_N)$ *relative to* L_N *and*

$$(T \circ R)(L_N) \cap (\bigcap_{d\in L_N} S(d)) \subseteq K.$$

Then $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d)) \neq \emptyset$*.*

Proof. We prove Theorem 3.1 in the following two cases:

Case (iii$_1$). Suppose that the conclusion of Theorem 3.1 does not hold. Then we have $\overline{T(X)} \cap K \subseteq \bigcup_{d\in D} S^c(d)$ and thus, $\overline{T(X)} \cap K = \bigcup_{d\in D}(S^c(d) \cap \overline{T(X)} \cap K)$. Since $\overline{T(X)} \cap K$ is compact and each $S^c(d)$ is compactly open, it follows that there exists $N_1 \in \langle D\rangle$ such that

$$\overline{T(X)} \cap K = \bigcup_{d\in N_1} (S^c(d) \cap \overline{T(X)} \cap K) \subseteq \bigcup_{d\in N_1} S^c(d). \tag{3.1}$$

By (iii$_1$), we have

$$\overline{T(X)} \setminus K \subseteq Y \setminus K \subseteq \bigcup_{d\in N_0} S^c(d) \text{ for some } N_0 \in \langle D\rangle. \tag{3.2}$$

Then it follows from (3.1) and (3.2) that

$$\overline{T(X)} = (\overline{T(X)} \setminus K) \cup (\overline{T(X)} \cap K) \subseteq \bigcup_{d\in N} S^c(d), \tag{3.3}$$

where $N = N_0 \cup N_1 = \{d_0, ..., d_n\} \in \langle D\rangle$. By the definition of a pseudo H-space, there exists an upper semicontinuous set-valued mapping $\varphi_N : \Delta_n \to 2^X$ with nonempty compact values. By Proposition 3.1.11 of Aubin and Ekeland [29], we know that $\varphi_N(\Delta_n)$ is compact subset of X. Since T is an upper semicontinuous set-valued mapping with compact values, it follows from Proposition 3.1.11 of Aubin and Ekeland [29] again that $T(\varphi_N(\Delta_n))$ is compact subset of Y. By (3.3), we have

$$T(\varphi_N(\Delta_n)) \subseteq \overline{T(X)} \subseteq \bigcup_{d\in N} S^c(d),$$

and thus, $T(\varphi_N(\Delta_n)) = \bigcup_{d\in N}(S^c(d) \cap T(\varphi_N(\Delta_n)))$, i.e., $\{S^c(d) \cap T(\varphi_N(\Delta_n)) : d \in N\}$ is an open cover of the compact set $T(\varphi_N(\Delta_n))$. Let $\{\lambda_i\}_{i=0}^n$ be the partition of unity subordinated to this cover. Define a continuous mapping $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$ by $\psi(y) = \sum_{i=0}^n \lambda_i(y)e_i$ for each $y \in T(\varphi_N(\Delta_n))$. Since $T \in \widetilde{\mathcal{B}}(X, D, Y)$, the composition mapping $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N$ has a fixed point $z_0 \in \Delta_n$; that is, $z_0 \in \psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N(z_0)$. Let $\overline{y} \in T(\varphi_N(z_0))$ such that $z_0 = \psi(\overline{y})$. Then we have

$$z_0 = \psi(\overline{y}) = \sum_{j\in J(\overline{y})} \lambda_j(\overline{y})e_j \in \Delta_{|J(\overline{y})|-1},$$

where $J(\overline{y})=\{j\in\{0,...,n\}:\lambda_j(\overline{y})\neq 0\}$. By (ii), we have the following:

$$\begin{aligned}\overline{y} &\in T(\varphi_N(\psi(\overline{y})))\\ &\subseteq T(\varphi_N(\Delta_{|J(\overline{y})|-1}))\\ &\subseteq \bigcup_{j\in J(\overline{y})} S(d_j).\end{aligned}$$

Hence, there exists $j\in J(\overline{y})$ such that $\overline{y}\in S(d_j)$. On the other hand, by the definitions of $J(\overline{y})$ and the partition $\{\lambda_i\}_{i=0}^n$, we have

$$\begin{aligned}\overline{y} &\in S^c(d_j)\cap T(\varphi_N(\Delta_n))\\ &= (Y\setminus S(d_j))\cap T(\varphi_N(\Delta_n))\\ &\subseteq Y\setminus S(d_j),\end{aligned}$$

which is a contradiction. Therefore, the conclusion of Theorem 3.1 holds.

Case (iii$_2$). Suppose that the conclusion of Theorem 3.1 is false. Then we have

$$\begin{aligned}\emptyset &= \overline{T(X)}\cap K\cap(\bigcap_{d\in D} S(d))\\ &= \bigcap_{d\in D}(\overline{T(X)}\cap K\cap S(d)).\end{aligned}$$

By (i), $\{\overline{T(X)}\cap K\cap S(d):d\in D\}$ is a family of closed sets in K. Then there exists $N\in\langle D\rangle$ such that

$$\begin{aligned}\emptyset &= \bigcap_{d\in N}(\overline{T(X)}\cap K\cap S(d))\\ &= \overline{T(X)}\cap K\cap(\bigcap_{d\in N} S(d)),\end{aligned}$$

that is, $\overline{T(X)}\cap(\bigcap_{d\in N}S(d))\subseteq Y\setminus K$. By (iv$_2$), there is a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X,D;\varphi_N)$ relative to L_N and

$$(T\circ R)(L_N)\cap(\bigcap_{d\in L_N} S(d))\subseteq K.$$

Since $(T\circ R)(L_N)\cap(\bigcap_{d\in L_N}S(d))\subseteq\overline{T(X)}\cap(\bigcap_{d\in N}S(d))\subseteq Y\setminus K$, it follows that $(T\circ R)(L_N)\cap(\bigcap_{d\in L_N}S(d))=\emptyset$. Therefore, we have

$$(T\circ R)(L_N)\subseteq\bigcup_{d\in L_N}S^c(d). \tag{3.4}$$

Since T is upper semicontinuous with compact values and $R(L_N)$ is compact, it follows from Proposition 3.1.11 of Aubin and Ekeland [29] that $(T\circ R)(L_N)$ is compact subset of Y. Thus, by (3.4), there exists $M=\{d_0,...,d_m\}\in\langle L_N\rangle$ such that

$$(T\circ R)(L_N)=\bigcup_{d\in M}((T\circ R)(L_N)\cap S^c(d)).$$

Observe that $R(L_N)$ is a pseudo H-subspace of $(X,D;\varphi_N)$ relative to L_N and so $(R(L_N),L_N;\varphi_N)$ is also a pseudo H-space. Then there exists an upper semicontinuous set-valued mapping $\varphi_M : \Delta_m \to 2^{R(L_N)}$ with nonempty compact values. We may assume that $\{\lambda_i\}_{i=0}^m$ is the partition of unity subordinated to the open cover $\{(T \circ R)(L_N) \cap S^c(d) : d \in M\}$, which implies that

$$\begin{cases} \text{for each } i \in \{0, 1, ..., m\},\ \lambda_i : (T \circ R)(L_N) \to [0, 1] \text{ is continuous;} \\ \text{for each } i \in \{0, 1, ..., m\}, \lambda_i(y) > 0 \Rightarrow y \in (T \circ R)(L_N) \cap S^c(d); \\ \sum_{i=0}^m \lambda_i(y) = 1 \ \text{ for each } y \in (T \circ R)(L_N). \end{cases}$$

