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NULL CONTROLLABILITY FOR A CLASS OF SEMILINEAR DEGENERATE/SINGULAR PARABOLIC EQUATIONS

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Abstract. In this paper we prove the null controllability results for a class of semilinear degenerate/singular one-dimensional parabolic equations. The proof is based on Schauder's fixed point theorem.
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1. Introduction

The study of controllability for non-degenerate parabolic equations has attracted the interest of several authors in the past few decades. After the pioneering works [19, 21, 22], there has been substantial progress in understanding the controllability properties of non-degenerate parabolic equations with variable coefficients. These results also have been extended to semilinear problems [16, 17, 18, 29, 30] and to parabolic equations in unbounded domains [7, 25]. This theory is now quite complete for uniformly parabolic equations in both bounded and unbounded domains.

More recently, several situations where the operator is not uniformly parabolic have been investigated. Such studies may be motivated by various physical problems such as boundary layer models [6], Fisher genetics population models, Bydyko-Sellers climate models,..., where degenerate parabolic operators coupling transport and diffusion phenomena. We refer the reader to [3, 9, 10, 11, 12, 13, 14, 24, 26] for the study of controllability of parabolic equations with degenerate diffusion at the boundary. In particular, new Carleman estimates (and consequently null controllability properties) were established in [13] for the operator

$$Pu = u_t - (x^\alpha u_x)_x, \quad x \in (0,1), \tag{1.1}$$

with suitable boundary conditions.

Another interesting situation that has not been largely studied is the case of parabolic operators with singular lower order terms. First results in this direction were obtained in [15, 28] for the heat operator with a singular potential

$$Pu = u_t - u_{xx} - \frac{\lambda}{x^\beta}u, \quad x \in (0,1),$$

with Dirichlet boundary conditions. In the case $\beta = 2$, we have the so-called inverse square potential that arises for example in quantum mechanics or in linearized combustion problems [5]. This potential is known to generate interesting phenomena since the work of Baras and Goldstein [4]. Indeed, global positive solutions exist (for any value $\lambda \in \mathbb{R}$) if $\beta < 2$ whereas instantaneous and complete blow-up occurs (for any value of λ) if $\beta > 2$. Therefore, the exponent $\beta = 2$ is critical. This makes the case of inverse-square potentials particularly interesting. And, when the exponent is critical, i.e., when $\beta = 2$, it is the value of parameter λ that determines the behavior of the equation. Indeed, global positive solutions exist when $\lambda \leq 1/4$ whereas instantaneous and complete blow-up occurs when $\lambda > 1/4$. The critical value $1/4$ of the parameter λ is the optimal value of the constant in the Hardy inequality

$$\int_0^1 z_x^2 dx \geq \frac{1}{4}\int_0^1 \frac{z^2}{x^2} dx \text{ for all } z \in H_0^1(0,1). \tag{1.2}$$

It is noticed that the long-time behavior of solutions and approximate controllability for semilinear parabolic equations involving an inverse-square potential were studied recently in [1, 2].

Recently, Vancostenoble [27] proved some new Carleman estimates, and consequently null controllability results, for the following linear degenerate/singular parabolic equation

$$u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^\beta}u = 1_\omega h, \tag{1.3}$$

with suitable boundary conditions. In the case of the purely degenerate operator (1.1) (one aslo can see (1.3) when $\beta = 0$), a key ingredient in the proof of Carleman estimates in [13] relies on the following Hardy inequality

$$\int_0^1 x^\alpha z_x^2 dx \geq \frac{(1-\alpha)^2}{4}\int_0^1 \frac{z^2}{x^{2-\alpha}} dx, \text{ for all } z \in C_c^\infty(0,1). \tag{1.4}$$

As in above, in the case $\alpha = 0$, the critical exponent of the singular potential λ/x^β is $\beta = 2$. This fact is a consequence of the Hardy inequality (1.2). More generally, for a given singular potential, Cabré and Martel [8] proved that existence of some Hardy inequalities involving the considered potential. Therefore (1.4) implies that the critical exponent becomes $\beta = 2 - \alpha$ when $\alpha \neq 0$. This leads us to assume that $\beta \leq 2-\alpha$. With no loss of generality, we assume that $\beta > 0$. Indeed, when $\beta \leq 0$, the potential is no more singular and the controllability result easily follows from [13]. In summary, we assume

$$0 < \beta \leq 2 - \alpha.$$

As in the case $\alpha = 0$, the critical value of the parameter λ when $\beta = 2-\alpha$ is given by the optimal constant in (1.4), that is $\lambda(\alpha) = \dfrac{(1-\alpha)^2}{4}$. Thus, we finally assume that

$$\lambda \leq \lambda(\alpha) \text{ when } \beta = 2-\alpha.$$

No condition on λ is assumed for sub-critical exponent β, i.e., when $\beta < 2-\alpha$.

