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ON AN A-BIFURCATION THEOREM WITH APPLICATION TO A PARAMETERIZED INTEGRO-DIFFERENTIAL SYSTEM

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Abstract. In this paper, we define a notion of *an A-bifurcation* for a system of differential equations in a separable Hilbert space. By using the methods of the topological degree theory and guiding functions, we prove the theorem on the existence and uniqueness of an A-bifurcation point. It is shown how the abstract result can be applied to study the global structure of the solution set of a feedback control system governed by integro-differential equations.
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1. Introduction

The necessity of studying differential equations in Banach spaces arises in many applied problems. For example, it is known (see, e.g. [12]) that a partial differential or integro-differential equations under some suitable conditions can be reduced to differential equations in appropriate Banach spaces which, in turn, are equivalent to certain operator equations. Furthermore, following this way, the corresponding topological degree theory can be applied to study the existence and qualitative behavior of solutions of differential equations in Banach spaces. However, from the practical point of view it is important to study *approximable solutions* rather than usual solutions since these solutions can be determined by using the *approximation method.* For operator equations containing $A-$proper maps in Banach spaces with projectional schemes (for example, in separable Hilbert spaces) F.E. Browder and W.V. Petryshyn defined the *generalized topological degree* for studying the existence of approximable solutions (see, [5, 18]). This approach was extended to operator inclusions in [15] and was applied to study the *global bifurcations of approximable solutions* in [1, 19, 20, 21]. Notice that in many cases the $A-$properness seems to be not a very natural property.

In this paper, we consider the systems of differential equations in separable Hilbert spaces which can be rewritten as operator inclusions without involving the $A-$properness property. We define a notion of the *A-bifurcation points* for these systems. It is shown that every bifurcation point of approximable solutions is an A-bifurcation point. By using the method of guiding functions in Hilbert spaces (see, [13, 14, 17]) we obtain sufficient conditions under which $(0,0)$ is the unique A-bifurcation point of the considered problem, and hence, if this problem has a bifurcation of approximable solutions, then it may occur only at $(0,0)$. The result on the existence of a global A-bifurcation (Theorem 3.5) is proved.

The paper is organized in the following way. In the next section we recall some notions and notation from multivalued analysis and theory of Fredholm operators. In Section 3, after the setting of the problem, by using the topological degree theory and the guiding function of the form $V(x,\mu) = \frac{1}{2}\mu\langle x, x\rangle_H$, where x belongs to a Hilbert space H, $\mu \in \mathbb{R}$, we prove the existence and uniqueness of an A-bifurcation point for the considered problem. As an application of the abstract result, we consider, in the last section, the bifurcation in a system of integro-differential equations which may be treated as a feedback control system.

2. Preliminaries

2.1. **Multimaps.** Let X, Y be metric spaces. Denote by $P(Y)\,[K(Y)]$ the collection of all nonempty [respectively, nonempty compact] subsets of Y.

Definition 2.1. (see, e.g. [3, 8, 11]). (i) A multivalued map (multimap) $F\colon X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$F_+^{-1}(V) = \{x \in X \colon F(x) \subset V\}$$

is open in X; (ii) A u.s.c. multimap F is said to be completely u.s.c., if it maps every bounded subset $X_1 \subset X$ into a relatively compact subset $F(X_1)$ of Y; (iii) A multimap F is said to be compact, if the set $\overline{F(X)}$ is compact in Y.

A set $M \in K(Y)$ is said to be *aspheric* (or UV^∞, or ∞-*proximally connected*) (see, e.g. [16, 7, 8]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\sigma\colon S^n \to O_\delta(M)$, $n = 0, 1, 2, \cdots$, can be extended to a continuous map $\widetilde{\sigma}\colon B^{n+1} \to O_\varepsilon(M)$, where $S^n = \{x \in \mathbb{R}^{n+1}\colon \|x\| = 1\}$, $B^{n+1} = \{x \in \mathbb{R}^{n+1}\colon \|x\| \le 1\}$, and $O_\delta(M)$ $[O_\varepsilon(M)]$ denotes the δ-neighborhood [resp. ε-neighborhood] of the set M.

Definition 2.2. (see [10]). A nonempty compact space A is said to be an R_δ-*set* if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.
Definition 2.3. (see [8]). A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J-*multimap* ($\Sigma \in J(X, Y)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.
Proposition 2.4. (see [8]). *Let Z be an ANR-space. In each of the following cases a u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap:*
for each $x \in X$ the value $\Sigma(x)$ is
(a) a convex set;
(b) a contractible set;
(c) an R_δ-set;
(d) an AR-space.
In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.
Definition 2.5. Let $\mathcal{O} \subseteq X$. By $CJ(\mathcal{O}, X)$ we will denote the collection of all multimaps $F\colon \mathcal{O} \to K(X)$ that may be represented in the form of composition $F = f \circ G$, where $G \in J(\mathcal{O}, Y)$ and $f\colon Y \to X$ is a continuous map. The composition $f \circ G$ will be called the decomposition of F. We will denote $F = (f \circ G)$.

It is worth noting that a multimap can admit different decompositions (see [8]).

Now, let X be a Banach space and $U \subset X$ be an open bounded subset and $F = (f \circ G) \in CJ(\overline{U}, X)$ be a compact $CJ-$multimap such that $x \notin F(x)$ for $x \in \partial U$. Then the topological degree $deg(i - F, \overline{U})$ of the corresponding compact multivalued $CJ-$vector field $(i - F)(x) = x - F(x)$ is well-defined and has all usual properties of the Leray-Schauder topological degree (see, e.g. [8]).