Furthermore, we define a continuous mapping $\psi : (T \circ R)(L_N) \to \Delta_m$ by $\psi(y) = \sum_{i=0}^m \lambda_i(y)e_i$ for each $y \in (T \circ R)(L_N)$. Let $T^{'} := T|_{R(L_N)}$. Since $T \in \widetilde{\mathcal{B}}(X, D, Y)$, it obviously follows that $T^{'} \in \widetilde{\mathcal{B}}(R(L_N), L_N, Y)$. Then the composition $\psi \circ T^{'}|_{\varphi_M(\Delta_m)} \circ \varphi_M : \Delta_m \to 2^{\Delta_m}$ has a fixed point $z_0 \in \Delta_m$; that is, $z_0 \in \psi \circ T^{'}|_{\varphi_M(\Delta_m)} \circ \varphi_M(z_0)$. Let $\overline{y} \in T^{'}(\varphi_M(z_0))$ such that $z_0 = \psi(\overline{y})$. Then we have

$$z_0 = \psi(\overline{y}) = \sum_{j \in J(\overline{y})} \lambda_j(\overline{y})e_j \in \Delta_{|J(\overline{y})|-1},$$

where $J(\overline{y}) = \{j \in \{0, ..., m\} : \lambda_j(\overline{y}) \neq 0\}$. Let $S^{'} := S|_{L_N}$. By (ii), it is easy to see that $S^{'}$ is a W-GPH-KKM mapping with respect to $T^{'}$. Thus, we have the following:

$$\begin{aligned} \overline{y} &\in T^{'}(\varphi_M(\psi(\overline{y}))) \\ &\subseteq T^{'}(\varphi_M(\Delta_{|J(\overline{y})|-1})) \\ &\subseteq \bigcup_{j \in J(\overline{y})} S^{'}(d_j). \end{aligned}$$

Hence, there exists $j \in J(\overline{y})$ such that $\overline{y} \in S^{'}(d_j)$. On the other hand, by the definitions of $J(\overline{y})$ and the partition $\{\lambda_i\}_{i=0}^m$, we have

$$\begin{aligned} \overline{y} &\in S^c(d_j) \cap (T \circ R)(L_N) \\ &= (Y \setminus S(d_j)) \cap (T \circ R)(L_N) \\ &\subseteq Y \setminus S(d_j) \\ &= Y \setminus S^{'}(d_j), \end{aligned}$$

which is a contradiction. Therefore, the conclusion of Theorem 3.1 holds.

Remark 3.1. (1) Theorem 3.1 generalizes Theorem 3.3 of Yang and Deng [30] in the following ways: (a) from FC-spaces to pseudo H-spaces; (b) from two set-valued mappings to three set-valued mappings; (c) from $\mathcal{U}_C^K(X, Y)$ to $\widetilde{\mathcal{B}}(X, D, Y)$; in turn, it also generalizes Theorem 3 of Park and Kim [21] in several aspects.

(2) (ii) of Theorem 3.1 can be replaced by the following equivalent condition:

(ii)$^{'}$ for each $y \in Y$, $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to $S^*(y)$.

Proof. (ii)$\Rightarrow$(ii)$^{'}$: Let $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and $\{d_{i_0}, ..., d_{i_k}\} \subseteq N \cap S^*(y)$. Then $\{d_{i_0}, ..., d_{i_k}\} \subseteq D \setminus S^{-1}(y)$. Thus, $y \notin \bigcup_{j=0}^k S(d_{i_j})$. Since $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^k S(d_{i_j})$, it follows that $y \notin T(\varphi_N(\Delta_k))$; that is, $T^{-1}(y) \cap \varphi_N(\Delta_k) = \emptyset$. Therefore, we have $\varphi_N(\Delta_k) \subseteq X \setminus T^{-1}(y) = T^*(y)$.

(ii)$'$⇒(ii): Let $N = \{d_0, ..., d_n\} \in \langle D\rangle$, $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$, and $y \in T(\varphi_N(\Delta_k))$. Then there exists $x \in \varphi_N(\Delta_k)$ such that $y \in T(x)$. So, we have

$$x \in T^{-1}(y) \cap \varphi_N(\Delta_k) = (Y \setminus T^*(y)) \cap \varphi_N(\Delta_k) \neq \emptyset.$$

This means that $\varphi_N(\Delta_k) \not\subseteq T^*(y)$. By (ii)$'$, we have $\{d_{i_0}, ..., d_{i_k}\} \not\subseteq N \cap S^*(y)$ and hence, $\{d_{i_0}, ..., d_{i_k}\} \cap (D \setminus S^*(y)) \neq \emptyset$, i.e., $\{d_{i_0}, ..., d_{i_k}\} \cap S^{-1}(y) \neq \emptyset$. Let $d \in \{d_{i_0}, ..., d_{i_k}\} \cap S^{-1}(y)$. Then $y \in S(d) \subseteq \bigcup_{j=0}^k S(d_{i_j})$, which implies that $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^k S(d_{i_j})$.

Theorem 3.2. *Let $(X, D; \varphi_N)$ be a pseudo H-space, Y be a Hausdorff topological space. Let $S : D \to 2^Y$, $R : D \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, D, Y)$ be three set-valued mappings such that*

(i) *S is transfer compactly closed-valued;*

(ii) *for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and each $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$, $T(\varphi_N(\Delta_k)) \subseteq \cup_{j=0}^k \mathrm{ccl}\ S(d_{i_j})$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii$_1$) *$\bigcap_{d\in N_0} \mathrm{ccl}\ S(d) \subseteq K$ for some $N_0 \in \langle D\rangle$; or*

(iii$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap (\bigcap_{d\in L_N} \mathrm{ccl}\ S(d)) \subseteq K.$$

Then $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d)) \neq \emptyset$.

Proof. Define a set-valued mapping $\widetilde{S} : D \to 2^Y$ by $\widetilde{S}(d) = \mathrm{ccl}\ S(d)$ for each $d \in D$. Then for each $d \in D$, $\widetilde{S}(d)$ is compactly closed. By using the same argument as that used in the proof of Theorem 3.1, we can easily prove that $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} \mathrm{ccl}\ S(d)) \neq \emptyset$. So, it suffices to prove $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} \mathrm{ccl}\ S(d)) = \overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d))$. Clearly, $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d)) \subseteq \overline{T(X)} \cap K \cap (\bigcap_{d\in D} \mathrm{ccl}\ S(d))$. If $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} \mathrm{ccl}\ S(d)) \not\subseteq \overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d))$, then there exists $y \in \overline{T(X)} \cap K \cap (\bigcap_{d\in D} \mathrm{ccl}\ S(d))$ such that $y \notin \overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d))$. Since S is transfer compactly closed-valued, there exists a $d' \in D$ such that $y \notin \mathrm{cl}_{\overline{T(X)}\cap K}((S(d') \cap \overline{T(X)} \cap K))$, which is a contradiction. Hence, $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d)) \neq \emptyset$, This completes the proof.

