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ON THE ROBUSTNESS OF EXPONENTIAL DICHOTOMIES

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Abstract. For linear evolution families in Banach spaces, we establish the robustness of a weaker notion of exponential dichotomy. This notion allows the contraction and expansion along the stable and unstable subspaces to be nonuniform and requires nothing from the angles between these subspaces.

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1. Introduction

The notion of an exponential dichotomy plays an important role in a substantial part of the theory of differential equations and dynamical systems, most notably in what concerns topological conjugacies and invariant manifolds. The local instability of the trajectories caused by the existence of an exponential dichotomy also influences the global behavior of the system and together with the nontrivial recurrence caused by the existence of a finite invariant measure turns out to be one of the main mechanisms for the occurrence of stochastic behavior. Due to the important role played by the notion of an exponential dichotomy it is important to understand how exponential dichotomies vary under perturbations. In particular, it is well know that they are robust. That is, any sufficiently small linear perturbation of an exponential dichotomy is still an exponential dichotomy. We refer to the books [4, 9, 15] for details and references on the theory of exponential dichotomies and its applications.

Nevertheless, the existence of an exponential dichotomy is often a stringent requirement for the dynamics and it is of interest to look for more general types of hyperbolic behavior. The notion of *nonuniform* exponential dichotomy turns out to be much more typical, in particular in view of its ubiquity in the context of ergodic theory. We refer to the books [1, 3] for details and reference on the measure-theoretical and nonuniform parts of the theory. It is also of utmost interest to obtain information on the persistence of the hyperbolic behavior under sufficiently small linear perturbations. While in general the Lyapunov exponents need not be continuous on the dynamics (see [1]), it is nevertheless possible to give reasonable assumptions under which one can establish a corresponding robustness property (see [3] for details, based on our work [2]).

In this paper, we would like to propose a slightly different and in fact broader perspective of the robustness problem. As a preliminary discussion, let us return to the notion of (uniform) exponential dichotomy. The main requirement of this notion is the existence of contraction and expansion bounds along subspaces that together generate the whole space. But there is a second aspect, many times crucial, that often is not stressed sufficiently: the angles between the stable and unstable spaces (or the norms of the corresponding projections, in the infinite-dimensional setting) are uniformly bounded away from zero. Indeed, the continuity of these subspaces on the base point guarantees automatically that the angles between them are bounded away from zero on compact hyperbolic sets and so along any trajectory in such sets. We note that the fact that the angle is bounded away from zero (together with a lower bound on the size of the stable and unstable manifolds) is the starting point for the construction of a local product structure and so of Markov partitions. However, this is not necessarily the case in noncompact invariant sets, such as for example when one considers geodesic flows on certain noncompact factors of manifolds of nonconstant negative curvature.

The outcome of the above discussion is that it is equally possible that:

(1) the contraction and expansion along the stable and unstable spaces are not uniform (which is the general situation even when all Lyapunov exponents are nonzero with respect to an ergodic invariant measure);
(2) the angle between the stable and unstable spaces goes to zero at some points (in the context of ergodic theory the angles go to zero at most subexponentially for almost all trajectories, although in a zero measure set that may be dense and have full topological entropy the angle can in principle go to zero at any prescribed speed).

In our former work [2], we established a version of the robustness property when the expansion and contraction rates may deteriorate exponentially and simultaneously the angles may go to zero exponentially (as explained before, this setting is strongly motivated by results of ergodic theory).

In contrast, in the present work, our main aim is to effect a corresponding study when the angle between the stable and unstable spaces is not known a priori. More precisely, we still require that there exist contraction and expansion along stable and unstable subspaces but no requirement is made on the relative position of these spaces. Certainly, there is less to prove in the corresponding robustness property since now we only need to show that any sufficiently small linear perturbation has again contraction and contraction along (perturbed) stable and unstable subspaces. On the other hand, one cannot use any a priori knowledge about the angles between the original stable and unstable spaces. We emphasize that this information was used in a crucial manner in [2] and, to the best of our knowledge, either in this manner or in some equivalent form (such as in terms of norms of projections) in all former works on the robustness property in the uniform case. Our proofs exhibit the dichotomies of the perturbed dynamics as explicitly as possible, in terms of fixed points of appropriate contractions. Some of our arguments are inspired by work of Popescu in [14] in the uniform case.

In the case of uniform exponential dichotomies the study of robustness has a long history. In particular, it was discussed by Massera and Schäffer [8] (building on earlier work of Perron [12]; see also [9]), Coppel [6], and in the case of Banach spaces by Dalec′kiĭ and Kreĭn [7], with different approaches and successive generalizations. The continuous dependence of the projections for the exponential dichotomies of the perturbed equations was obtained by Palmer [11]. For more recent works we refer to [5, 10, 13, 14] and the references therein.

2. Robustness of exponential dichotomies

In this section we establish the robustness of a weak version of exponential dichotomy. We consider the general case of a linear dynamics that need not be invertible.

We denote by $\mathcal{B}(X)$ the space of bounded linear operators in a Banach space X. Let $T(t,s)$ be an evolution family of linear operators in $\mathcal{B}(X)$ for $t,s \in \mathbb{R}$ with $t \geq s$. This means that:

(1) $T(t,t) = \mathrm{Id}$ and
$$T(t,\tau)T(\tau,s) = T(t,s), \quad t \geq \tau \geq s;$$
(2) $(t,s,x) \mapsto T(t,s)x$ is continuous in $\{(t,s,x) \in \mathbb{R}^2 \times X : t \geq s\}$.