The aim of this paper is to extend the null controllability results for the linear problem in [27] to the following semilinear problem

$$\begin{cases} u_t - (x^\alpha u_x)_x - \dfrac{\lambda}{x^\beta}u + f(t,x,u) = 1_\omega h, & (t,x) \in Q_T = (0,T)\times(0,1), \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0, & x \in (0,1), \end{cases} \tag{1.5}$$

where $u_0 \in L^2(0,1), h \in L^2(Q_T), 0 \leq \alpha < 1$, and ω is a nonempty subinterval of $(0,1)$. Here, 1_ω is the characteristic function of the set ω and it is assumed that the function $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies $f(t,x,0) = 0$ and there exist two positive constants C_1, C_2 such that for all $(t,x) \in Q_T$,

$$|f(t,x,u) - f(t,x,v)| \leq C_1(1 + |u|^\theta + |v|^\theta)|u-v| \ \forall u,v \in \mathbb{R}, \text{ for some } \theta \geq 0, \tag{1.6}$$

and

$$\big(f(t,x,u+v) - f(t,x,u)\big)v \geq -C_2 v^2 \text{ for all } u,v \in \mathbb{R}. \tag{1.7}$$

We denote $\lambda(\alpha) = \dfrac{(1-\alpha)^2}{4}$ and consider the operator

$$Au = (x^\alpha u_x)_x + \frac{\lambda}{x^\beta}u$$

with sub-critical potentials:

$$\begin{cases} \alpha \in [0,1), & 0 < \beta < 2-\alpha, \ \lambda \in \mathbb{R}, \\ \alpha \in [0,1), & \beta = 2-\alpha, \ \lambda < \lambda(\alpha). \end{cases} \tag{1.8}$$

Let us explain contents and methods used in the paper. First, we prove that problem (1.1) is well-posed by using the compactness method when initial data $u_0 \in L^2(0,1)$, and the fixed point method when $u_0 \in H^1_{\alpha,0}(0,1)$. Second, under above conditions on α, β, λ, by repeating arguments in [27], we easily obtain the null controllability results for the following associated linear degenerate/singular parabolic problem.

$$u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^\beta} + c(t,x)u = 1_\omega h,$$

where $c(\cdot,\cdot) \in L^\infty(Q_T)$. To study the null controllability of semilinear problem (1.5), following the general lines of the approach in [17], we exploit the fixed point method. We first prove the approximately null controllability result for problem (1.5) by using Schauder's fixed point theorem. Then we show that problem (1.5) is null controllable by passing to the limits.

The paper is organized as follows. In Section 2, we prove the well-posedness of problem (1.1), and as a consequence, the well-posedness of the linear problem associated to problem (1.1). In Section 3, we prove the null controllability for the semilinear degenerate/singular problem (1.5).

2. Well-posedness of the problem

2.1. **Function spaces and operator.** For sub-critical potentials, i.e., when (1.8) holds, the domain of the operator A is defined by

$$D(A) := \Big\{u \in H^1_{\alpha,0}(0,1) \cap H^2_{loc}((0,1]) \mid (x^\alpha u_x)_x + \frac{\lambda}{x^\beta} u \in L^2(0,1)\Big\},$$

where

$$H^1_{\alpha,0}(0,1) := \Big\{u \in L^2(0,1) \mid u \text{ is absolutely continuous in } [0,1], \\ \sqrt{x^\alpha} u_x \in L^2(0,1) \text{ and } u(0) = u(1) = 0\Big\}.$$

Then $H^1_{\alpha,0}(0,1)$ is a Banach space endowed with the norm

$$\|u\|_{H^1_{\alpha,0}} = \Big(\int_0^1 x^\alpha u_x^2 dx\Big)^{1/2}.$$

Proposition 2.1. [27] *Assume that* (1.8) *holds. Then there exist* $\eta \ge 0$ *and* $C = C(\alpha,\beta,\lambda) > 0$ *such that*

$$\forall u \in H^1_{\alpha,0}(0,1), \quad \int_0^1 x^\alpha u_x^2 - \frac{\lambda}{x^\beta} u^2 + \eta u^2 \ge C\|u\|^2_{H^1_{\alpha,0}}. \tag{2.1}$$

We have the following embedings (see [9, Theorems 6.1-6.4]).