Remark 3.2. (1) Theorem 3.2 generalizes and improves Theorem 2.1 of Kalmoun and Riahi [14] in the following ways: (a) from G-convex spaces to pseudo H-spaces; (b) D needs not to be a subset of X; (c) from $\mathcal{U}_C^K(X, Y)$ to $\widetilde{\mathcal{B}}(X, D, Y)$; (d) from TG-KKM mappings to W-GPH-KKM mappings; (e) (iii$_1$) and (iii$_2$) are weaker than (i) and (ii) of Theorem 2.1 of Kalmoun and Riahi [14], respectively. Theorem 3.2 also generalizes Theorem 3.2 of Fang and Huang [31] in several aspects.

(2) We have shown that Theorem 3.1 implies Theorem 3.2. It is obvious that Theorem 3.2 implies Theorem 3.1. Therefore, Theorem 3.1 is equivalent to Theorem 3.2.

4. Coincidence and maximal element theorems

Theorem 4.1. *Let $(X, D; \varphi_N)$ be a pseudo H-space and Y be a Hausdorff topological space. Let $G,\ H : Y \to 2^D$, $R : D \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, D, Y)$ be four set-valued mappings such that*

(i) *for each $d \in D$, $H^{-1}(d)$ is compactly open;*

(ii) *for each $x \in X$, $y \in T(x)$, $R(G(y))$ is a pseudo H-space of $(X, D; \varphi_N)$ relative to $H(y)$;*

(iii) *there exists a nonempty compact subset K of Y such that $\overline{T(X)} \cap K \subseteq H^{-1}(D)$;*

(iv) *either*

(iv$_1$) *$\bigcap_{d\in N_0}(Y \setminus H^{-1}(d)) \subseteq K$ for some $N_0 \in \langle D\rangle$; or*

(iv$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap \Big(\bigcap_{d\in L_N} (Y \setminus H^{-1}(d))\Big) \subseteq K.$$

Then there exists $(\overline{x}, \overline{d}, \overline{y}) \in X \times D \times Y$ such that $\overline{x} \in R(\overline{d})$, $\overline{d} \in G(\overline{y})$, $\overline{y} \in T(\overline{x})$.

Proof. Define a set-valued mapping $S : D \to 2^Y$ by $S(d) = Y \setminus H^{-1}(d)$ for each $d \in D$. By (i), each $S(d)$ is compactly closed. Thus, (i) of Theorem 3.1 is satisfied. It is easy to see that (iv$_1$) and (iv$_2$) imply that (iii$_1$) and (iii$_2$) of Theorem 3.1 hold, respectively. By (iii), we have

$$\overline{T(X)} \cap K \cap \Big(\bigcap_{d\in D} S(d)\Big) \subseteq H^{-1}(D) \cap \Big(\bigcap_{d\in D} S(d)\Big) = \emptyset.$$

Hence, by Theorem 3.1, (ii) of this theorem must be violated; that is, S is not W-GPH-KKM mapping with respect to T, which implies that there exist $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$ such that

$$T(\varphi_N(\Delta_k)) \not\subseteq \bigcup_{j=0}^{k} (Y \setminus H^{-1}(d_{i_j})).$$

Therefore, we can choose some point $\overline{x} \in \varphi_N(\Delta_k)$ and $\overline{y} \in T(\overline{x})$ such that $\overline{y} \notin Y \setminus H^{-1}(d_{i_j})$ for all $j = 0, ..., k$; that is, $\overline{y} \in H^{-1}(d_{i_j})$ for all $j = 0, ..., k$. By (ii), we have

$$\overline{x} \in \varphi_N(\Delta_k) \subseteq R(G(\overline{y})).$$

Hence, there exists $\overline{d} \in G(\overline{y})$ such that $\overline{x} \in R(\overline{d})$. This completes the proof.

When $X = D$ and R is the identity mapping on X, we can obtain the following theorem.

Theorem 4.2. *Let $(X; \varphi_N)$ be a pseudo H-space and Y be a Hausdorff topological space. Let $G,\ H : Y \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, Y)$ be three set-valued mappings such that*

(i) *for each $x \in X$, $H^{-1}(x)$ is compactly open;*

(ii) *for each $x \in X$, $y \in T(x)$, $G(y)$ is a pseudo H-space of $(X; \varphi_N)$ relative to $H(y)$;*

(iii) *there exists a nonempty compact subset K of Y such that $\overline{T(X)} \cap K \subseteq H^{-1}(X)$;*

(iv) *either*

(iv$_1$) *$\bigcap_{x\in N_0}(Y \setminus H^{-1}(x)) \subseteq K$ for some $N_0 \in \langle X\rangle$; or* (iv$_2$) *for each* $N \in \langle X\rangle$, *there exists a compact pseudo H-subspace* L_N *of* $(X;\varphi_N)$ *containing* N *such that*

$$T(L_N)\cap\Big(\bigcap_{x\in L_N}(Y\setminus H^{-1}(x))\Big)\subseteq K.$$

Then there exists $(\overline{x},\overline{y})\in X\times Y$ *such that* $\overline{x}\in G(\overline{y})$ *and* $\overline{y}\in T(\overline{x})$.

Remark 4.1. Theorem 4.2 generalizes Theorem 1 of Browder [32], Theorem 1 of Tarafdar [33], Theorem 2.3 of Mehta and Sessa [34], and Theorem 4.3 of Balaj and Lin [10] from topological vector spaces to pseudo H-spaces without any linear and convex structure. By setting $F(x)=H^{-1}(x)$ for each $x\in X$, Theorem 4.2 can be restated with the conclusion that there exists a point $(\overline{x},\overline{y})\in X\times Y$ such that $\overline{y}\in T(\overline{x})\cap F(\overline{x})$. Thus, Theorem 4.2 generalizes Theorem 3.1 of Yang and Deng [30] in several aspects: (a) from FC-spaces to pseudo H-spaces; (b) from $\mathcal{U}_C^K(X,Y)$ to $\widetilde{\mathcal{B}}(X,Y)$; (c) from two set-valued mappings to three set-valued mappings.

Theorem 4.3. *Let* $(X,D;\varphi_N)$ *be a pseudo H-space and* Y *be a Hausdorff topological space. Let* $G,\ H:Y\to 2^D$, $R:D\to 2^X$, *and* $T\in\widetilde{\mathcal{B}}(X,D,Y)$ *be four set-valued mappings such that*

(i) *for each* $d\in D$, $H^{-1}(d)$ *is compactly open;*

(ii) *for each* $x\in X$, $y\in T(x)$, $R(G(y))$ *is a pseudo H-space of* $(X,D;\varphi_N)$ *relative to* $(\text{cint } H^{-1})^{-1}(y)$*;*

(iii) *there exists a nonempty compact subset* K *of* Y *such that* $\overline{T(X)}\cap K\subseteq \text{cint } H^{-1}(D)$*;*

(iv) *either*

(iv$_1$) $\bigcap_{d\in N_0}(Y\setminus \text{cint } H^{-1}(d))\subseteq K$ *for some* $N_0\in\langle D\rangle$*; or*

(iv$_2$) *for each* $N\in\langle D\rangle$, *there exists a subset* L_N *of* D *containing* N *such that* $R(L_N)$ *is a compact pseudo H-subspace of* $(X,D;\varphi_N)$ *relative to* L_N *and*

$$(T\circ R)(L_N)\cap\Big(\bigcap_{d\in L_N}(Y\setminus \text{cint } H^{-1}(d))\Big)\subseteq K.$$

Then there exists $(\overline{x},\overline{d},\overline{y})\in X\times D\times Y$ *such that* $\overline{x}\in R(\overline{d})$, $\overline{d}\in G(\overline{y})$, $\overline{y}\in T(\overline{x})$.