The evolution family $T(t,s)$ is said to admit a *nonuniform dichotomy* if:

(1) there exist decompositions $X = E(t) \oplus F(t)$ for each $t \in \mathbb{R}$ satisfying
$$T(t,s)E(s) = E(t) \quad \text{and} \quad T(t,s)F(s) = F(t) \tag{2.1}$$
for $t \geq s$, such that the map
$$\overline{T}(t,s) := T(t,s)|F(s) \colon F(s) \to F(t)$$
is invertible for each $t \geq s$;

(2) there exist constants $\lambda, D > 0$ and $a \geq 0$ such that
$$\|T(t,s)y\| \leq De^{-\lambda(t-s)+a|s|}\|y\|, \quad t \geq s, \ y \in E(s)$$
and
$$\|T(t,s)z\| \leq De^{-\lambda(s-t)+a|s|}\|z\|, \quad s \geq t, \ z \in F(s),$$
where
$$T(t,s) = \overline{T}(s,t)^{-1}|F(s), \quad t \leq s.$$

Now let $P(t)$ and $Q(t) = \mathrm{Id} - P(t)$ be the projections associated respectively to the spaces $E(t)$ and $F(t)$ in the decomposition $X = E(t) \oplus F(t)$. It follows readily from (2.1) that
$$T(t,s)P(s) = P(t)T(t,s), \quad t \geq s.$$

We also consider the perturbed equation
$$u(t) = T(t,s)u(s) + \int_s^t T(t,\tau)B(\tau)u(\tau)\,d\tau, \quad t \geq s, \tag{2.2}$$

for some function $B\colon \mathbb{R} \to \mathcal{B}(X)$ such that $t \mapsto B(t)x$ is continuous for each $x \in X$. We always assume that equation (2.2) defines an evolution family $\hat{T}(t,s)$ of bounded linear operators.

The following is our robustness result for nonuniform dichotomies.

Theorem 2.1. *Let $T(t,s)$ be an evolution family admitting a nonuniform dichotomy with $\lambda > 2a > 0$ and assume that*

$$\|B(t)\| \leq \frac{\delta e^{-2a|t|}}{\|P(t)\| + \|Q(t)\|}, \quad t \in \mathbb{R}.$$

If δ is sufficiently small, then the evolution family $\hat{T}(t,s)$ defined by equation (2.2) *admits a nonuniform dichotomy, with the constants λ and a replaced respectively by λ and $2a$.*

Proof. We divide the proof of the theorem into several steps.

Step 1. Construction of bounded solutions I. We first construct bounded solutions into the future. Given $s \in \mathbb{R}$, let

$$I_s = \{t \in \mathbb{R} : t \geq s\}.$$

We consider the Banach space

$$\mathcal{C} = \big\{U\colon I_s \to \mathcal{B}(X) \text{ continuous: } U(t)Q(s) = 0 \text{ for } t \geq s \text{ and } \|U\| < +\infty\big\},$$

with the norm

$$\|U\| = \sup\big\{\|U(t)\|e^{\lambda(t-s)-a|s|} : t \in I_s\big\},$$

where $\mathcal{B}(E(s), X)$ is the space of bounded linear operators from $E(s)$ to X.

Lemma 2.2. *If δ is sufficiently small, then for each $s \in \mathbb{R}$ there exists a unique $U = U_s \in \mathcal{C}$ such that*

$$\begin{aligned} U(t) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau)\,d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau)\,d\tau \end{aligned} \tag{2.3}$$

for $t \in I_s$. Moreover, for each $\xi \in X$ the function

$$u(t) = U(t)\xi = U(t)P(s)\xi, \quad t \geq s$$

is a solution of equation (2.2).

Proof of the lemma. We show that the operator L defined for each $U \in \mathcal{C}$ by

$$\begin{aligned} (LU)(t) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau)\,d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau)\,d\tau \end{aligned} \tag{2.4}$$

has a unique fixed point in $\mathcal{C}$. We first note that

$$\int_t^\infty \|T(t,\tau)Q(\tau)B(\tau)U(\tau)\|\,d\tau \le D\delta e^{-\lambda(t-s)+a|s|}\|U\| \int_t^\infty e^{-2\lambda(\tau-t)}\,d\tau$$
$$= \frac{D}{2\lambda}\delta e^{-\lambda(t-s)+a|s|}\|U\| < +\infty.$$

Therefore, $(LU)(t)$ is well defined and

$$\begin{aligned}
\|(LU)(t)\| &\le \|T(t,s)|E(s)\|\cdot\|P(s)\| + \int_s^t \|T(t,\tau)P(\tau)\|\cdot\|B(\tau)\|\cdot\|U(\tau)\|\,d\tau \\
&\quad + \int_t^\infty \|T(t,\tau)Q(\tau)\|\cdot\|B(\tau)\|\cdot\|U(\tau)\|\,d\tau \\
&\le De^{-\lambda(t-s)+a|s|}\|P(s)\| + D\delta e^{-\lambda(t-s)+a|s|}\|U\|\int_s^t e^{-2a|\tau|}\,d\tau \\
&\quad + D\delta e^{-\lambda(t-s)+a|s|}\|U\|\int_t^\infty e^{-2\lambda(\tau-t)}\,d\tau \\
&\le De^{-\lambda(t-s)+a|s|}\|P(s)\| + \frac{D}{a}\delta e^{-\lambda(t-s)+a|s|}\|U\| \\
&\quad + \frac{D}{2\lambda}\delta e^{-\lambda(t-s)+a|s|}\|U\|.
\end{aligned} \tag{2.5}$$

This implies that

$$\|LU\| \le D\|P(s)\| + \delta D\Big(\frac{1}{a} + \frac{1}{2\lambda}\Big)\|U\| < +\infty, \tag{2.6}$$

and we obtain a well defined operator $L\colon \mathcal{C} \to \mathcal{C}$. Using (2.4) and proceeding in a similar manner to that in (2.5), we also obtain

$$\|LU_1 - LU_2\| \le \delta D\Big(\frac{1}{a} + \frac{1}{2\lambda}\Big)\|U_1 - U_2\|$$

for $U_1, U_2 \in \mathcal{C}$. Therefore, for any sufficiently small δ the operator L is a contraction and there exists a unique $U_s \in \mathcal{C}$ such that $LU_s = U_s$.