Now let us recall (see, e.g. [4]) that a metric space X is called *the absolute retract (the AR-space)* [resp., *the absolute neighborhood retract (the ANR-space)]* provided for each homeomorphism h taking it onto a closed subset $h(X)$ of a metric space X', the set $h(X)$ is the retract of X' [resp., of its open neighborhood in X']. Notice that the class of ANR-spaces is broad enough: in particular, a finite-dimensional compact set is the ANR-space if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the ANR-spaces. The union of a finite number of convex closed subsets in a normed space is also the ANR-space.

2.2. **Fredholm Operators.** Now we recall some notions from the Fredholm operators theory.
Definition 2.6. (see, e.g. [6]). A linear bounded map $\ell\colon X \to Y$ is said to be a Fredholm operator of index zero, if

(*i*) $Im\ell$ is closed in Y;
(*ii*) $Ker\ell$ and $Coker\ell$ have the finite dimensions and

$$dim\ Ker\ell = dim\ Coker\ell.$$

Throughout this paper, symbol H denotes a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. For every $n \in \mathbb{N}$, let H_n be an $n-$dimensional subspace of H with the basis $\{e_k\}_{k=1}^{n}$ and P_n be a projection of H onto H_n. By $\langle \cdot, \cdot \rangle_H$ we denote the inner product in H. The symbol I denotes the interval $[0, T]$. By $C(I, H)$

$[L^2(I,H)]$ we denote the spaces of all continuous [respectively, square summable] functions $u\colon I \to H$ with usual norms

$$\|u\|_C = \max_{t\in I} \|u(t)\|_H \quad \text{and} \quad \|u\|_2 = \left(\int_0^T \|u(t)\|_H^2 dt\right)^{\frac{1}{2}}.$$

A closed ball of radius r centered at 0 in $C(I,H)$ is denoted by $B_C(0,r)$. Consider the space of all absolutely continuous functions $u : I \to H$ whose derivatives belong to $L^2(I,H)$. It is known (see, e.g. [2]) that this space can be identified with the Sobolev space $W^{1,2}(I,H)$ endowed with the norm

$$\|u\|_W = \left(\|u\|_2^2 + \|u'\|_2^2\right)^{\frac{1}{2}}.$$

The embedding $W^{1,2}(I,H) \hookrightarrow C(I,H)$ is continuous, and for every $n \geq 1$ the space $W^{1,2}(I,H_n)$ is compactly embedded in $C(I,H_n)$. By $W_T^{1,2}(I,H)$ we denote the subspace of all functions $x \in W^{1,2}(I,H)$ satisfying the boundary condition $x(0) = x(T)$.

Let $n \in \mathbb{N}$, and $\ell\colon W_T^{1,2}(I,H_n) \to L^2(I,H_n)$ be a linear Fredholm operator of index zero. Then there exist the projections (see, e.g. [6]):

$$C_n\colon W_T^{1,2}(I,H_n) \to W_T^{1,2}(I,H_n)$$

and

$$Q_n\colon L^2(I,H_n) \to L^2(I,H_n)$$

such that $Im\ C_n = Ker\,\ell$ and $Ker\ Q_n = Im\,\ell$. If the operator

$$\ell_{C_n}\colon dom\,\ell \cap Ker\,C_n \to Im\,\ell$$

is defined as the restriction of ℓ on $dom\,\ell \cap Ker\,C_n$, then ℓ_{C_n} is a linear isomorphism and we can define the operator $K_{C_n}\colon Im\,\ell \to dom\,\ell$, $K_{C_n} = \ell_{C_n}^{-1}$. Now, set $Coker\,\ell = L^2(I,H_n)/Im\,\ell$; and let $\Pi_n\colon L^2(I,H_n) \to Coker\,\ell$ be the canonical projection

$$\Pi_n(z) = z + Im\,\ell$$

and $\Lambda_n\colon Coker\,\ell \to Ker\,\ell$ be the linear continuous isomorphism. Then the equation

$$\ell x = y,\ y \in L^2(I,H_n)$$

is equivalent to

$$(i - C_n)x = (\Lambda_n\Pi_n + K_{C_n,Q_n})y,$$

where $K_{C_n,Q_n}\colon L^2(I,H_n) \to W_T^{1,2}(I,H_n)$ is given as

$$K_{C_n,Q_n} = K_{C_n}(i - Q_n).$$

3. A-bifurcation theorem

3.1. **The setting of the problem.** Let a separable Hilbert space H be compactly embedded in a Banach space Y and

$$\|z\|_Y \leq h\|z\|_H, \text{ for all } z \in H, \tag{3.1}$$

for some $h > 0$.

By $\mathcal{L}(H)$ we denote the Banach space of all linear bounded operators on H with the usual norm.

Consider the following family of differential equations

$$\begin{cases} x'(t) = \mu A x(t) + f\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in I, \\ y'(t) = g\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in I, \\ x(0) = x(T), y(0) = 0, \end{cases} \tag{3.2}$$

where $A \in \mathcal{L}(H)$; $f\colon I \times H \times Y \times \mathbb{R} \to H$ and $g\colon I \times H \times Y \times \mathbb{R} \to H$ are continuous maps.

We shall assume the following conditions:

(A) there exists $a > 0$ such that

$$\big\langle w, Aw \big\rangle_H \geq a\big\langle w, w\big\rangle_H$$

for all $w \in H$;

(f) there exists $0 < c < a$ such that

$$\|f(t, w, z, \mu)\|_H \leq c\|w\|_H(|\mu| + \|z\|_Y)$$

for all $(t, w, z, \mu) \in I \times H \times Y \times \mathbb{R}$;

(g) there exists $d > 0$ such that $a > c(1 + dhTe^{dhT})$ and

$$\|g(t, w, z, \mu)\|_H \leq d\Big(\|w\|_H + \|z\|_Y + |\mu|\Big)$$

for all $(t, w, z, \mu) \in I \times H \times Y \times \mathbb{R}$.