Proof. Define a set-valued mapping $\widetilde{H}:Y\to 2^D$ by $\widetilde{H}(y)=(\text{cint } H^{-1})^{-1}(y)$ for each $y\in Y$. Then by (i), we have $\widetilde{H}^{-1}(d)=\text{cint } H^{-1}(d)=H^{-1}(d)$ for each $d\in D$, which is compactly open. Hence, (i) of Theorem 4.1 is satisfied. By (ii), for each $x\in X$, $y\in T(x)$, $R(G(y))$ is a pseudo H-space of $(X,D;\varphi_N)$ relative to $\widetilde{H}(y)$. By (iii), there exists a nonempty compact subset K of Y such that $\overline{T(X)}\cap K\subseteq \widetilde{H}^{-1}(D)$. Suppose that (iv$_1$) holds. Then by (iv$_1$), we have $\bigcap_{d\in N_0}(Y\setminus \widetilde{H}^{-1}(d))\subseteq K$ for some $N_0\in\langle D\rangle$. If (iv$_2$) is satisfied, then by (iv$_2$), for each $N\in\langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X,D;\varphi_N)$ relative to L_N and

$$(T\circ R)(L_N)\cap\Big(\bigcap_{d\in L_N}(Y\setminus \widetilde{H}^{-1}(d))\Big)\subseteq K.$$

Thus, all the requirements of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1, there exists $(\overline{x},\overline{d},\overline{y})\in X\times D\times Y$ such that $\overline{x}\in R(\overline{d})$, $\overline{d}\in G(\overline{y})$, $\overline{y}\in T(\overline{x})$.

Remark 4.2. (1) We have shown that Theorem 4.1 implies Theorem 4.3. It is obvious that Theorem 4.3 implies Theorem 4.1. Therefore, Theorem 4.1 is equivalent to Theorem 4.3.

(2) Theorem 4.3 generalizes Theorem 3.1 of Fang and Huang [31] in the following aspects: (a) from FC-spaces to pseudo H-spaces; (b) from $\mathcal{B}(Y,X)$ to $\widetilde{\mathcal{B}}(X,D,Y)$; (c) from two set-valued mappings to four set-valued mappings; (d) (iii) and (iv$_2$) are weaker than (ii) and (iii) of Theorem 3.1 of Fang and Huang [31], respectively.

Theorem 4.4. *Let $(X,D;\varphi_N)$ be a pseudo H-space, Y be a Hausdorff topological space, and $K\subseteq Y$ be a nonempty and compact set. Let $H:Y\to 2^D$, $R:D\to 2^X$, and $T\in\widetilde{\mathcal{B}}(X,D,Y)$ be three set-valued mappings such that*

(i) *for each $d\in D$, $H^{-1}(d)$ is compactly open;*

(ii) *for each $N=\{d_0,...,d_n\}\in\langle D\rangle$ and each $\{d_{i_0},...,d_{i_k}\}\subseteq N$,*

$$T(\varphi_N(\Delta_k))\cap(\bigcap_{j=0}^{k}H^{-1}(d_{i_j}))=\emptyset;$$

(iii) *one of the following conditions holds:*

(iii$_1$) *there exists $N_0\in\langle D\rangle$ such that $Y\setminus K\subseteq\bigcup_{d\in N_0}H^{-1}(d)$;*

(iii$_2$) *for each $N\in\langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X,D;\varphi_N)$ relative to L_N and*

$$(T\circ R)(L_N)\setminus K\subseteq\bigcup_{d\in L_N}H^{-1}(d).$$

Then there exists a point $\overline{y}\in\overline{T(X)}\cap K$ such that $H(\overline{y})=\emptyset$.

Proof. Define a set-valued mapping $S:D\to 2^Y$ by $S(d)=Y\setminus H^{-1}(d)$ for each $d\in D$. By (ii), we have $T(\varphi_N(\Delta_k))\subseteq\bigcup_{j=0}^{k}S(d_{i_j})$ for each $N=\{d_0,...,d_n\}\in\langle D\rangle$ and each $\{d_{i_0},...,d_{i_k}\}\subseteq N$, which implies that S is a W-GPH-KKM mapping with respect to T. It follows from (iii$_1$) that

$$\begin{aligned}\bigcap_{d\in N_0}S(d)&=\bigcap_{d\in N_0}(Y\setminus H^{-1}(d))\\&=Y\setminus\bigcup_{d\in N_0}H^{-1}(d)\\&\subseteq Y\setminus(Y\setminus K)=K.\end{aligned}$$

By (iii$_2$), we have, for each $N\in\langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X,D;\varphi_N)$ relative to L_N and

$$\begin{aligned}(T\circ R)(L_N)\cap(\bigcap_{d\in L_N}S(d))&=(T\circ R)(L_N)\cap(Y\setminus\bigcup_{d\in L_N}H^{-1}(d))\\&\subseteq(T\circ R)(L_N)\cap(Y\setminus((T\circ R)(L_N)\setminus K))\\&\subseteq K.\end{aligned}$$

It is clear that (i) implies that (i) of Theorem 3.1 holds. Therefore, the set-valued mappings S, R, and T satisfy all the requirements of Theorem 3.1, and hence, by

Theorem 3.1, we have

$$\begin{aligned}\emptyset &\neq \overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d))\\ &= \overline{T(X)} \cap K \cap (Y \setminus \bigcup_{d\in D} H^{-1}(d)).\end{aligned}$$

Then there exists $\overline{y} \in \overline{T(X)} \cap K$ such that $\overline{y} \notin H^{-1}(d)$ for each $d \in D$; that is, $d \notin H(\overline{y})$ for each $d \in D$, which implies that $H(\overline{y}) = \emptyset$. This completes the proof.

By using Theorem 3.2 and the similar argument as in the proof of Theorem 4.4, we can obtain the following maximal element theorem. We omit the proof.

Theorem 4.5. *Let $(X, D; \varphi_N)$ be a pseudo H-space, Y be a Hausdorff topological space, and $K \subseteq Y$ be nonempty and compact. Let $H : Y \to 2^D$, $R : D \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, D, Y)$ be three set-valued mappings such that*

(i) *$H^{-1} : D \to 2^Y$ is transfer compactly open-valued;*

(ii) *for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and each $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$,*

$$T(\varphi_N(\Delta_k)) \cap (\bigcap_{j=0}^{k} \text{cint } H^{-1}(d_{i_j})) = \emptyset;$$

(iii) *one of the following conditions holds:*

(iii$_1$) *there exists $N_0 \in \langle D\rangle$* such that *$Y \setminus K \subseteq \bigcup_{d\in N_0} \text{cint } H^{-1}(d)$;*

(iii$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \setminus K \subseteq \bigcup_{d\in L_N} \text{cint } H^{-1}(d).$$

Then there exists a point $\overline{y} \in \overline{T(X)} \cap K$ such that $H(\overline{y}) = \emptyset$.

Remark 4.3. (1) Even $X = D$ and R is the identity mapping on X, Theorem 4.5 generalizes Theorem 2.2 of Ding [24] from FC-spaces to pseudo H-spaces; in turn, Theorem 4.5 also generalizes Theorem 2.1 of Shen [35], Theorem 2 of Tulcea [36], and Theorem 5.1 of Yannelis and Prabhakar [37] in several aspects.