Finally, we note that

$$\begin{aligned}
U(t) - T(t,s)U(s) &= T(t,s)P(s) - T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau)\,d\tau \\
&\quad + \int_s^t T(t,\tau)Q(\tau)B(\tau)U(\tau)\,d\tau \\
&= \int_s^t T(t,\tau)B(\tau)U(\tau)\,d\tau
\end{aligned}$$

for each $t \ge s$. This completes the proof of the lemma. □

Now let $\bar U(t,s)\colon X \to X$ be the linear operator $\bar U(t,s) = U_s(t)$.

Lemma 2.3. *If δ is sufficiently small, then*

$$\bar U(t,\tau)\bar U(\tau,s) = \bar U(t,s), \quad t \ge \tau \ge s.$$

Proof of the lemma. We first note that

$$\begin{aligned}\bar{U}(t,\tau)\bar{U}(\tau,s) = T(t,s)P(s) &+ \int_s^\tau T(t,\sigma)P(\sigma)B(\sigma)\bar{U}(\sigma,\tau)\bar{U}(\tau,s)\,d\sigma \\ &+ \int_\tau^t T(t,\sigma)P(\sigma)B(\sigma)\bar{U}(\sigma,\tau)\bar{U}(\tau,s)\,d\sigma \\ &- \int_t^\infty T(t,\sigma)Q(\sigma)B(\sigma)\bar{U}(\sigma,\tau)\bar{U}(\tau,s)\,d\sigma.\end{aligned}$$

Given $\tau, s \in \mathbb{R}$ with $\tau \geq s$, we consider the Banach space

$$\mathcal{C}_\tau = \big\{H\colon I_\tau \to \mathcal{B}(X) : H \text{ is continuous and } \|H\|_\tau < +\infty\big\},$$

with the norm

$$\|H\|_\tau = \sup\big\{\|H(t)\|e^{-2a|t|} : t \in I_\tau\big\}.$$

Writing

$$h(t) = \bar{U}(t,\tau)\bar{U}(\tau,s) - \bar{U}(t,s)$$

for $t \geq \tau$, one can show that $L_1 h = h$, where

$$(L_1 H)(t) = \int_\tau^t T(t,\sigma)P(\sigma)B(\sigma)H(\sigma)\,d\sigma - \int_t^\infty T(t,\sigma)Q(\sigma)B(\sigma)H(\sigma)\,d\sigma.$$

for each $H \in \mathcal{C}_\tau$ and $t \in I_\tau$. We have

$$\begin{aligned}&\int_\tau^t \|T(t,\sigma)P(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H(\sigma)\|\,d\sigma \\ &+ \int_t^\infty \|T(t,\sigma)Q(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H(\sigma)\|\,d\sigma \\ &\leq \frac{D}{\lambda}\delta\|H\|_\tau + \frac{D}{\lambda}\delta\|H\|_\tau = \frac{2D}{\lambda}\delta\|H\|_\tau.\end{aligned}$$

This shows that $(L_1 H)(t)$ is well defined and that

$$\|L_1 H\|_\tau \leq \frac{2D}{\lambda}\delta\|H\|_\tau < +\infty.$$

We thus obtain an operator $L_1\colon \mathcal{C}_\tau \to \mathcal{C}_\tau$. Moreover, for each $H_1, H_2 \in \mathcal{C}_\tau$ and $t \geq \tau$, we have

$$\begin{aligned}\|(L_1 H_1)(t) - (L_1 H_2)(t)\| &\leq \int_\tau^t \|T(t,\sigma)P(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H_1(\sigma) - H_2(\sigma)\|\,d\sigma \\ &+ \int_t^\infty \|T(t,\sigma)Q(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H_1(\sigma) - H_2(\sigma)\|\,d\sigma \\ &\leq \frac{D}{\lambda}\delta\|H_1 - H_2\|_\tau + \frac{D}{\lambda}\delta\|H_1 - H_2\|_\tau = \frac{2D}{\lambda}\delta\|H_1 - H_2\|_\tau.\end{aligned}$$

Therefore,

$$\|L_1 H_1 - L_1 H_2\|_\tau \leq \frac{2D}{\lambda}\delta\|H_1 - H_2\|_\tau.$$

This shows that for δ sufficiently small the operator L_1 is a contraction, and hence, there exists a unique $H \in \mathcal{C}_\tau$ such that $L_1 H = H$. Since $0 \in \mathcal{C}_\tau$ also satisfies this identity, we have $H = 0$. Now we show that $h \in \mathcal{C}_\tau$. It follows from Lemma 2.2 that

$$\begin{aligned}
\|\bar{U}(t,\tau)\bar{U}(\tau,s)\| &\leq \|\bar{U}(t,\tau)\| \cdot \|\bar{U}(\tau,s)\| \\
&\leq \|U\|^2 e^{-\lambda(t-s)+a(|\tau|+|s|)} \\
&\leq \|U\|^2 e^{-\lambda(t-s)} e^{a(t-\tau)} e^{a(t-s)} e^{2a|t|} \\
&\leq \|U\|^2 e^{(2a-\lambda)(t-s)} e^{2a|t|}
\end{aligned}$$

and

$$\|\bar{U}(t,s)\| \leq \|U\| e^{-\lambda(t-s)+a|s|} \leq \|U\| e^{(a-\lambda)(t-s)+a|t|} \leq \|U\| e^{2a|t|}$$

for $t \geq \tau \geq s$. Since $\lambda \geq 2a$, this shows that $h \in \mathcal{C}_\tau$ and it follows from the uniqueness of the fixed point of L_1 that $h = 0$. ☐