Theorem 2.2. *(i) The embeding* $H^1_{\alpha,0}(0,1) \hookrightarrow L^2(0,1)$ *is compact.*

(ii) The embeding $D(A) \hookrightarrow H^1_{\alpha,0}(0,1)$ *is compact.*

(iii) The embeding

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(A)) \hookrightarrow C([0,T];L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,0}(0,1))$$

is compact.

Furthermore, from Proposition 4 in [27] we have that when $0 \le \alpha < 1$, the following injections are continuous

$$H^1_{\alpha,0}(0,1) \hookrightarrow W^{1,1}(0,1) \hookrightarrow L^\infty(0,1). \tag{2.2}$$

The above properties guarantee that the bilinear form associated to $-(A-\eta I)$ is coercive in $H^1_{\alpha,0}(0,1)$. This implies the following result.

Proposition 2.3. *Assume that* (1.8) *holds and consider the constant* $\eta \ge 0$ *that is given by Proposition 2.1. Then* $A - \eta I$ *is a self-adjoint negative operator.*

2.2. **Well-posedness of the semilinear problem.**

Theorem 2.4. *Assume that* (1.6), (1.7) *and* (1.8) *hold, and that* $u_0 \in L^2(0,1), T > 0$ *given. Then problem* (1.1) *has a unique weak solution* u *satisfying*

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_{\alpha,0}(0,1)),$$
$$\frac{du}{dt} \in L^2(0,T; H^{-1}_{\alpha,0}(0,1)),$$

where $H^{-1}_{\alpha,0}(0,1)$ *is the dual space of* $H^1_{\alpha,0}(0,1)$. *Furthermore, if* $u_0 \in H^1_{\alpha,0}(0,1)$, *then problem* (1.1) *has a unique mild solution*

$$u \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A)) \cap C([0,T]; H^1_{\alpha,0}(0,1)).$$

Proof. We consider two cases of initial data.
Case 1: $u_0 \in L^2(0,1)$**.**
We look for an approximate solution $u_n(t)$ that belongs to the finite-dimensional space spanned by the first n eigenfunctions of $-A$ such that

$$u_n(t) = \sum_{j=1}^{n} u_{nj}(t) e_j,$$

and solves the problem

$$\begin{cases} u_{nt} - (x^\alpha u_{nx})_x - \dfrac{\lambda}{x^\beta} u_n + f(t,x,u_n) = 1_\omega h, & (t,x) \in Q_T, \\ u_n(t,1) = u_n(t,0) = 0, & t \in (0,T), \\ u_n(0,x) = P_n u_0(x), & x \in (0,1), \end{cases} \tag{2.3}$$

where $P_n : L^2(0,1) \to \operatorname{span}\{e_1, \ldots, e_n\}$ is the canonical projector. Hence we have a system of first-order ordinary differential equations for the functions $u_{n1}, \ldots, u_{nn}$,

$$u'_{nj} + \lambda_j u_{nj} + \langle f(t,x,u_n), e_j \rangle = (h, e_j),$$
$$u_{nj}(0) = (u_0, e_j), \quad j = \overline{1,n}.$$

According to theory of ODEs, we obtain the existence of approximate solutions $u_n(t)$.

Multiplying the first equation in (2.3) by $u_n(t)$ and integrating in $[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(0,1)} + \int_0^1 \left(x^\alpha u_{nx}^2 - \frac{\lambda}{x^\beta} u_n^2 + \eta u_n^2\right) dx = -\int_0^1 f(t,x,u_n) u_n dx + \int_0^1 \eta u_n^2 dx + \int_0^1 h u_n\, dx. \tag{2.4}$$

Using hypothesis (1.7) and inequality (2.1), we get from (2.4)

$$\frac{d}{dt}\|u_n\|^2_{L^2(0,1)} + 2C(\alpha,\lambda,\beta)\|u_n\|^2_{H^1_{\alpha,0}} \le (2C_2 + 2\eta + 1)\|u\|^2_{L^2(0,1)} + \|h\|^2_{L^2(0,1)}. \tag{2.5}$$