Let us denote $\widehat{W}^{1,2}(I, H) = \{y \in W^{1,2}(I, H)\colon y(0) = 0\}$. For each $n \in \mathbb{N}$ consider the approximation problem

$$\begin{cases} x'(t) = \mu P_n A x(t) + P_n f\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in I, \\ y'(t) = g\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in I, \\ x(0) = x(T), y(0) = 0. \end{cases} \tag{3.3}$$

By a solution of (3.3) we mean a pair $(x, \mu) \in W_T^{1,2}(I, H_n) \times \mathbb{R}$ for which there is a function $y \in \widehat{W}^{1,2}(I, H)$ such that the triplet (x, y, μ) satisfies (3.3).

Definition 3.1. (cf. [5, 19]) By an *approximable solution* to problem (3.2) we mean a pair $(x, \mu) \in W_T^{1,2}(I, H) \times \mathbb{R}$ with the following property:

(i) there is a function $y \in \widehat{W}^{1,2}(I, H)$ such that the triplet (x, y, μ) satisfies (3.2).

(ii) there are sequences $\{n_k\}$ and $\{(x_{n_k}, \mu_{n_k})\}$, $x_{n_k} \in W_T^{1,2}(I, H_{n_k})$, such that (x_{n_k}, μ_{n_k}) are solutions to (3.3) for all n_k and

$$x_{n_k} \overset{W^{1,2}(I,H)}{\longrightarrow} x \text{ and } \mu_{n_k} \to \mu.$$

From (f) it follows that $f(t,0,z,\mu)=0$ for all $(t,z,\mu)\in I\times Y\times\mathbb{R}$. Therefore, $(0,\mu)$ are approximable solutions to (3.2) for all $\mu\in\mathbb{R}$.

Definition 3.2. We will say that a point $(0,\mu_0)$ is a *bifurcation point of approximable solutions* to problem (3.2) if for every open subset $U\subset W_T^{1,2}(I,H)\times\mathbb{R}$ containing $(0,\mu_0)$ there exists $(x,\mu)\in U$, $x\neq 0$, such that (x,μ) is an approximable solution to (3.2).

The search for a bifurcation point of approximable solutions of problem (3.2) can be a difficult problem, especially, if we do not impose any compactness or condensivity assumptions on its right hand side. At least, the following question seems to be natural: are there any *candidates* for bifurcation points of approximable solutions? To answer this question we introduce the following notion.

Definition 3.3. A point $(0,\mu_0)$ is said to be an *A-bifurcation point* to problem (3.2) if for every open subset $U\subset W_T^{1,2}(I,H)\times\mathbb{R}$ containing $(0,\mu_0)$ there exist sequences $\{n_k\}_{k=1}^{\infty}$, $\{(x_{n_k},\mu_{n_k})\}$, $(x_{n_k},\mu_{n_k})\in U_{n_k}$, $x_{n_k}\neq 0$, such that (x_{n_k},μ_{n_k}) is a solution to (3.3), where $U_{n_k}=U\cap\big(W_T^{1,2}(I,H_{n_k})\times\mathbb{R}\big)$.

It is clear that if $(0,\mu_0)$ is a bifurcation point of approximable solutions of (3.2), then it is an A-bifurcation point of this problem. Moreover, an $A-$bifurcation point will be a bifurcation point of approximable solutions if problem (3.2) has *the $A-$properness property*, i.e., from the existence of a sequence of solutions $\{(x_{n_k},\mu_{n_k})\}$ to approximation problem (3.3) it follows that problem (3.2) has a solution (x,μ) such that $(x_{n_k},\mu_{n_k})\to(x,\mu)$ (see, [5, 19] for more detail about $A-$properness property).

In the sequel, we need the following assertion which easily follows from Theorem 70.12 [8].

Lemma 3.4. *Let E be a separable Banach space and $\varphi\colon I\times E\to E$ be a map satisfying the following conditions:*

($\varphi 1$) *φ is completely continuous, i.e. φ is continuous and maps every bounded subset $\Omega\subset I\times E$ into a relatively compact subset $\varphi(\Omega)$ of E;*

($\varphi 2$) *there is $q>0$ such that*

$$\|\varphi(t,y)\|_E\leq q(1+\|y\|_E),$$

for all $(t,y)\in I\times E$.

Then the set of all solutions of the Cauchy's problem

$$\begin{cases} u'(t)=\varphi(t,u(t)) \text{ for a.e. } t\in I,\\ u(0)=u_0\in E,\end{cases}$$

is an $R_\delta-$set in $C(I,E)$.

3.2. **Main result.** To formulate the main result let us denote by $\mathcal{S}_n$ $(n\in\mathbb{N})$ the set of all non-trivial solutions of (3.3).

Theorem 3.5. *Let conditions (A), (f) and (g) hold. Then $(0,0)$ is a unique A-bifurcation point of problem (3.2). Moreover, there exists a sequence of connected sets $\mathcal{C}_{2n-1}\subset\mathcal{S}_{2n-1}$, such that $(0,0)\in\overline{\mathcal{C}_{2n-1}}$ and $\mathcal{C}_{2n-1}$ are unbounded for all $n=1,2,\cdots$.*

Proof. For each $(x,\mu)\in C(I,H)\times\mathbb{R}$ consider the map

$$g^{(x,\mu)}\colon I\times Y\to Y,\ g^{(x,\mu)}(t,z)=g(t,x(t),z,\mu).$$

It is easy to verify that the map $g^{(x,\mu)}$ satisfies all conditions of Lemma 3.4. Therefore, for each $(x,\mu) \in C(I,H) \times \mathbb{R}$ the set $\Pi^{(x,\mu)}$ of all solutions of the Cauchy problem

$$\begin{cases} y'(t) = g(t, x(t), y(t), \mu), \text{ for a.e. } t \in I, \\ y(0) = 0, \end{cases}$$

is an $R_\delta-$set in $C(I,Y)$.
Define the multimap $\Pi\colon C(I,H) \times \mathbb{R} \to K(C(I,Y))$, $\Pi(x,\mu) = \Pi^{(x,\mu)}$. Applying the known result (see, e.g., Theorem 5.2.5 [11]) on continuous dependence of solution sets of differential inclusions (in particular, differential equations) we obtain that multimap Π is upper semicontinuous, and hence, it is a $J-$multimap.