Step 2. Construction of bounded solutions II. Now we construct bounded solutions into the past. Consider the set $J_s = \{t \in \mathbb{R} : t \leq s\}$ and the Banach space

$$\mathcal{D} = \bigl\{V : J_s \to \mathcal{B}(X) \text{ continuous: } V(t)P(s) = 0 \text{ for } t \leq s \text{ and } \|V\| < +\infty\bigr\},$$

with the norm

$$\|V\| = \sup\bigl\{\|V(t)\| e^{-\lambda(t-s)-a|s|} : t \in J_s\bigr\}.$$

Lemma 2.4. *If δ is sufficiently small, then for each $s \in \mathbb{R}$ there exists a unique $V = V_s \in \mathcal{D}$ such that*

$$\begin{aligned}
V(t) = T(t,s)Q(s) &+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau)\,d\tau \\
&- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau)\,d\tau.
\end{aligned} \tag{2.7}$$

for $t \in J_s$. Moreover, for each $\xi \in X$ the function

$$V(t) = V(t)\xi = V(t)Q(s)\xi, \quad t \leq s$$

is a solution of equation (2.2).

Proof of the lemma. We show that the operator M defined for each $V \in \mathcal{D}$ by

$$\begin{aligned}
(MV)(t) = T(t,s)Q(s) &+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau)\,d\tau \\
&- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau)\,d\tau
\end{aligned} \tag{2.8}$$

has a unique fixed point in $\mathcal{D}$. We first note that

$$\begin{aligned}
\int_{-\infty}^{t} \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau)\|\,d\tau &\leq D\delta e^{\lambda(t-s)+a|s|}\|V\| \int_{-\infty}^{t} e^{2\lambda(\tau-t)}\,d\tau \\
&\leq \frac{D}{2\lambda}\delta e^{\lambda(t-s)+a|s|}\|V\|.
\end{aligned}$$

Therefore, $(MV)(t)$ is well defined, and

$$\begin{aligned}\|(MV)(t)\| &\le \|T(t,s)|F(s)\| \cdot \|Q(s)\| + \int_{-\infty}^{t} \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau)\| \, d\tau \\ &+ \int_{t}^{s} \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau)\| \, d\tau \\ &\le De^{\lambda(t-s)+a|s|}\|Q(s)\| + D\delta e^{\lambda(t-s)+a|s|}\|V\| \int_{-\infty}^{t} e^{2\lambda(\tau-t)} \, d\tau \\ &+ D\delta e^{\lambda(t-s)+a|s|}\|V\| \int_{t}^{s} e^{-2a|\tau|} \, d\tau \\ &\le De^{\lambda(t-s)+a|s|}\|Q(s)\| + \frac{D}{2\lambda}\delta e^{\lambda(t-s)+as}\|V\| \\ &+ \frac{D}{a}\delta e^{\lambda(t-s)+a|s|}\|V\|.\end{aligned} \tag{2.9}$$

This implies that

$$\|MV\| \le D\|Q(s)\| + \delta D\Big(\frac{1}{2\lambda} + \frac{1}{a}\Big)\|V\| < +\infty, \tag{2.10}$$

and we obtain a well defined operator $M\colon \mathcal{D} \to \mathcal{D}$. Using (2.8) and proceeding in a similar manner to that in (2.9), we also obtain

$$\|MV_1 - MV_2\| \le \delta D\Big(\frac{1}{2\lambda} + \frac{1}{a}\Big)\|V_1 - V_2\|$$

for $V_1, V_2 \in \mathcal{D}$. Therefore, for any sufficiently small δ the operator M is a contraction and there exists a unique $V_s \in \mathcal{D}$ such that $MV_s = V_s$.

Moreover, we have

$$\begin{aligned}V(s) - T(s,t)V(t) &= Q(s) + \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau) \, d\tau - T(s,t)T(t,s)Q(s) \\ &-T(s,t)\bigg(\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau) \, d\tau + \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau) \, d\tau\bigg) \\ &= Q(s) + \int_{-\infty}^{t} T(s,\tau)P(\tau)B(\tau)V(\tau) \, d\tau + \int_{t}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau) \, d\tau \\ &-Q(s) - \int_{-\infty}^{t} T(s,\tau)P(\tau)B(\tau)V(\tau) \, d\tau + \int_{t}^{s} T(s,\tau)Q(\tau)B(\tau)V(\tau) \, d\tau \\ &= \int_{t}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau) \, d\tau + \int_{t}^{s} T(s,\tau)Q(\tau)B(\tau)V(\tau) \, d\tau \\ &= \int_{t}^{s} T(s,\tau)B(\tau)V(\tau) \, d\tau\end{aligned}$$

for each $t \le s$. This completes the proof of the lemma. ☐

Now let $\bar{V}(t,s)\colon X \to X$ be the linear operator $\bar{V}(t,s) = V_s(t)$.

Lemma 2.5. *If δ is sufficiently small, then*

$$\bar V(t,\tau)\bar V(\tau,s)=\bar V(t,s), \quad t\le\tau\le s.$$

Proof of the lemma. The argument is analogous to that in the proof of Lemma 2.3. We have

$$\begin{aligned}\bar V(t,\tau)\bar V(\tau,s)&=T(t,s)Q(s)-\int_\tau^s T(t,\sigma)Q(\sigma)B(\sigma)\bar V(\sigma,s)\,d\sigma\\&\quad+\int_{-\infty}^t T(t,\sigma)P(\sigma)B(\sigma)\bar V(\sigma,\tau)\bar V(\tau,s)\,d\sigma\\&\quad-\int_t^\tau T(t,\sigma)Q(\sigma)B(\sigma)\bar V(\sigma,\tau)\bar V(\tau,s)\,d\sigma.\end{aligned}$$