(2) Theorem 4.4 is equivalent to Theorem 4.5. Firstly, we show that Theorem 4.5 implies Theorem 4.4. Suppose that all the conditions of Theorem 4.4 are satisfied. Since H^{-1} is compactly open-valued, it follows that H^{-1} is transfer compactly open-valued and $H^{-1}(y) = \text{cint } H^{-1}(y)$ for each $y \in Y$. It is easy to check that the other conditions of Theorem 4.5 hold and hence, by Theorem 4.5, there exists a point $\overline{y} \in \overline{T(X)} \cap K$ such that $H(\overline{y}) = \emptyset$. Secondly, we prove that Theorem 4.4 implies Theorem 4.5. Suppose that all the conditions of Theorem 4.5 are fulfilled. Define a set-valued mapping $\widetilde{H} : Y \to 2^D$ by $\widetilde{H}(y) = (\text{cint } H^{-1})^{-1}(y)$ for each $y \in Y$. Then $\widetilde{H}^{-1}(d) = \text{cint } H^{-1}(d)$ for each $d \in D$ and so, each $\widetilde{H}^{-1}(d)$ is compactly open. It is easy to see that the other conditions of Theorem 4.4 are satisfied. Hence, by Theorem 4.4, there exists a point $\overline{y} \in \overline{T(X)} \cap K$ such that $\widetilde{H}(\overline{y}) = (\text{cint } H^{-1})^{-1}(\overline{y}) = \emptyset$. Now, we prove that $H(\overline{y}) = \emptyset$. Suppose the contrary. Then $H(\overline{y}) \neq \emptyset$. Taking $\overline{d} \in H(\overline{y})$ leads to $\overline{y} \in H^{-1}(d) \cap K$. Since H^{-1} is transfer compactly open-valued, there exists

$d^{'} \in D$ such that

$$\overline{y} \in \text{cint } H^{-1}(d^{'}) \cap K \subseteq \text{cint } H^{-1}(d^{'});$$

that is, $d^{'} \in (\text{cint } H^{-1})^{-1}(\overline{y})$, which contradicts $(\text{cint } H^{-1})^{-1}(\overline{y}) = \emptyset$. Therefore, we have $H(\overline{y}) = \emptyset$.

Theorem 4.6. *Let I be any index set, Y be a Hausdorff topological space, and $K \subseteq Y$ be nonempty and compact. For each $i \in I$, let $(X_i, D_i; \varphi_{N_i})$ be a pseudo H-space. Let $X = \prod_{i\in I} X_i$, $D = \prod_{i\in I} D_i$, and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$ such that $(X, D; \varphi_N)$ is a pseudo H-space defined as in Lemma 2.3. Let $R : D \to 2^X$, $T \in \widetilde{\mathcal{B}}(X, D, Y)$, and for each $i \in I$, $H_i : Y \to 2^{D_i}$ be a set-valued mapping such that*

(i) *$H_i^{-1} : D_i \to 2^Y$ is transfer compactly open-valued;*

(ii) *for each $y \in Y$, $I(y) = \{i \in I : H_i(y) \neq \emptyset\}$ is finite;*

(iii) *for each $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and each $\{d_{m_0}, ..., d_{m_k}\} \subseteq N$,*

$$T(\varphi_N(\Delta_k)) \cap (\bigcap_{j=0}^{k} \text{cint } H_i^{-1}(\pi_i(d_{m_j}))) = \emptyset;$$

(iv) *one of the following conditions holds:*

(iv$_1$) *there exists $N_0 \in \langle D\rangle$ such that for each $y \in Y \setminus K$, there exists $\widetilde{d} \in N_0$ such that for each $i \in I(y)$, $y \in \text{cint } H_i^{-1}(\pi_i(\widetilde{d}))$;*

(iv$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and for each $y \in (T \circ R)(L_N) \setminus K$, there exists $\widetilde{d} \in L_N$ such that for each $i \in I(y)$, $y \in \text{cint } H_i^{-1}(\pi_i(\widetilde{d}))$.*

Then there exists $\overline{y} \in \overline{T(X)} \cap K$ such that $H_i(\overline{y}) = \emptyset$ for each $i \in I$.

Proof. Define a set-valued mapping $H : Y \to 2^D$ by

$$H(y) = \begin{cases} \bigcap_{i\in I(y)} H_i^{'}(y), & \text{if } I(y) \neq \emptyset, \\ \emptyset, & \text{if } I(y) = \emptyset, \end{cases}$$

where $H_i^{'}(y) = \pi_i^{-1}(H_i(y))$ for each $y \in Y$. Then $I(y) \neq \emptyset$ if and only if $H(y) \neq \emptyset$. We shall show that $H^{-1} : D \to 2^Y$ is transfer compactly open-valued. Indeed, for each $d \in D$ and for each nonempty compact subset C of Y, if $y \in H^{-1}(d) \cap C$, then for each $i \in I(y)$, $y \in H_i^{-1}(\pi_i(d)) \cap C$. Since each H_i^{-1} is transfer compactly open-valued, there exists $\overline{d}_i \in D_i$ such that

$$y \in \text{int}_C\ (H_i^{-1}(\overline{d}_i) \cap C) \text{ for each } i \in I(y). \tag{4.1}$$

For each $i \in I(y)$, let $\overline{d} = (d^i, \overline{d}_i)$, where $d^i \in D^i = \prod_{j\neq i} D_j$ is a fixed element. Now, we have

$$\begin{aligned} y \in H_i^{'-1}(\overline{d}) \quad &\Leftrightarrow \quad \overline{d} \in H_i^{'}(y) = \pi_i^{-1}(H_i(y)) \\ &\Leftrightarrow \quad \overline{d}_i \in H_i(y) \text{ and } d^i \in D^i \\ &\Leftrightarrow \quad y \in H_i^{-1}(\overline{d}_i) \text{ and } d^i \in D^i. \end{aligned}$$

This shows that $H_i^{'-1}(\overline{d}) = H_i^{-1}(\overline{d}_i)$ for each fixed $d^i \in D^i$. Therefore, combining (4.1), we have

$$y \in \bigcap_{i\in I(y)} \text{int}_C\ (H_i^{'-1}(\overline{d}) \cap C). \tag{4.2}$$

By (ii), $I(y)$ is finite and hence, by (4.2), we have

$$y \in \text{int}_C \ (\bigcap_{i \in I(y)} H_i^{'-1}(\overline{d}) \cap C) = \text{int}_C \ (H^{-1}(\overline{d}) \cap C).$$

Hence, H^{-1} is transfer compactly open-valued and so (i) of Theorem 4.5 holds.

Let $y \in Y$ with $H(y) \neq \emptyset$. Then there exists $\widehat{i} \in I(y)$ such that $H_{\widehat{i}}(y) \neq \emptyset$. For each $d \in D$, we have

$$\begin{aligned} H^{-1}(d) &= \{y \in Y : d \in \bigcap_{i \in I(y)} H_i^{'}(y)\} \\ &= \{y \in Y : \pi_i(d) \in H_i(y), \text{ for each } i \in I(y)\} \\ &\subseteq \{y \in Y : y \in H_{\widehat{i}}^{-1}(\pi_{\widehat{i}}(d))\} = H_{\widehat{i}}^{-1}(\pi_{\widehat{i}}(d)). \end{aligned}$$

For each $N = \{d_0, ..., d_n\} \in \langle D \rangle$ and each $\{d_{m_0}, ..., d_{m_k}\} \subseteq N$, if

$$y \in \bigcap_{j=0}^{k} \text{cint } H^{-1}(d_{m_j}),$$

then

$$y \in \bigcap_{j=0}^{k} \text{cint } H_{\widehat{i}}^{-1}(\pi_{\widehat{i}}(d_{m_j})).$$

By (iii), we have $y \notin T(\varphi_N(\Delta_k))$, which implies that

$$T(\varphi_N(\Delta_k)) \cap (\bigcap_{j=0}^{k} \text{cint } H^{-1}(d_{m_j})) = \emptyset.$$

So, (ii) of Theorem 4.5 is satisfied.