Define now the multimap

$$\widetilde{\Pi}\colon C(I,H) \times \mathbb{R} \to K\big(C(I,H) \times C(I,Y) \times \mathbb{R}\big),$$
$$\widetilde{\Pi}(x,\mu) = \{x\} \times \Pi(x,\mu) \times \{\mu\},$$

and the map $\widetilde{f}\colon C(I,H) \times C(I,Y) \times \mathbb{R} \to L^2(I,H)$,

$$\widetilde{f}(x,y,\mu)(t) = \mu Ax(t) + f(t, x(t), y(t), \mu), \ t \in I.$$

Then we can replace problem (3.2) with the following operator inclusion

$$Lx \in Q(x,\mu), \tag{3.4}$$

where $L\colon W_T^{1,2}(I,H) \to L^2(I,H)$, $Lx = x'$ and

$$Q\colon C(I,H) \times \mathbb{R} \to K(L^2(I,H)),$$
$$Q(x,\mu) = \widetilde{f} \circ \widetilde{\Pi}(x,\mu).$$

It is easy to see that Q is a $CJ-$multimap and for each $n \in \mathbb{N}$ the restriction

$$L_n = L_{|_{W_T^{1,2}(I,H_n)}}\colon W_T^{1,2}(I,H_n) \to L^2(I,H_n)$$

is the linear Fredholm operator of index zero and

$$Ker\, L_n \cong H_n \cong Coker\, L_n.$$

The space $L^2(I,H_n)$ can be decomposed as:

$$L^2(I,H_n) = \mathcal{L}_0^{(n)} \oplus \mathcal{L}_1^{(n)},$$

where $\mathcal{L}_0^{(n)} = Coker\, L_n$, and $\mathcal{L}_1^{(n)} = Im L_n$.
For every $u \in L^2(I,H_n)$ we denote its corresponding decompositions by

$$u = u_0^{(n)} + u_1^{(n)}.$$

Notice that a pair $(x,\mu) \in W_T^{1,2}(I,H_n) \times \mathbb{R}$ is a solution of (3.3), or equivalently of the inclusion

$$L_n x \in \mathbb{P}_n Q(x,\mu),$$

if and only if it is a fixed point

$$x \in G_n(x,\mu), \tag{3.5}$$

of the $CJ-$multimap

$$G_n\colon C(I,H_n)\times\mathbb{R}\to K(C(I,H_n)),$$
$$G_n(x,\mu)=C_nx+(\Lambda_n\Pi_n+K_{C_n,Q_n})\circ\mathbb{P}_nQ(x,\mu),$$

where projection $\Pi_n\colon L^2(I,H_n)\to H_n$ is defined as

$$\Pi_n u=\frac{1}{T}\int\limits_0^T u(s)\,ds$$

and the homomorphism $\Lambda_n\colon H_n\to H_n$ is the identity operator and $\mathbb{P}_n\colon L^2(I,H)\to L^2(I,H_n)$ is a continuous map defined by

$$(\mathbb{P}_nf)(t)=P_nf(t),\ t\in I.$$

Step 1. We will show that for every $n\geq 1$ the multimap G_n is completely u.s.c. Indeed, for every bounded subset $U\subset C(I,H_n)\times\mathbb{R}$ and for every $(x,\mu)\in U$ choose an arbitrarily $q\in Q(x,\mu)$. Then there exists $y\in\Pi(x,\mu)$ such that

$$q(t)=\mu Ax(t)+f(t,x(t),y(t),\mu),\ t\in I.$$

Since $y\in\Pi(x,\mu)$ and (g) we have

$$\begin{aligned}\|y(t)\|_Y&=\left\|\int_0^t g(s,x(s),y(s),\mu)ds\right\|_Y\\&\leq h\int_0^t\|g(s,x(s),y(s),\mu)\|_Hds\\&\leq h\int_0^t d\big(\|x(s)\|_H+\|y(s)\|_Y+|\mu|\big)ds\\&\leq dhT\|x\|_C+dh|\mu|T+\int_0^t dh\|y(s)\|_Yds.\end{aligned}$$

Applying the Gronwall lemma (see, e.g. [9]) we obtain

$$\|y(t)\|_Y\leq dhT\big(\|x\|_C+|\mu|\big)e^{dhT},\ t\in I. \tag{3.6}$$

From the boundedness of the set U it follows that there is $M_U>0$ such that $|\mu|\leq M_U$ for all $(x,\mu)\in U$. From the continuity of the map A it follows that there is $M_1>0$ such that $\|\mu Aw\|_H\leq M_1\|w\|_H$ for all $(t,w,\mu)\in I\times H\times[-M_U,M_U]$, and hence, for every $t\in I$ the following relation holds true:

$$\begin{aligned}\|q(t)\|_H&=\|\mu Ax(t)+f(t,x(t),y(t),\mu)\|_H\\&\leq M_1\|x(t)\|_H+\|f(t,x(t),y(t),\mu)\|_H\\&\leq M_1\|x\|_C+c\|x(t)\|_H\big(\|y(t)\|_Y+|\mu|\big).\end{aligned} \tag{3.7}$$

Consequently, the set $Q(U)$ is bounded in $L^2(I,H)$. Then the set $(\Lambda_n\Pi_n+K_{C_n,Q_n})\circ\mathbb{P}_nQ(U)$ is bounded in $W_T^{1,2}(I,H_n)$ and by the compact embedding property, it is relatively compact in $C(I,H_n)$. Finally, our assertion follows from the fact that the operator C_n is continuous and takes values in a finite dimensional space.