Given $\tau,s\in\mathbb{R}$ with $\tau\le s$, consider the Banach space

$$\mathcal{D}_\tau=\big\{\bar H\colon J_\tau\to\mathcal{B}(X):\bar H\text{ is continuous and }\|\bar H\|_\tau<+\infty\big\},$$

with the norm

$$\|\bar H\|_\tau=\sup\big\{\|\bar H(t)\|e^{-2a|t|}:t\in J_\tau\big\}.$$

Writing

$$\bar h(t)=\bar V(t,\tau)\bar V(\tau,s)-\bar V(t,s)$$

for $t\le\tau$, one can show that $M_1\bar h=\bar h$, where

$$(M_1\bar H)(t)=\int_{-\infty}^t T(t,\sigma)P(\sigma)B(\sigma)\bar H(\sigma)\,d\sigma-\int_t^\tau T(t,\sigma)Q(\sigma)B(\sigma)\bar H(\sigma)\,d\sigma$$

for each $\bar H\in\mathcal{D}_\tau$ and $t\in J_\tau$. Proceeding in a similar manner to that in the proof of Lemma 2.3, one can show that 0 is the unique fixed point of M_1 in $\mathcal{D}_\tau$. Since $\bar h\in\mathcal{D}_\tau$, we conclude that $\bar h=0$. □

Step 3. Characterization of the bounded solutions. In the following two lemmas we show that all bounded solutions of equation (2.2) with a certain growth are those constructed in Lemmas 2.2 and 2.4.

We start with the solutions into the future.

Lemma 2.6. *Given $s\in\mathbb{R}$, if $y\colon[s,+\infty)\to X$ is a bounded solution of equation* (2.2) *with $y(s)=\xi$ and $t\mapsto Q(t)y(t)$ is bounded for $t\ge s$, then $y(t)=\bar U(t,s)\xi$ for $t\ge s$.*

Proof of the lemma. For each $p\ge t\ge s$, we have

$$P(t)y(t)=T(t,s)P(s)\xi+\int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{2.11}$$

and

$$Q(p)y(p)=T(p,t)Q(t)y(t)+\int_t^p T(p,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

The last identity can be written in the form

$$Q(t)y(t)=T(t,p)Q(p)y(p)-\int_t^p T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{2.12}$$

Since $Q(t)y(t)$ is bounded, we have

$$\|T(t,p)Q(p)y(p)\| \le CDe^{-\lambda(p-t)+a|p|}$$

for some constant $C>0$. Since $\lambda > 2a$, taking limits in (2.12) when $p\to+\infty$, we obtain

$$Q(t)y(t) = -\int_t^{+\infty} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

Adding this identity to (2.11) yields the identity

$$\begin{aligned} y(t) &= T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \\ &\quad - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau \end{aligned} \tag{2.13}$$

for $t\ge s$. Now let $z(t)=y(t)-\bar U(t,s)\xi$. It follows from Lemma 2.2 and (2.13) that

$$\begin{aligned} z(t) &= -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)\big(y(\tau)-\bar U(\tau,s)\xi\big)\,d\tau \\ &= -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)z(\tau)\,d\tau. \end{aligned}$$

Letting

$$\|z\|_\infty = \sup_{t\ge s}\|z(t)\|,$$

we obtain

$$\begin{aligned} \|z(t)\| &\le \int_t^\infty \|T(t,\tau)|F(\tau)\|\cdot\|Q(\tau)\|\cdot\|B(\tau)\|\cdot\|z(\tau)\|\,d\tau \\ &\le \delta D\|z\|_\infty \int_t^\infty e^{-\lambda(\tau-t)+a|\tau|}e^{-2a|\tau|}\,d\tau \\ &\le \frac{\delta D}{\lambda}\|z\|_\infty \end{aligned}$$

and thus,

$$\|z\|_\infty \le \frac{\delta D}{\lambda}\|z\|_\infty.$$

Taking δ sufficiently small, we conclude that $\|z\|_\infty=0$ and $z(t)=0$ for all $t\ge s$. This yields that $y(t)=\bar U(t,s)\xi$ for $t\ge s$. □

Now we consider the solutions into the past.

Lemma 2.7. *Given $s\in\mathbb{R}$, if $y\colon(-\infty,s]\to X$ is a bounded solution of equation* (2.2) *with $y(s)=\xi$ and $t\mapsto P(t)y(t)$ is bounded for $t\le s$, then $y(t)=\bar V(t,s)\xi$ for $t\le s$.*

Proof of the lemma. For each $p\le t\le s$, we have

$$P(t)x(t) = T(t,p)P(p)y(p) + \int_p^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{2.14}$$

and

$$Q(s)\xi = T(s,t)Q(t)y(t) + \int_t^s T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{2.15}$$

Since

$$\|T(t,p)P(p)y(p)\| \leq CDe^{-\lambda(t-p)+a|p|},$$

for some constant $C > 0$, and $\lambda > 2a$, taking limits in (2.14) when $p \to -\infty$, we obtain

$$P(t)y(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau. \tag{2.16}$$

On the other hand, by (2.15), we have

$$Q(t)y(t) = T(t,s)Q(s)\xi - \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{2.17}$$

Adding (2.16) and (2.17) yields the identity

$$\begin{aligned} y(t) = T(t,s)Q(s)\xi &+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \\ &- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau \end{aligned}$$

for $t \leq s$. Now let $w(t) = y(t) - \bar{V}(t,s)\xi$. It follows from Lemma 2.4 that

$$\begin{aligned} w(t) &= \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)\big(y(\tau) - \bar{V}(\tau,s)\xi\big)\,d\tau \\ &= \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)w(\tau)\,d\tau. \end{aligned}$$