Using the Gronwall inequality, in particular, we deduce from (2.5) that

$$\|u_n(t)\|^2_{L^2(0,1)} \le e^{(2(C_2+\eta)+1)t}\|u_0\|^2_{L^2(0,1)} + \int_0^t e^{(2(C_2+\eta)+1)(t-s)}\|h(s)\|^2_{L^2(0,1)} ds.$$

Hence

$$\|u_n(t)\|^2_{L^2(0,1)} \leq e^{(2(C_2+\eta)+1)T}\left(\|u_0\|^2_{L^2(0,1)} + \int_0^T \|h(s)\|^2_{L^2(0,1)}ds\right) \text{ for all } t \in (0,T]. \tag{2.6}$$

Now, integrating (2.5) from 0 to T and using (2.6) we have

$$\|u_n(T)\|^2_{L^2(0,1)} + 2C(\alpha,\lambda,\beta)\int_0^T \|u_n\|^2_{H^1_{\alpha,0}}dt \leq C(C_2,\eta,T)\left(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2(Q_T)}\right).$$

Hence, $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,0}(0,1))$. Using (1.6), the boundedness of $\{u_n\}$ in $L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,0}(0,1))$ and from (2.2), we infer the boundedness of $f(t,x,u_n)$ in $L^2(0,T;L^2(\Omega))$. On the other hand, we have

$$\frac{du_n}{dt} = (x^\alpha u_{nx})_x + \frac{\lambda}{x^\beta}u_n - f(t,x,u_n).$$

So, $\{u_{nt}\}$ is bounded in $L^2(0,T;H^{-1}_{\alpha,0}(0,1))$. Hence by choosing a subsequence we can assume that

$$\begin{aligned} u_{nt} &\rightharpoonup u_t \text{ in } L^2(0,T;H^{-1}_{\alpha,0}(0,1)),\\ u_n &\rightharpoonup u \text{ in } L^2(0,T;H^1_{\alpha,0}(0,1)) \text{ and in } L^2(Q_T),\\ f(t,x,u_n) &\rightharpoonup \kappa \text{ in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

From the fact that $u \in L^2(0,T;H^1_{\alpha,0}(0,1)) \cap L^q(Q_T)$ and $u_t \in L^2(0,T;H^{-1}_{\alpha,0}(0,1)) + L^2(Q_T)$, we infer that $u \in C([0,T];L^2(0,1))$. And therefore, $u \in C([0,T];L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,0}(0,1))$.

Since the boundedness of $\{u_n\}$ and $\{u_{nt}\}$, it follows from the Compacness Lemma [23, p. 58] that

$$u_n \to u \text{ in } L^2(0,T;L^2(0,1)) \text{ up to a subsequence.}$$

Hence we can choose a subsequence u_{n_k} such that

$$u_{n_k}(t,x) \to u(t,x) \text{ for a.e. } (t,x) \in Q_T.$$

It follows from the continuity of f with respectively the third variable that

$$f(t,x,u_{n_k}(t,x)) \to f(t,x,u(t,x)) \text{ for a.e. } (t,x) \in Q_T.$$

In view of the boundedness of $f(t,x,u_{n_k})$ in $L^2(Q_T)$, by [23, Lemma 1.3], we conclude that

$$f(t,x,u_{n_k}) \rightharpoonup f(t,x,u) \text{ in } L^2(Q_T).$$

Taking in to account the uniqueness of weak limits, we obtain $\kappa = f(t,x,u)$. By a standard argument, one can show that $u(0) = u_0$. This implies that u is a weak solution to problem (1.5).

To prove the uniqueness of weak solutions, we assume u, v are two solutions of (1.5). Putting $w = u - v$, we have

$$\begin{cases} w_t - (x^\alpha w_x)_x - \dfrac{\lambda}{x^\beta} w + f(t,x,u) - f(t,x,v) = 0, & (t,x) \in Q_T = (0,T) \times (0,1), \\ w(t,1) = w(t,0) = 0, & t \in (0,T), \\ w(0,x) = 0, & x \in (0,1), \end{cases} \tag{2.7}$$

Multiplying the first equation in (2.7) by w, then integrating over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(0,1)} + \|w\|^2_{H^1_{\alpha,0}} + \int\limits_0^1 \big(f(t,x,u) - f(t,x,v)\big)\big(u - v\big)dx = 0.$$

Using (1.7), we get

$$\frac{d}{dt}\|w\|^2_{L^2(0,1)} \le 2C_2\|w\|^2_{L^2(0,1)}.$$

By the Gronwall inequality, we have

$$\|w(t)\|^2_{L^2(0,1)} \le e^{2C_2 t}\|w(0)\|^2_{L^2(0,1)} \equiv 0.$$

This implies the desired result.