STEP 2. Now, for each $\mu \neq 0$ let us show that there exists $\delta_\mu > 0$ such that inclusion (3.5) has no non-trivial solutions on $B_C^{(n)}(0, \delta_\mu) \times \{\mu\}$ for all $n \in \mathbb{N}$, where $B_C^{(n)}(0, \delta_\mu) = B_C(0, \delta_\mu) \cap C(I, H_n)$.
Toward this goal, assume that $(x, \mu) \in W_T^{1,2}(I, H_n) \times \{\mu\}$ is a non-trivial solution of (3.5). Then there is a function $y \in \Pi(x, \mu)$ such that

$$x'(t) = \mu P_n Ax(t) + P_n f(t, x(t), y(t), \mu), \text{ for a.e. } t \in I.$$

Therefore,

$$\int_0^T \big\langle \mu x(t), \mu P_n Ax(t) + P_n f(t, x(t), y(t), \mu)\big\rangle_H dt = \int_0^T \big\langle \mu x(t), x'(t)\big\rangle_H dt = 0.$$

On the other hand, since $x(t) \in H_n$ for all $t \in I$, conditions (A), (f), (g) and (3.6) imply

$$\begin{aligned}
&\int_0^T \big\langle \mu x(t), \mu P_n Ax(t) + P_n f(t, x(t), y(t), \mu)\big\rangle_H dt \\
&\quad = \int_0^T \big\langle \mu x(t), \mu Ax(t) + f(t, x(t), y(t), \mu)\big\rangle_H dt \\
&\quad \geq a\mu^2 \|x\|_2^2 - |\mu| \int_0^T \|x(t)\|_H \, \|f(t, x(t), y(t), \mu)\|_H dt \\
&\quad \geq a\mu^2 \|x\|_2^2 - |\mu| c \int_0^T \|x(t)\|_H \|x(t)\|_H \big(|\mu| + \|y(t)\|_Y\big) dt \\
&\quad \geq a\mu^2 \|x\|_2^2 - c|\mu| \int_0^T \|x(t)\|_H^2 \Big(|\mu| + dhT\big(\|x\|_C + |\mu|\big)e^{dhT}\Big) dt \\
&\quad = |\mu| \|x\|_2^2 \Big(|\mu|\big(a - c(1 + dhTe^{dhT})\big) - cdhTe^{dhT}\|x\|_C\Big) > 0
\end{aligned}$$

provided

$$0 < \|x\|_C < \frac{\big(a - c(1 + dhTe^{dhT})\big)|\mu|}{cdhTe^{dhT}}. \tag{3.8}$$

So, inclusion (3.5) has no solutions (x, μ) if the norm $\|x\|_C$ satisfies estimate (3.8).

STEP 3. For a fixed $n \in \mathbb{N}$ and $r, \varepsilon > 0$ define a multimap

$$G_n^r \colon B_{r,\varepsilon}^{(n)} \to K(C(I, H_n) \times \mathbb{R}),$$
$$G_n^r(x, \mu) = \Big\{x - G_n(x, \mu), \|x\|_C^2 - r^2\Big\},$$

where

$$B_{r,\varepsilon}^{(n)} = \big\{(x, \mu) \in C(I, H_n) \times \mathbb{R} \colon \|x\|_C^2 + \mu^2 \leq r^2 + \varepsilon^2\big\}.$$

It is easy to verify that G_n^r is a compact multivalued $CJ-$vector field. For every numbers $r, \varepsilon > 0$ such that

$$r < \frac{\varepsilon\big(a - c(1 + dhTe^{dhT})\big)}{cdhTe^{dhT}}, \tag{3.9}$$

let us show that $0 \notin G_n^r(x, \mu)$ for all $(x, \mu) \in \partial B_{r,\varepsilon}^{(n)}$.

Indeed, to the contrary assume that there is $(x,\mu) \in \partial B^{(n)}_{r,\varepsilon}$ such that $0 \in G^r_n(x,\mu)$. Then

$$\begin{cases} x \in G_n(x,\mu), \\ \|x\|_C = r. \end{cases}$$

Consequently, $\mu = \pm\varepsilon$. So, inclusion (3.5) has a non-trivial solution (x,μ) where $\mu = \pm\varepsilon$ and $\|x\|_C$ satisfies (3.8). That is a contradiction.
Thus, the topological degree

$$\omega_n = deg(G^r_n, B^{(n)}_{r,\varepsilon})$$

is well-defined and does not depend on the choice of $r > 0$ satisfying (3.9).

STEP 4. Now, we will evaluate ω_n. Toward this goal, consider the multimap

$$\Sigma_n \colon B^{(n)}_{r,\varepsilon} \times [0,1] \to K\big(C(I,H_n) \times \mathbb{R}\big),$$
$$\Sigma_n(x,\mu,\lambda) = \Big\{x - C_n x + (\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \alpha_n\big(\mathbb{P}_n Q(x,\mu),\lambda\big),\ \tau\Big\},$$
$$\tau = \lambda\big(\|x\|^2_C - r^2\big) + (1-\lambda)\big(\varepsilon^2 - \mu^2\big),$$

where $\alpha_n \colon L^2(I,H_n) \times [0,1] \to L^2(I,H_n)$ is defined as

$$\alpha_n\big(\psi^{(n)}_0 + \psi^{(n)}_1, \lambda\big) = \psi^{(n)}_0 + \lambda\psi^{(n)}_1;\ \psi^{(n)}_0 \in \mathcal{L}^{(n)}_0,\ \psi^{(n)}_1 \in \mathcal{L}^{(n)}_1$$

It is easy to see that Σ_n is a compact multivalued $CJ-$vector field. Let us show that

$$0 \notin \Sigma_n(x,\mu,\lambda)$$

for all $(x,\mu,\lambda) \in \partial B^{(n)}_{r,\varepsilon} \times [0,1]$.