Letting

$$\|w\|_\infty = \sup_{t\leq s} \|w(t)\|,$$

we obtain

$$\begin{aligned} \|w(t)\| &\leq \int_{-\infty}^{t} \|T(t,\tau)|E(\tau)\| \cdot \|P(\tau)\| \cdot \|B(\tau)\| \cdot \|w(\tau)\|\,d\tau \\ &\leq \delta D\|w\|_\infty \int_{-\infty}^{t} e^{-\lambda(t-\tau)+a|\tau|}e^{-2a|\tau|}\,d\tau \\ &\leq \frac{\delta D}{\lambda}\|w\|_\infty \end{aligned}$$

and thus,

$$\|w\|_\infty \leq \frac{\delta D}{\lambda}\|w\|_\infty.$$

Taking δ sufficiently small, we conclude that $\|w\|_\infty = 0$ and $w(t) = 0$ for all $t \leq s$. This yields that $y(t) = \bar{V}(t,s)\xi$ for $t \leq s$. $\square$

Step 4. Construction of invariant subspaces. Now we construct stable and unstable invariant subspaces for the perturbed equation. For this we observe that the stable and unstable subspaces should correspond to the bounded solutions respectively into the future and into the past. However, since the perturbed dynamics need not be invertible this requires a special care when establishing the invariance of the subspaces.

For each $t \in \mathbb{R}$, we consider the linear subspaces

$$\hat{E}(t) = \operatorname{Im} \bar{U}(t,t) \quad \text{and} \quad \hat{F}(t) = \operatorname{Im} \bar{V}(t,t).$$

Lemma 2.8. *For each $t, s \in \mathbb{R}$ with $t \geq s$, we have*

$$\hat{E}(t) = \hat{T}(t,s)\hat{E}(s) \quad \textit{and} \quad \hat{F}(t) = \hat{T}(t,s)\hat{F}(s),$$

provided that δ is sufficiently small.

Proof of the lemma. By Lemma 2.2, for each $\xi \in X$ the function $\bar{U}(t,s)\xi$, $t \geq s$ is a solution of equation (2.2) with initial condition at time s equal to $\bar{U}(s,s)\xi$. Therefore, $\bar{U}(t,s) = \hat{T}(t,s)\bar{U}(s,s)$, where $\hat{T}(t,s)$ is the evolution operator associated to equation (2.2). Hence, by Lemma 2.3,

$$\begin{aligned}
\hat{T}(t,s)\hat{E}(s) &= \operatorname{Im} \bar{U}(t,s) \\
&= \operatorname{Im} \big(\bar{U}(t,t)\bar{U}(t,s)\big) \\
&= \bar{U}(t,t) \operatorname{Im} \bar{U}(t,s) \subset \hat{E}(t)
\end{aligned}$$

for $t \geq s$. Similarly, by Lemma 2.4, the function $\bar{V}(t,s)\xi$, $t \leq s$ is a solution of equation (2.2), and hence,

$$\bar{V}(s,s) = \hat{T}(s,t)\bar{V}(t,s). \tag{2.18}$$

This implies that

$$\begin{aligned}
\hat{F}(s) &= \hat{T}(s,t) \operatorname{Im} \bar{V}(t,s) \\
&= \hat{T}(s,t) \operatorname{Im} \big(\bar{V}(t,t)\bar{V}(t,s)\big) \\
&\subset \hat{T}(s,t)\hat{F}(t)
\end{aligned}$$

for $t \leq s$.

Now we establish the reverse inclusions. For this we use Lemmas 2.6 and 2.7. Take $x \in \hat{E}(t)$ and $y \in \hat{T}(t,s)^{-1}x$. Then $\hat{T}(\tau,s)y = \hat{T}(\tau,t)x$ for each $\tau \geq t$. Since $x \in \hat{E}(t) = \operatorname{Im} \bar{U}(t,t)$, we have $x = \bar{U}(t,t)z$ for some $z \in X$ and hence,

$$\hat{T}(\tau,s)y = \hat{T}(\tau,t)\bar{U}(t,t)z = \bar{U}(\tau,t)z.$$

By (2.3), we have

$$Q(\tau)\hat{T}(\tau,s)y = Q(\tau)\bar{U}(\tau,t)z = -\int_\tau^\infty T(\tau,r)Q(r)B(r)\bar{U}(r,t)z\,dr.$$

Thus, for $\tau \geq t \geq s$ we obtain

$$\begin{aligned}\|Q(\tau)\hat{T}(\tau,s)y\| &\leq \int_\tau^\infty \|T(\tau,r)|F(r)\| \cdot \|Q(r)\| \cdot \|B(r)\| \cdot \|\bar{U}(r,t)\| \cdot \|z\|\,dr\\ &\leq D\delta e^{-\lambda(\tau-t)+a|t|}\|U\| \cdot \|z\| \int_\tau^\infty e^{-2\lambda(r-\tau)}\,dr\\ &\leq \frac{D}{2\lambda}\delta e^{-\lambda(\tau-t)+a|t|}\|U\| \cdot \|z\| < +\infty.\end{aligned}$$

It follows from Lemma 2.6 that $\hat{T}(\tau,s)y = \bar{U}(\tau,s)y$ for $\tau \geq s$. In particular,

$$y = \hat{T}(s,s)y = \bar{U}(s,s)y \in \hat{E}(s).$$

Therefore, $x = \hat{T}(t,s)y \in \hat{T}(t,s)\hat{E}(s)$ and we obtain $\hat{E}(t) \subset \hat{T}(t,s)\hat{E}(s)$. This establishes the first identity in the lemma. For the second identity, take $x \in \hat{T}(s,t)\hat{F}(t)$ and $y \in \hat{T}(s,t)^{-1}x$ with $y = \bar{V}(t,t)z$ for some $z \in X$. Then the function $V(\tau,t)z$, $\tau \leq t$ satisfies the hypothesis of Lemma 2.7 and the same happens with

$$(-\infty,s] \ni \tau \mapsto \begin{cases}\bar{V}(\tau,t)z, & \tau \leq t,\\ \hat{T}(\tau,t)\bar{V}(t,t)z, & t \leq \tau \leq s.\end{cases} \tag{2.19}$$