Case 2: $u_0 \in H^1_{\alpha,0}(0,1)$**.**
From Lemma 2.3 we see that $\tilde{A} = (A - \eta I)$, where η is the constant in Proposition 2.1, is a sectorial operator and $Re\sigma_{\tilde{A}} > \delta > 0$. Then problem (1.5) can be rewritten as an abstract evolutionary equation in $H^1_{\alpha,0}(0,1)$:

$$\frac{du}{dt}(t) + \tilde{A}(u(t)) = \tilde{f}(u),\ u(0) = u_0 \in H^1_{\alpha,0}(0,1), \tag{2.8}$$

where the Nemytskii map

$$\tilde{f}(u)(x) = -f(t,x,u(t,x)) - \eta u(t,x) + 1_\omega h. \tag{2.9}$$

Under the assumption of f we have

$$\begin{aligned} \|\tilde{f}(u) - \tilde{f}(v)\|^2_{L^2(0,1)} &= \int_0^1 \left(|f(t,x,u) - f(t,x,v)| + \eta|u - v|\right)^2 x \\ &\le C\int_0^1 (|f(t,x,u) - f(t,x,v)|^2 + |u - v|^2)dx \\ &\le C\int_0^1 (1 + |u|^\theta + |v|^\theta)^2|u - v|^2 dx \\ &\le C\left(1 + \|u\|^{2\theta}_{L^\infty(0,1)} + \|v\|^{2\theta}_{L^\infty(0,1)}\right)\int_0^1 |u - v|^2 dx. \end{aligned}$$

Hence, we deduce from (2.2) that

$$\|\tilde{f}(u) - \tilde{f}(v)\|_{L^2(0,1)} \le C(\|u\|_{H^1_{\alpha,0}(0,1)}, \|v\|_{H^1_{\alpha,0}(0,1)})\|u - v\|_{H^1_{\alpha,0}(0,1)},$$

that is, the Nemytskii $\tilde{f}$ is a locally Lipschitzian map from $H^1_{\alpha,0}(0,1)$ to $L^2(0,1)$. Noting that $\tilde{A}$ is a sectorial operator on $L^2(0,1)$ with fractional power spaces $X^\alpha = D(\tilde{A}^\alpha)$, where $D(\tilde{A}^\alpha)$ is the domain of $\tilde{A}^\alpha$, and $X^{1/2} = H^1_{\alpha,0}(0,1)$, $X^0 = L^2(0,1)$. By Theorem 3.3.3 in [20], we obtain the existence of a unique mild solution u, i.e. u is a continuous solution of the following integral equation

$$u(t) = e^{-\tilde{A}t}u_0 + \int_0^t e^{-\tilde{A}(t-s)}\tilde{f}(u(s))ds,\ t > 0,$$

where $e^{-\tilde{A}t}$ is the semigroup generated by $\tilde{A}$. This ends the proof. □

Remark 2.5. The above theorem is also true for the linear degenerate/singular parabolic problem associated to problem (1.5), that is, when $f(t,x,u) = c(t,x)u$ with $c(t,x) \in L^\infty(Q_T)$.

3. Null controllability results

3.1. **Null controllability results for the linear degenerate/singular parabolic problem.** In this subsection, we consider the linear degenerate/singular parabolic problem associated to problem (1.5)

$$\begin{cases} u_t - (x^\alpha u_x)_x - \dfrac{\lambda}{x^\beta}u + c(t,x)u = 1_\omega h, & (t,x) \in Q_T, \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0, & x \in (0,1), \end{cases} \tag{3.1}$$

where $c(t,x)$ satisfies

$$c(t,x) \in L^\infty(Q_T). \tag{3.2}$$

The well-posedness of this problem is obtained from Theorem 2.4. Using the Carleman estimates [27, Theorem 5.1] and repeating the arguments in [27], we easily obtain the following result.