Suppose that (iv$_1$) holds. Then there exists $N_0 \in \langle D \rangle$ such that for each $y \in Y \setminus K$, there exists $\widetilde{d} \in N_0$ such that for each $i \in I(y)$, $y \in \text{cint } H_i^{-1}(\pi_i(\widetilde{d}))$. Therefore, $y \in \bigcap_{i \in I(y)} \text{cint } H_i^{-1}(\pi_i(\widetilde{d}))$. Since $H^{-1}(d) = \{y \in Y : \pi_i(d) \in H_i(y), \text{ for each } i \in I(y)\}$ for each $d \in D$ and $I(y)$ is finite, we have

$$y \in \bigcap_{i \in I(y)} \text{cint } H_i^{-1}(\pi_i(\widetilde{d})) \subseteq \text{cint } H^{-1}(\widetilde{d}),$$

which implies that $Y \setminus K \subseteq \bigcup_{d \in N_0} \text{cint } H^{-1}(d)$. Hence, (iii$_1$) of Theorem 4.5 holds. Suppose that (iv$_2$) is fulfilled. Then it is easy to see that for each $N \in \langle D \rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and $(T \circ R)(L_N) \setminus K \subseteq \bigcup_{d \in L_N} \text{cint } H^{-1}(d)$. Therefore, (iii$_2$) of Theorem 4.5 is satisfied. By Theorem 4.5, there exists $\overline{y} \in \overline{T(X)} \cap K$ such that $H(\overline{y}) = \emptyset$. This implies that $I(y) = \emptyset$ and hence, $H_i(\overline{y}) = \emptyset$ for each $i \in I$.

Remark 4.4. Theorem 4.6 generalizes Theorem 4.1 of Lin, Yu and Lai [38] from topological vector spaces to pseudo H-spaces without any linear and convex structure.

5. Generalized equilibrium theorems

In recent years, many authors (see, for example, [8,14,31] and references therein) studied one or more of the following generalized equilibrium problems.

Let D and Z be two nonempty sets, Y be a topological space. Let $L : Y \times D \to 2^Z$ and $W : Y \to 2^Z$ be two set-valued mappings. Find $\overline{y} \in Y$ such that one of the following conditions occurs:

$$L(\overline{y}, d) \subseteq W(\overline{y}) \text{ for each } d \in D;$$

$$L(\overline{y}, d) \cap W(\overline{y}) \neq \emptyset \text{ for each } d \in D;$$

$$L(\overline{y}, d) \nsubseteq W(\overline{y}) \text{ for each } d \in D;$$

$$L(\overline{y}, d) \cap W(\overline{y}) = \emptyset \text{ for each } d \in D.$$

Recently, Balaj [39-40] adopted a unified approach for all these problems mentioned above considering a (binary) relation ρ on 2^Z and looking for a point $\overline{y} \in Y$ such that $L(\overline{y}, d)\rho\ W(\overline{y})$ for all $d \in D$. Denote by ρ^c the complementary relation of ρ, which implies that for any $U,\ V \subseteq Z$, exactly one of the following relations $U\rho\ V$, $U\rho^c\ V$ holds.

Motivated and inspired by these recent works on generalized equilibrium problems, in this section, we shall prove some new existence theorems of solutions to generalized equilibrium problems in the setting of noncompact pseudo H-spaces without any linear and convex structure.

Theorem 5.1. *Let $(X, D; \varphi_N)$ be a pseudo H-space, Y be a Hausdorff topological space, Z be a nonempty set. Let $L : Y \times D \to 2^Z$, $F,\ W : Y \to 2^Z$, $R : D \to 2^X$, and $T \in \widetilde{\mathcal{B}}(X, D, Y)$ be five set-valued mappings such that*

(i) *for each $d \in D$, the set $\{y \in Y : L(y, d)\rho\ W(y)\}$ is compactly closed;*

(ii) *for each $y \in Y$, the set $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to the set $\{d \in D : L(y, d)\rho^c\ W(y)\}$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii$_1$) *$\bigcap_{d\in N_0}\{y \in Y : L(y, d)\rho\ W(y)\} \subseteq K$ for some $N_0 \in \langle D\rangle$; or*

(iii$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap \Big(\bigcap_{d\in L_N}\{y \in Y : L(y, d)\rho\ W(y)\}\Big) \subseteq K.$$

Then there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d)\rho\ W(\overline{y})$ for each $d \in D$.

Proof. Define a set-valued mapping $S : D \to 2^Y$ by

$$S(d) = \{y \in Y : L(y, d)\rho\ W(y)\} \text{ for each } d \in D.$$

Then it follows from (i) that each $S(d)$ is compactly closed. Now, we show that (ii) of Theorem 3.1 is satisfied. Suppose the contrary. Then there exists $N = \{d_0, ..., d_n\} \in \langle D\rangle$ and $\{d_{i_0}, ..., d_{i_k}\} \subseteq N$ such that

$$T(\varphi_N(\Delta_k)) \nsubseteq \bigcup_{j=0}^{k} S(d_{i_j}).$$

Thus, there exists $y^* \in T(\varphi_N(\Delta_k))$ such that $y^* \notin S(d_{i_j})$ for each $j \in \{0, ..., k\}$; that is, $d_{i_j} \in \{d \in D : L(y^*, d)\rho^c\, W(y^*)\}$, By (ii), we have $\varphi_N(\Delta_k) \subseteq T^*(y^*)$, which implies that

$$x \in T^*(y^*) \text{ for each } x \in \varphi_N(\Delta_k). \tag{5.1}$$

Since $y^* \in T(\varphi_N(\Delta_k))$, it follows that there exists $\overline{x} \in \varphi_N(\Delta_k)$ such that $y^* \in T(\overline{x})$, i.e., $\overline{x} \notin T^*(y^*)$, which contradicts (5.1). Therefore, (ii) of Theorem 3.1 holds. Suppose that (iii_1) of Theorem 5.1 is fulfilled. Then it follows from the definition of S that there exists $N_0 \in \langle D\rangle$ such that $\bigcap_{d\in N_0} S(d) \subseteq K$, which implies that ($\text{iii}_1$) of Theorem 3.1 is satisfied. If (iii_2) of Theorem 5.1 holds, then by the definition of S again, we know that for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and

$$(T \circ R)(L_N) \cap (\bigcap_{d\in L_N} S(d)) \subseteq K.$$

Therefore, (iii_2) of Theorem 3.1 is satisfied. Thus, all the conditions of Theorem 3.1 are fulfilled. By Theorem 3.1, we have

$$\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d)) \neq \emptyset.$$

Hence, there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d)\rho\, W(\overline{y})$ for each $d \in D$. This completes the proof.