By (2.7), we have

$$P(\tau)\hat{T}(\tau,s)y = P(\tau)\bar{V}(\tau,t)z = \int_{-\infty}^\tau T(\tau,r)P(r)B(r)\bar{V}(r,t)z\,dr.$$

Thus, for $\tau \leq t \leq s$ we obtain

$$\begin{aligned}\|P(\tau)\hat{T}(\tau,s)y\| &\leq \int_{-\infty}^\tau \|T(\tau,r)|E(r)\| \cdot \|P(r)\| \cdot \|B(r)\| \cdot \|\bar{V}(r,t)\| \cdot \|z\|\,dr\\ &\leq D\delta e^{-\lambda(t-\tau)+a|t|}\|V\| \cdot \|z\| \int_{-\infty}^\tau e^{2\lambda(r-\tau)}\,dr\\ &\leq \frac{D}{2\lambda}\delta e^{-\lambda(t-\tau)+a|t|}\|V\| \cdot \|z\| < +\infty.\end{aligned}$$

Hence, it follows from Lemma 2.7 that $\bar{V}(\tau,t)z = \bar{V}(\tau,s)w$, $\tau \leq t$ for some $w \in X$. In particular, by (2.18),

$$x = \hat{T}(s,t)\bar{V}(t,t)z = \hat{T}(s,t)\bar{V}(t,s)w = \bar{V}(s,s)w.$$

This shows that $x \in \hat{F}(s)$, and hence $\hat{T}(s,t)\hat{F}(t) \subset \hat{F}(s)$. This completes the proof of the lemma. $\square$

We first show that the dynamics is invertible along the spaces $\hat{F}(t)$. Since $\bar{V}(s,s)^2 = \bar{V}(s,s)$, restricting identity (2.18) to $\hat{F}(s)$ yields the identity

$$\mathrm{Id}_{\hat{F}(s)} = \bar{V}(s,s)|\hat{F}(s) = \hat{T}(s,t)\bar{V}(t,s)|\hat{F}(s). \tag{2.20}$$

Now we show that $\bar{V}(t,s)\hat{F}(s) = \hat{F}(t)$ for $t \leq s$. Since

$$\bar{V}(t,s) = \bar{V}(t,t)\bar{V}(t,s),$$

we have

$$\bar{V}(t,s)\hat{F}(s) \subset \operatorname{Im}\bar{V}(t,t) = \hat{F}(t).$$

For the reverse inclusion, we use a particular case of the argument in the former lemma. Indeed, take $x \in \hat{F}(t)$ and $z \in X$ such that $x = \bar{V}(t,t)z$. Then the function in (2.19) satisfies the hypotheses of Lemma 2.7 and $\bar{V}(\tau,t)z = \bar{V}(\tau,s)w$, $\tau \le t$ for some $w \in X$. This shows that

$$x = \bar{V}(t,t)z = \bar{V}(t,s)w = \bar{V}(t,s)\bar{V}(s,s)w \in \bar{V}(t,s)\hat{F}(s).$$

Therefore, $\bar{V}(t,s)\hat{F}(s) = \hat{F}(t)$. Hence it follows from (2.20) that the operator $\hat{T}(s,t)|\hat{F}(t)$ is invertible with

$$\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1} = \bar{V}(t,s)|\hat{F}(s).$$

It follows from Lemma 2.8 that

$$\hat{T}(t,s)|\hat{E}(s) = \bar{U}(t,s)|\hat{E}(s)\colon \hat{E}(s) \to \hat{E}(t), \quad t \ge s \tag{2.21}$$

and

$$\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1} = \bar{V}(t,s)|\hat{F}(s)\colon \hat{F}(s) \to \hat{F}(t), \quad t \le s. \tag{2.22}$$

Step 5. Exponential bounds along $\hat{E}(t)$ and $\hat{F}(t)$. Now we obtain the required exponential bounds for the perturbed dynamics.

Lemma 2.9. *For δ sufficiently small, there exists $K > 0$ such that*

$$\|\hat{T}(t,s)|\hat{E}(s)\| \le Ke^{-\lambda(t-s)+a|s|}, \quad t \ge s. \tag{2.23}$$

Proof of the lemma. It follows from (2.21) together with Lemma 2.3 that

$$\|\hat{T}(t,s)|\hat{E}(s)\| = \sup \frac{\|\bar{U}(t,s)y\|}{\|\bar{U}(s,s)y\|},$$

with the supremum taken over all $y \in E(s)$ such that $\bar{U}(s,s)y \ne 0$. By Lemma 2.2, we have

$$\|\bar{U}(t,s)y\| \le \|U\|e^{-\lambda(t-s)+a|s|}\|y\| \tag{2.24}$$

for every $y \in E(s)$. Moreover,

$$\bar{U}(t,t) = P(t) - \int_s^\infty T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)\,d\tau,$$

which yields the identity

$$\bar{U}(t,t)y = y - \int_s^\infty T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)y\,d\tau.$$

Using (2.24), we obtain

$$\begin{aligned}\int_s^\infty \big\|T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)y\big\|\,d\tau &\le D\delta\|U\|\cdot\|y\|\int_s^\infty e^{-2\lambda(\tau-s)}\,d\tau\\ &\le \frac{D\delta}{2\lambda}\|U\|\|y\|\end{aligned}$$

and thus,

$$\begin{aligned}\|\bar{U}(t,t)y\| &\geq \|y\| - \int_s^\infty \big\|T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)y\big\|\,d\tau \\ &\geq \|y\| - \frac{D\delta\|U\|}{2\lambda}\|y\| = \left(1 - \frac{D\delta\|U\|}{2\lambda}\right)\|y\|.\end{aligned}$$