To the contrary, assume that there is $(\widetilde{x},\widetilde{\mu},\widetilde{\lambda}) \in \partial B^{(n)}_{r,\varepsilon} \times [0,1]$ such that

$$0 \in \Sigma_n(\widetilde{x},\widetilde{\mu},\widetilde{\lambda}).$$

Then, $\lambda\big(\|x\|^2_C - r^2\big) + (1-\lambda)\big(\varepsilon^2 - \mu^2\big) = 0$, and there exists $\widetilde{q} \in Q(\widetilde{x},\widetilde{\mu})$ such that

$$\begin{cases} L_n\widetilde{x} = \widetilde{\lambda}\widetilde{q}^{(n)}_1 \\ 0 = \widetilde{q}^{(n)}_0, \end{cases}$$

where $\widetilde{q}^{(n)}_0 + \widetilde{q}^{(n)}_1 = \widetilde{q}^{(n)} = \mathbb{P}_n\widetilde{q}$; $\widetilde{q}^{(n)}_0 \in \mathcal{L}^{(n)}_0$ $\widetilde{q}^{(n)}_1 \in \mathcal{L}^{(n)}_1$.
From $(\widetilde{x},\widetilde{\mu}) \in \partial B^{(n)}_{r,\varepsilon}$ it follows that

$$\|\widetilde{x}\|^2_C - r^2 = \varepsilon^2 - \mu^2.$$

Therefore, $\|x\|_C = r$ and $\mu = \pm\varepsilon$.

If $\widetilde{\lambda} > 0$, then from the choice of r and the fact that $\widetilde{x} \in C(I,H_n)$ we have

$$\begin{aligned} 0 < \int_0^T \big\langle \widetilde{\mu}\widetilde{x}(s), \widetilde{q}(s)\big\rangle_H\, ds &= \int_0^T \big\langle \widetilde{\mu}\widetilde{x}(s), P_n\widetilde{q}(s)\big\rangle_H\, ds \\ &= \int_0^T \big\langle \widetilde{\mu}\widetilde{x}(s), \widetilde{q}^{(n)}(s)\big\rangle_H\, ds \\ &= \int_0^T \big\langle \widetilde{\mu}\widetilde{x}(s), \frac{1}{\widetilde{\lambda}}\widetilde{x}'(s)\big\rangle_H\, ds = 0, \end{aligned}$$

giving the contradiction.

If $\widetilde{\lambda} = 0$, then $L_n\widetilde{x} = 0$, i.e. $\widetilde{x}(t) = w \in H_n \cong \mathbb{R}^n$ for all $t \in I$. Since $\|w\|_C = \|w\|_H = r$ and from the choice of r we have

$$\int_0^T \big\langle \widetilde{\mu} w, q(s) \big\rangle_H \, ds > 0,$$

for all $q \in Q(w, \widetilde{\mu})$.
On the other hand

$$\begin{aligned}\int_0^T \big\langle \widetilde{\mu} w, q(s) \big\rangle_H \, ds &= \int_0^T \big\langle \widetilde{\mu} w, P_n q(s) \big\rangle_H \, ds \\ &= T\big\langle \widetilde{\mu} w, \Pi_n q^{(n)} \big\rangle_H \cong T\big\langle \widetilde{\mu} w, \Pi_n q^{(n)} \big\rangle_{\mathbb{R}^n},\end{aligned}$$

where $q^{(n)} = \mathbb{P}_n q \in \mathbb{P}_n Q(w, \widetilde{\mu})$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the inner product in $\mathbb{R}^n$.
Therefore,

$$\big\langle \widetilde{\mu} w, \Pi_n q^{(n)} \big\rangle_{\mathbb{R}^n} > 0, \tag{3.10}$$

and hence, $\Pi_n q^{(n)} \neq 0$. In particular, $\Pi_n \widetilde{q}^{(n)} \neq 0$. But $\Pi_n \widetilde{q}^{(n)} = \Pi_n \widetilde{q}_0^{(n)} = 0$, that is the contradiction.

Thus, Σ_n is a homotopy connecting the vector fields $\Sigma_n(x, \mu, 1) = G_n^r(x, \mu)$ and

$$\Sigma_n(x, \mu, 0) = \{x - C_n x - \Pi_n \mathbb{P}_n Q(x, \mu),\ \varepsilon^2 - \mu^2\}.$$

By virtue of the homotopy invariance property of the topological degree we have

$$deg\big(G_n^r, B_{r,\varepsilon}^{(n)}\big) = deg\big(\Sigma_n(\cdot, \cdot, 0), B_{r,\varepsilon}^{(n)}\big).$$

The operator $C_n + \Pi_n \mathbb{P}_n Q$ takes values in $H_n \cong \mathbb{R}^n$, so

$$deg\big(\Sigma_n(\cdot, \cdot, 0), B_{r,\varepsilon}^{(n)}\big) = deg\big(\Sigma_n(\cdot, \cdot, 0), \overline{U}_{r,\varepsilon}^{(n)}\big),$$

where $\overline{U}_{r,\varepsilon}^{(n)} = B_{r,\varepsilon}^{(n)} \cap (\mathbb{R}^n \times \mathbb{R})$.
In the space $H_n \times \mathbb{R} \cong \mathbb{R}^n \times \mathbb{R}$ the vector field $\Sigma(\cdot, \cdot, 0)$ has the form

$$\Sigma_n(x, \mu, 0) = \big\{-\Pi_n \mathbb{P}_n Q(x, \mu),\ \varepsilon^2 - \mu^2\big\},$$