Remark 5.1. The solution set of the generalized equilibrium problem considered in Theorem 5.1 is a compact subset of $\overline{T(X)} \cap K$. Indeed, the solution set is $\overline{T(X)} \cap K \cap (\bigcap_{d\in D} S(d))$, which is compactly closed subset of the compact set $\overline{T(X)} \cap K$.

For each $y \in Y$ and each $d \in D$, let ρ denote the relations between $L(y, d)$ and $W(y)$ represented by $L(y, d) \subseteq W(y)$, $L(y, d) \nsubseteq W(y)$, $L(y, d) \cap W(y) \neq \emptyset$, and $L(y, d) \cap W(y) = \emptyset$, respectively. Then by Theorem 5.1, we can obtain the following results.

Corollary 5.1. *Let $(X, D; \varphi_N)$, Y, Z, L, F, W, R, and T be as in Theorem 5.1. Suppose that the following conditions are fulfilled:*

(i) *for each $d \in D$, the set $\{y \in Y : L(y, d) \subseteq W(y)\}$ is compactly closed;*

(ii) *for each $y \in Y$, the set $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to the set $\{d \in D : L(y, d) \nsubseteq W(y)\}$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii_1) *$\bigcap_{d\in N_0}\{y \in Y : L(y, d) \subseteq W(y)\} \subseteq K$ for some $N_0 \in \langle D\rangle$; or*

(iii_2) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap (\bigcap_{d\in L_N} \{y \in Y : L(y, d) \subseteq W(y)\}) \subseteq K.$$

Then there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d) \subseteq W(\overline{y})$ for each $d \in D$.

Remark 5.2. (1) Corollary 5.1 generalizes Theorem 4.1 of Fang and Huang [31] in the following aspects: (a) from FC-spaces to pseudo H-spaces; (b) from $\mathcal{B}(Y, X)$ to $\widetilde{\mathcal{B}}(X, Y)$; (c) from three set-valued mappings to five set-valued mappings; (d) (ii) and (iii$_2$) are weaker than (iii) and (iv) of Theorem 4.1 of Fang and Huang [31], respectively.

(2) When Z in Corollary 5.1 is a topological space, (i) of Corollary 5.1 can be replaced by the following condition:

(i)$'$ for each $d \in D$, $y \mapsto L(y,d)$ is lower semicontinuous on each compact subset of Y and the graph of W is closed in $Y \times Z$.

In fact, for each $d \in D$ and each compact subset C of Y, let $\{y_\nu\}$ be a net in $\{y \in Y : L(y,d) \subseteq W(y)\} \cap C$ such that $\{y_\nu\}$ converges to y_0. Since Y is a Hausdorff topological space, it follows that C is closed and hence, $y_0 \in C$. Since $\{y_\nu\} \subseteq \{y \in Y : L(y,d) \subseteq W(y)\} \cap C$, we have $L(y_\nu, d) \subseteq W(y_\nu)$ for all ν. Let $z \in L(y_0,d)$. Then by Lemma 2.1 and the fact that for each $d \in D$, $y \mapsto L(y,d)$ is lower semicontinuous on each compact subset of Y, there exists a net $\{z_\nu\} \subseteq L(y_\nu,d)$ such that $\{z_\nu\}$ converges to z. Thus, $z_\nu \in W(y_\nu)$ for all ν. Since the graph of W is closed in $Y \times Z$, we have $z \in W(y_0)$ and hence, $L(y_0,d) \subseteq W(y_0)$. Therefore, $y_0 \in \{y \in Y : L(y,d) \subseteq W(y)\} \cap C$, which implies that for each $d \in D$, the set $\{y \in Y : L(y,d) \subseteq W(y)\}$ is compactly closed.

Corollary 5.2. *Let $(X, D; \varphi_N)$, Y, Z, L, F, W, R, and T be as in Theorem 5.1. Suppose that the following conditions are fulfilled:*

(i) *for each $d \in D$, the set $\{y \in Y : L(y,d) \nsubseteq W(y)\}$ is compactly closed;*

(ii) *for each $y \in Y$, the set $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to the set $\{d \in D : L(y,d) \subseteq W(y)\}$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii$_1$) $\bigcap_{d\in N_0}\{y \in Y : L(y,d) \nsubseteq W(y)\} \subseteq K$ *for some $N_0 \in \langle D\rangle$; or*

(iii$_2$) *for each $N \in \langle D\rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap \Big(\bigcap_{d\in L_N} \{y \in Y : L(y,d) \nsubseteq W(y)\}\Big) \subseteq K.$$

Then there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d) \nsubseteq W(\overline{y})$ for each $d \in D$.

Remark 5.3. When Z in Corollary 5.2 is a topological space, (i) of Corollary 5.2 can be replaced by the following condition:

(i)$'$ L has nonempty compact values and for each $d \in D$, $y \mapsto L(y,d)$ is upper semicontinuous on each compact subset of Y;

(i)$''$ The graph of W is open in $Y \times Z$.

In fact, for each $d \in D$ and each compact subset C of Y, let $\{y_\nu\}$ be a net in $\{y \in Y : L(y,d) \nsubseteq W(y)\} \cap C$ such that $\{y_\nu\}$ converges to y_0. Then we have $L(y_\nu,d) \nsubseteq W(y_\nu)$ for all ν; that is, there exists $z_\nu \in L(y_\nu,d)$ such that $z_\nu \notin W(y_\nu)$, or $z_\nu \in Z \setminus W(y_\nu)$ for all ν. By Lemma 2.2, there exists $z \in L(y_0,d)$ and a subnet of $\{z_\nu\}$ such that this subnet converges to z. Without loss of generality, we may assume that $\{z_\nu\}$ converges to z. Also since W has open graph in $Y \times Z$, we have $z \notin W(y_0)$. Therefore, $L(y_0,d) \nsubseteq W(y_0)$. By the fact that Y is a Hausdorff topological space, we know that C is closed subset of Y and hence, $y_0 \in C$. Consequently, $y_0 \in \{y \in Y : L(y,d) \nsubseteq W(y)\} \cap C$, which implies that the set $\{y \in Y : L(y,d) \nsubseteq W(y)\}$ is compactly closed for each $d \in D$.

Corollary 5.3. *Let $(X, D; \varphi_N)$, Y, Z, L, F, W, R, and T be as in Theorem 5.1. Suppose that the following conditions are fulfilled:*

(i) *for each $d \in D$, the set $\{y \in Y : L(y,d) \cap W(y) \neq \emptyset\}$ is compactly closed;*

(ii) *for each $y \in Y$, the set $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to the set $\{d \in D : L(y,d) \cap W(y) = \emptyset\}$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii$_1$) *$\bigcap_{d \in N_0} \{y \in Y : L(y,d) \cap W(y) \neq \emptyset\} \subseteq K$ for some $N_0 \in \langle D \rangle$; or*

(iii$_2$) *for each $N \in \langle D \rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap \Big(\bigcap_{d \in L_N} \{y \in Y : L(y,d) \cap W(y) \neq \emptyset\}\Big) \subseteq K.$$

Then there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d) \cap W(\overline{y}) \neq \emptyset$ for each $d \in D$.