Taking δ sufficiently small so that $D\delta\|U\|/(2\lambda) \leq 1/2$, we obtain

$$\|\bar{U}(s,s)y\| \geq \frac{1}{2}\|y\|.$$

Together with (2.24) this implies that

$$\frac{\|\bar{U}(t,s)y\|}{\|\bar{U}(s,s)y\|} \leq \frac{\|U\|e^{-\lambda(t-s)+a|s|}\|y\|}{\|\bar{U}(s,s)y\|} \leq 2\|U\|e^{-\lambda(t-s)+a|s|}.$$

This concludes the proof of the lemma. □

Lemma 2.10. *For δ sufficiently small, there exists $K > 0$ such that*

$$\|\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1}\| \leq Ke^{-\lambda(s-t)+a|s|}, \quad t \leq s. \tag{2.25}$$

Proof of the lemma. It follows from (2.22) together with Lemma 2.5 that

$$\|\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1}\| = \sup \frac{\|\bar{V}(t,s)z\|}{\|\bar{V}(s,s)z\|},$$

with the supremum taken over all $z \in F(s)$ such that $\bar{V}(s,s)z \neq 0$. By Lemma 2.4, we have

$$\|\bar{V}(t,s)z\| \leq \|V\|e^{\lambda(t-s)+a|s|}\|z\| \tag{2.26}$$

for every $z \in F(s)$. Moreover,

$$\bar{V}(s,s) = Q(s) + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)\bar{V}(\tau,s)\,d\tau,$$

which yields the identity

$$\bar{V}(s,s)z = z + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)\bar{V}(\tau,s)z\,d\tau.$$

Proceeding as in the proof of Lemma 2.9, we obtain

$$\int_{-\infty}^s \big\|T(s,\tau)P(\tau)B(\tau)\bar{V}(\tau,s)z\big\|\,d\tau \leq \frac{D\delta\|V\|}{2\lambda}\|z\|$$

and thus,

$$\|\bar{V}(s,s)z\| \geq \left(1 - \frac{D\delta\|V\|}{2\lambda}\right)\|z\|.$$

Taking δ sufficiently small so that $D\delta\|U\|/(2\lambda) \leq 1/2$, we obtain

$$\|\bar{V}(s,s)z\| \geq \frac{1}{2}\|z\|.$$

Together with (2.26) this implies that

$$\frac{\|\bar{V}(t,s)z\|}{\|\bar{V}(s,s)z\|} \le \frac{\|V\|e^{\lambda(t-s)+a|s|}\|z\|}{\|\bar{V}(s,s)z\|} \le 2\|V\|e^{\lambda(t-s)+a|s|}.$$

This concludes the proof of the lemma. □

Step 6. Construction of projections. Now we use the results in the former lemmas to show that $\hat{E}(t)$ and $\hat{F}(t)$ form a direct sum.

We start with an auxiliary statement about the operators

$$S_s = \bar{U}(s,s) + \bar{V}(s,s).$$

Lemma 2.11. *If δ is sufficiently small, then S_s is invertible for every $s \in \mathbb{R}$.*

Proof of the lemma. We have

$$\begin{aligned} S_s = \bar{U}(s,s) + \bar{V}(s,s) = P(s) - \int_s^\infty T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)\,d\tau \\ + Q(s) + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)\bar{V}(\tau,s)\,d\tau \end{aligned}$$

and hence,

$$S_s - \mathrm{Id} = -\int_s^\infty T(s,\tau)Q(\tau)B(\tau)\bar{U}(\tau,s)\,d\tau + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)\bar{V}(\tau,s)\,d\tau.$$

Therefore, using Lemmas 2.2 and 2.4, we obtain

$$\begin{aligned} \|S_s - \mathrm{Id}\| &\le \int_s^\infty \|T(s,\tau)|F(\tau)\| \cdot \|Q(\tau)\| \cdot \|B(\tau)\| \cdot \|\bar{U}(\tau,s)\|\,d\tau \\ &\quad + \int_{-\infty}^s \|T(s,\tau)|E(\tau)\| \cdot \|P(\tau)\| \cdot \|B(\tau)\| \cdot \|\bar{V}(\tau,s)\|\,d\tau \\ &\le D\delta\|U\| \int_s^\infty e^{-2\lambda(\tau-s)}\,d\tau + D\delta\|V\| \int_{-\infty}^s e^{2\lambda(\tau-s)}\,d\tau \\ &\le \frac{\delta D}{\lambda}\big(\|U\| + \|V\|\big). \end{aligned}$$

Moreover, it follows from (2.6) and (2.10) that

$$\|U\| \le D\Big/\Big(1 - \delta D\Big(\frac{1}{a} + \frac{1}{2\lambda}\Big)\Big)$$

and

$$\|V\| \le D\Big/\Big(1 - \delta D\Big(\frac{1}{a} + \frac{1}{2\lambda}\Big)\Big).$$

This implies that for δ sufficiently small (independent of s), the operator S_s is invertible. □

Lemma 2.12. *If δ is sufficiently small, then $\hat{E}(t) \oplus \hat{F}(t) = X$ for each $t \in \mathbb{R}$.*

Proof of the lemma. Take $\xi \in \hat{E}(t) \cap \hat{F}(t)$. It follows from (2.23) and (2.25) that

$$\frac{1}{K} e^{\lambda(t-s)-a|t|} \|\xi\| \leq \|\hat{T}(t,s)\xi\| \leq K e^{-\lambda(t-s)+a|s|} \|\xi\|$$

for each $t \geq s$. Since $a < \lambda$ this implies that $\xi = 0$. Therefore, $\hat{E}(t) \cap \hat{F}(t) = \{0\}$. Moreover, since the operator S_t is invertible, we have

$$X = S_t X \subset \operatorname{Im} \bar{U}(t,t) + \operatorname{Im} \bar{V}(t,t) = \hat{E}(t) + \hat{F}(t).$$

This concludes the proof of the lemma. □

The statements in the theorem follow now readily from the previous lemmas. □

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