Consider now the multimap: $\Gamma \colon \overline{U}_{r,\varepsilon}^{(n)} \times [0, 1] \to K(\mathbb{R}^n \times \mathbb{R})$ defined by

$$\Gamma(z, \mu, \lambda) = \big\{-\lambda \Pi_n \mathbb{P}_n Q(z, \mu) + (\lambda - 1)\mu z,\ \varepsilon^2 - \mu^2\big\}.$$

It is clear that Γ is a compact multivalued $CJ-$vector field. Assume that there exists $(z, \mu, \lambda) \in \partial U_{r,\varepsilon}^{(n)} \times [0, 1]$ such that $0 \in \Gamma(z, \mu, \lambda)$. Then there exists $w \in Q(z, \mu)$ such that

$$\begin{cases} \mu = \pm\varepsilon \\ (\lambda - 1)\mu z = \lambda \Pi_n \mathbb{P}_n w, \end{cases}$$

and by virtue of (3.10) we get the contradiction. So, Γ is a homotopy connecting $\Sigma_n(\cdot, \cdot, 0)$ and $V^\sharp(z, \mu) = \{-\mu z, \varepsilon^2 - \mu^2\}$. Therefore,

$$\omega_n = deg\big(\Sigma_n(\cdot, \cdot, 0), \overline{U}_{r,\varepsilon}^{(n)}\big) = deg\big(V^\sharp, \overline{U}_{r,\varepsilon}^{(n)}\big) = \begin{cases} 2 \text{ if } n \text{ is an odd number} \\ 0 \text{ if } n \text{ is an even number.} \end{cases}$$

Since $\omega_{2n-1} \neq 0$ for all $n \in \mathbb{N}$ there exists the corresponding sequence

$$\{(x_{2n-1}, \mu_{2n-1})\} \subset B^{(2n-1)}_{r,\varepsilon}$$

such that

$$\begin{cases} x_{2n-1} \in G_{2n-1}(x_{2n-1}, \mu_{2n-1}) \\ \|x_{2n-1}\|_C = r. \end{cases}$$

Therefore, $(0,0)$ is an A-bifurcation point of problem (3.2). Moreover, relation (3.8) holds true for all $\mu \neq 0$. Thus, $(0,0)$ is a unique A-bifurcation point of problem (3.2).

STEP 5. Now for each $m \in \mathbb{N}$ consider the global structure of solutions of problem (3.3) with $n = 2m - 1$. From $\omega_n \neq 0$ and relation (3.8) it follows that $(0,0)$ is a unique bifurcation point (of usual solutions) of (3.3) (or equivalently, of (3.5)).

Let $\mathcal{O}_n \subset C(I, H_n) \times \mathbb{R}$ be an open subset defined as

$$\mathcal{O}_n = \big(C(I, H_n) \times \mathbb{R}\big) \setminus \big(\{0\} \times \{\mathbb{R} \setminus (-1,1)\}\big).$$

Let us denote by $\mathcal{C}_n \subset \Big(\mathcal{S}_n \cup \{(0,0)\}\Big) \subset \mathcal{O}_n$ the component of $(0,0)$. Assume that $\mathcal{C}_n$ is compact. Then there exists an open bounded subset $U_n \subset \mathcal{O}_n$ such that

$$\overline{U}_n \subset \mathcal{O}_n,\ \mathcal{C}_n \subset U_n \text{ and } \partial U_n \cap \mathcal{S}_n = \emptyset.$$

Further, for every $r, R > 0$ consider the compact multivalued $CJ-$vector field $G_n^{\lambda r + (1-\lambda)R}$ on $\overline{U}_n \times [0,1]$.

Assume that there exists $(x_n, \mu_n, \lambda_n) \in \partial U_n \times [0,1]$ such that

$$0 \in G_n^{\lambda_n r + (1-\lambda_n)R}(x_n, \mu_n, \lambda_n).$$

Then,

$$\begin{cases} x_n \in G_n(x_n, \mu_n) \\ \|x_n\|_C = \lambda_n r + (1-\lambda_n)R, \end{cases}$$

that is the contradiction since $\partial U_n \cap \mathcal{S}_n = \emptyset$.

Therefore, the vector fields G_n^r and G_n^R are homotopic on ∂U_n. For sufficiently large R, G_n^R has no zeros on $\overline{U}_n$, so

$$deg(G_n^R, \overline{U}_n) = 0.$$

Consequently, $deg(G_n^r, \overline{U}_n) = 0$ for every $r > 0$.

Let $\Lambda = \{\mu \in \mathbb{R}\colon (0,\mu) \in \overline{U}_n\}$. From $\overline{U}_n \subset \mathcal{O}_n$ it follows that

$$\Lambda \subset (-1,1). \tag{3.11}$$

Now choose sufficiently small $r, \varepsilon > 0$ such that $B^{(n)}_{r,\varepsilon} \subset U_n$ and

$$x \notin G_n(x,\mu) \ \ \textit{provided} \ \ x \in B^{(n)}_C(0,r) \setminus \{0\},$$

for all $\mu \in (-1,1) \setminus (-\varepsilon, \varepsilon)$, where $B^{(n)}_C(0,r) = B_C(0,r) \cap C(I,H_n)$. From (3.11) and the choice of r, ε we have that

$$\Big\{(x,\mu) : (x,\mu) \in \overline{U}_n \ \ \textit{and} \ \ 0 \in G_n^r(x,\mu)\Big\} \subset B^{(n)}_{r,\varepsilon}.$$