Corollary 5.4. *Let $(X, D; \varphi_N)$, Y, Z, L, F, W, R, and T be as in Theorem 5.1. Suppose that the following conditions are fulfilled:*

(i) *for each $d \in D$, the set $\{y \in Y : L(y,d) \cap W(y) = \emptyset\}$ is compactly closed;*

(ii) *for each $y \in Y$, the set $T^*(y)$ is a pseudo H-subspace of $(X, D; \varphi_N)$ relative to the set $\{d \in D : L(y,d) \cap W(y) \neq \emptyset\}$;*

(iii) *there exists a nonempty compact subset K of Y such that either*

(iii$_1$) *$\bigcap_{d \in N_0} \{y \in Y : L(y,d) \cap W(y) = \emptyset\} \subseteq K$ for some $N_0 \in \langle D \rangle$; or*

(iii$_2$) *for each $N \in \langle D \rangle$, there exists a subset L_N of D containing N such that $R(L_N)$ is a compact pseudo H-subspace of $(X, D; \varphi_N)$ relative to L_N and*

$$(T \circ R)(L_N) \cap \Big(\bigcap_{d \in L_N} \{y \in Y : L(y,d) \cap W(y) = \emptyset\}\Big) \subseteq K.$$

Then there exists a point $\overline{y} \in K \cap \overline{T(X)}$ such that $L(\overline{y}, d) \cap W(\overline{y}) = \emptyset$ for each $d \in D$.

Acknowledgments. The authors thank the referee for his or her many valuable suggestions and helpful comments which improve the exposition of the paper. This work was supported by the Planning Foundation for Humanities and Social Sciences of Ministry of Education of China ("Research on utilizing conflict of water resource and initial water right allocation in a river basin based on game theory", no. 12YJAZH084), the Young & Middle-Aged Academic Learners Program of the "Qinglan project" of Jiangsu Providence and by the Natural Science Foundation of Jiangsu University of Technology.

References

[1] B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunksatzes für n-dimensionale Simplexe*, Fund. Math., **14**(1929), 132-137.

[2] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142**(1961), 305-310.

[3] H. Brézis, L. Nirenberg, G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital., **4**(1972), 293-300.

[4] G.Q. Tian, *Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity*, J. Math. Anal. Appl., **170**(1992), 457-471.
[5] Q.H. Ansari, Y.C. Lin, J.C. Yao, *General KKM theorem with applications to minimax and variational inequalities*, J. Optim. Theory Appl., **104**(2000), 41-57.
[6] C.Z. Cheng, *A minimax inequality and variational inequalities*, Prog. Nat. Sci., **7**(1997), 92-99.
[7] X.Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, NY, 1999.
[8] L.J. Lin, W.P. Wan, *KKM type theorems and coincidence theorems with applications to the existence of equilibria*, J. Optim. Theory Appl., **123**(2004), 105-122.
[9] H.K. Pathak, M.S. Khan, *On D-KKM theorem and its applications*, Bull. Austral. Math. Soc., **67**(2003), 67-77.
[10] M. Balaj, L.J. Lin, *Equivalent forms of a generalized KKM theorem and their applications*, Nonlinear Anal., **73**(2010), 673-682.
[11] C.D. Horvath, *Some results on multivalued mappings and inequalities without convexity*, in "Nonlinear and Convex Analysis" (B.L. Lin and S. Simons, Eds.), Marcel Dekker, New York, 1987, 99-106.
[12] C. Bardaro, R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl., **132**(1988), 484-490.
[13] R.U. Verma, *I-G-H-KKM mappings and minimax inequalities*, Math. Comput. Model., **30**(1999), 147-152.
[14] E.M. Kalmoun, H. Riahi, *Topological KKM theorems and generalized vector equilibria on G-convex spaces with applications*, Proc. Amer. Math. Soc., **129**(2001), 1335-1348.
[15] Y.C. Lin, J.-C. Yao, *A generalization of Peleg's theorem with applications to a system of variational inequalities*, Inter. J. Pure Appl. Math., **5**(2003), 217-228.
[16] C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, New York, 1994.
[17] N.X. Tan, *Quasi-variational inequalities in topological linear locally convex Hausdorff spaces*, Math. Nachr., **122**(1985), 231-245.
[18] X.P. Ding, *Generalized G-KKM theorems in generalized convex spaces and their applications*, J. Math. Anal. Appl., **266**(2002), 21-37.
[19] T.C. Lai, Y.C. Lin, J.C. Yao, *Existence of equilibrium for abstract economies on pseudo H-spaces*, Appl. Math. Lett., **17**(2004), 691-696.
[20] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl., **97**(1983), 151-201.
[21] S. Park, H. Kim, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl., **209**(1997), 551-571.
[22] H. Ben-Ei-Mechaiekh, S. Chebbi, M. Florenzano, J.V. LIinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl., **222**(1998), 138-150.
[23] P.Q. Khanh, N.H. Quan, J.C. Yao, *Generalized KKM-type theorems in GFC-spaces and applications*, Nonlinear Anal., **71**(2009), 1227-1234.
[24] X.P. Ding, *Maximal element theorems in product FC-spaces and generalized games*, J. Math. Anal. Appl., **305**(2005), 29-42.
[25] G.L. Cain Jr., L. González, *The Knaster-Kuratowski-Mazurkiewicz theorem and abstract convexities*, J. Math. Anal. Appl., **338**(2008), 563-571.
[26] S. Plubtieng, T. Thammathiwat, *Existence of solutions of systems of generalized implicit vector quasi-equilibrium problems in G-convex spaces*, Comput. Math. Appl., **62**(2011), 124-130.
[27] K. Fan, *Fixed points and minimax theorems in locally convex spaces*, Proc. Nat. Acad. Sci. U.S.A., **38**(1952), 121-126.
[28] L. Deng, X. Xia, *Generalized R-KKM theorems in topological space and their applications*, J. Math. Anal. Appl., **285**(2003), 679-690.
[29] J.P. Aubin, I. Ekland, *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.
[30] M.G. Yang, L. Deng, *Coincidence theorem for admissible set-valued mappings and its applications in FC-spaces*, Acta. Math. Appl. Sin-E., **26**(2010), 455-462.

[31] M. Fang, N. J. Huang, *KKM type theorems with applications to generalized vector equilibrium problems in FC-spaces*, Nonlinear Anal., **67**(2007), 809-817.
[32] F.E. Browder, *Coincidence theorems, minimax theorems, and variational inequalities*, Contemp. Math., **26**(1984), 47-80.
[33] E. Tarafdar, *On nonlinear variational inequalities*, Proc. Amer. Math. Soc., **67**(1977), 95-98.
[34] G. Mehta, S. Sessa, *Coincidence theorems and maximal elements in topological vector spaces*, Math. Japon., **47**(1992), 839-845.
[35] Z.F. Shen, *Maximal element theorems of H-majorized correspondence and existence of equilibrium for abstract economies*, J. Math. Anal. Appl., **256**(2001), 67-79.
[36] C.I. Tulcea, *On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games*, J. Math. Anal. Appl., **136**(1988), 267-289.
[37] N.C. Yannelis, N.D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom., **12**(1983), 233-245.
[38] L.J. Lin, Z.T. Yu, Q.H. Ansari, L.P. Lai, *Fixed point and maximal element theorems with applications to abstract economies and minimax inequalities*, J. Math. Anal. Appl., **284**(2003), 656-671.
[39] M. Balaj, *An intersection theorem with applications in minimax theory and equilibrium problem*, J. Math. Anal. Appl., **336**(2007), 363-371.
[40] M. Balaj, *Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities*, Nonlinear Anal., **68**(2008), 3962-3971.

Received: December 22, 2012; Accepted: February 10, 2013.