So, we obtain

$$deg(G_n^r, B^{(n)}_{r,\varepsilon}) = deg(G_n^r, \overline{U}_n) = 0,$$

that is the contradiction.
Thus, $\mathcal{C}_n$ is a non-compact component, i.e. either $\mathcal{C}_n$ is unbounded or $\overline{\mathcal{C}_n} \cap \overline{\mathcal{O}_n} \neq \emptyset$. Since $(0,0)$ is the unique bifurcation point of (3.3), there is only one possible case, that is $\mathcal{C}_n$ is unbounded.
Remark 3.6. Theorem 3.5 gives us an important information about the bifurcation of approximable solutions in problem (3.2). If there is a bifurcation point of approximable solutions of this problem, then it should be $(0,0)$. For example, if problem (3.2) has the $A-$properness property, then $(0,0)$ is the unique bifurcation point of approximable solutions to (3.2) and there exists an unbounded connected subset of non-trivial approximable solutions of (3.2) which bifurcates from $(0,0)$. Moreover, an A-bifurcation point can be a bifurcation point of usual solutions. For example, if $Aw = aw, \forall w \in H$, and there is an odd number n such that $f(t,w,z,\mu) \in H_n$ for all $(t,w,z,\mu) \in I \times H_n \times Y \times \mathbb{R}$, then $(0,0)$ is a *bifurcation point* of (3.2) and the unbounded branch $\mathcal{C}_n$ is a branch of non-trivial solutions of (3.2).

4. Application to a parameterized integro - differential system

Consider the following parameterized integro-differential system

$$\begin{cases} \frac{\partial u(t,s)}{\partial t} = \mu a u(t,s) + \int_0^1 K(t,s)u(t,\sigma)\big(\mu + v(t,\sigma)\big)d\sigma, \\ \frac{\partial v(t,s)}{\partial t} = \int_0^1 G(t,s)\big(u(t,\sigma) + v(t,\sigma) + \mu\big)d\sigma, \\ u(0,s) = u(1,s);\ v(0,s) = 0, \end{cases} \tag{4.1}$$

where the first and the second equations hold for for a.e. $t \in [0,1]$ and all $s \in [0,1]$; $a > 0$; and $K, G\colon [0,1] \times [0,1] \to \mathbb{R}$ are continuous maps.

Let us denote by $Y = C[0,1]$ and $H = W^{1,2}[0,1]$. It is clear that Y is a Banach space, H is a separable Hilbert space, the embedding $H \hookrightarrow Y$ is compact and $\|z\|_Y \leq \|z\|_H$ for all $z \in H$.

By a solution to problem (4.1) we mean a triplet (u,v,μ) consisting of continuous functions $u, v\colon [0,1] \times [0,1] \to \mathbb{R}$ whose partial derivatives $\frac{\partial u(t,s)}{\partial t}$ and $\frac{\partial v(t,s)}{\partial t}$ exist and satisfy (4.1). Moreover, we can consider relations (4.1) as the law of evolution of a feedback control system with the *state function* $u(t,s)$ and the *control function* $v(t,s)$. Our goal can be formulated as the finding of the state and control as continuous functions $u(t,s)$ and $v(t,s)$ such that at every moment t the functions $u(t,\cdot)$ and $v(t,\cdot)$ belong to the Sobolev space $W^{1,2}[0,1]$.

An equivalent approach to (4.1) is the following: by a solution to (4.1) we mean a pair (u,μ) consisting of a continuous function u and a number μ, for which there is a continuous function v such that the triplet (u,v,μ) satifies (4.1).

It is easy to see that $(0,\mu)$ are solutions to (4.1) for all $\mu \in \mathbb{R}$. Let us denote by $\mathcal{S}$ the set of all non-trivial solutions of (4.1), i.e., the set of all solutions (u,μ) of (4.1) such that $u \neq 0$. We will study the global structure of $\mathcal{S}$ under the following hypotheses:

- $(A1)$ for every $t \in [0,1]$ the partial derivatives $\frac{\partial K(t,s)}{\partial s}$ and $\frac{\partial G(t,s)}{\partial s}$ exist for a.e. $s \in [0,1]$ and belong to space $L^2[0,1]$;

$(A2)$ $a > K(1 + Ge^G)$, where

$$K = \max_{t,s\in[0,1]} K(t,s) \text{ and } G = \max_{t,s\in[0,1]} G(t,s).$$

For each $t \in [0,1]$ let us denote by $x(t) = u(t,\cdot)$ and $y(t) = v(t,\cdot)$. Replace (4.1) with the following problem

$$\begin{cases} x'(t) = \mu Ax(t) + f(t,x(t),y(t),\mu), \text{ for a.e. } t \in [0,1], \\ y'(t) = g(t,x(t),y(t),\mu), \text{ for a.e. } t \in [0,1], \\ x(0) = x(1),\ y(0) = 0, \end{cases} \tag{4.2}$$

where $A\colon H \to H,\ Aw = aw$,

$$f\colon [0,1] \times H \times Y \times \mathbb{R} \to H,$$

$$f(t,w,z,\mu)(s) = \int_0^1 K(t,s)w(\sigma)\big(\mu + z(\sigma)\big)d\sigma,$$

and

$$g\colon [0,1] \times H \times Y \times \mathbb{R} \to H,$$

$$g(t,w,z,\mu) = \int_0^1 G(t,s)\big(w(\sigma) + z(\sigma) + \mu\big)d\sigma.$$

Notice that if (x,y,μ) is a solution to (4.2), then the corresponding triplet (u,v,μ) is a solution to (4.1).

It is easy to verify that the maps A, f and g satisfy conditions (A), (f) and (g), respectively. Applying Theorem 3.5 and Remark 3.6 we obtain the following assertion.

Theorem 4.1. *Let conditions* $(A1)-(A2)$ *hold. Then* $(0,0)$ *is a unique A-bifurcation point of problem* (4.1)*. Moereover, if for all* $t \in [0,1]$ *the function* $K(t,\cdot)$ *takes values in* H_n *with* n *an odd number, then* $(0,0)$ *is a bifurcation point of usual solutions of problem* (4.1) *and there exists a unbounded connected subset* $\mathcal{R} \subset \mathcal{S}$ *such that* $(0,0) \in \overline{\mathcal{R}}$.

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