Theorem 3.1. *Assume that* (1.8) *and* (3.2) *hold. Then for any* $T > 0$ *and* $u_0 \in L^2(0,1)$ *given, there exists* $h \in L^2(\omega \times (0,T))$ *such that the solution* u *of* (3.1) *satisfies*

$$u(T,x) = 0 \quad x \in (0,1),$$

that is, problem (3.1) *is null controllable. Moreover, there exists a positive constant* C_T *such that*

$$\int_0^T \int_\omega h^2 dxdt \le C_T \int_0^1 u_0^2 dx.$$

In order to prove the null controllability results for the semilinear problem (1.5), we need the following lemma.

Lemma 3.2. *Under assumptions* (1.6)-(1.7)*, then*

$$\int_0^1 f_v(t,x,\xi v)d\xi \in L^\infty(Q_T) \text{ for all } v \in X = C([0,T]; H^1_{\alpha,0}(0,1)). \tag{3.3}$$

Proof. We have

$$\int_0^1 f_v(t,x,\xi v)d\xi = \begin{cases} \dfrac{f(t,x,v)}{v} & \text{if } v \neq 0, \\ f_v(t,x,0) & \text{if } v = 0. \end{cases} \tag{3.4}$$

Note that

$$f_v(t,v,0) = \lim_{v\to 0} \frac{f(t,x,v) - f(t,x,0)}{v-0} = \lim_{v\to 0} \frac{f(t,x,v)}{v}.$$

Hence, we have only to prove that $\int_0^1 f_v(t,x,\xi v)d\xi$ belongs to $L^\infty(Q_T)$ as $v \neq 0$. When $v \neq 0$, using (1.6), we have

$$\sup_{Q_T} \left| \int_0^1 f_v(t,x,\xi v)d\xi \right| = \sup_{Q_T} \left| \frac{f(t,x,v)}{v} \right| \leq C_1(1 + \sup_{Q_T} |v|^\theta). \tag{3.5}$$

Since $0 \leq \alpha < 1$, we have from (2.2) that the embedding

$$X = C([0,T]; H^1_{\alpha,0}(0,1)) \hookrightarrow L^\infty(Q_T) \text{ is continuous.}$$

Thus,

$$\int_0^1 f_v(t,x,\xi v)d\xi \in L^\infty(Q_T).$$

This completes the proof. □

3.2. **Null controllability for semilinear degenerate/singular parabolic problem.** In this subsection, we consider the main problem (1.5). First, we consider problem (1.5) when $u_0 \in H^1_{\alpha,0}(0,1)$. Using the null controllability result of the associated linear problem and the fixed point method using Schauder's theorem, we will prove the following approximately null controllability result.

Theorem 3.3. *Let $T > 0$ and $u_0 \in H^1_{\alpha,0}(0,1)$. Under assumptions (1.6), (1.7) and (1.8), problem (1.5) is approximately null controllable, that is, for any $\varepsilon > 0$, there exists $h^\varepsilon \in L^2(\omega \times (0,T))$ such that*

$$\|u^{h^\varepsilon}(T)\|_{L^2(0,1)} \leq \varepsilon, \tag{3.6}$$

and

$$\int_0^T \int_\omega |h^\varepsilon|^2 dxdt \leq C_T \int_0^1 u_0^2 dx, \text{ for some positive constant } C_T. \tag{3.7}$$

Proof. Let $\varepsilon > 0$ and consider the mapping

$$\tau_\varepsilon : X \ni v \mapsto u^{\varepsilon,v} \in X.$$

Here $X := C([0,T]; H^1_{\alpha,0}(0,1))$ and $u^{\varepsilon,v}$ is the unique solution of the problem

$$\begin{cases} u_t^{\varepsilon,v} - (x^\alpha u_x^{\varepsilon,v})_x - \frac{\lambda}{x^\beta} u^{\varepsilon,v} + c^v(t,x) u^{\varepsilon,v} = 1_\omega h^{\varepsilon,v}, & (t,x) \in Q_T, \\ u^{\varepsilon,v}(t,1) = u^{\varepsilon,v}(t,0) = 0 & t \in (0,T), \\ u^{\varepsilon,v}(0,x) = u_0, & x \in (0,1), \end{cases} \tag{3.8}$$

where $c^v(t,x) = \int_0^1 f_v(t,x,\xi v)d\xi$. From Lemma 3.2 we have $c^v(t,x) \in L^\infty(Q_T)$ for all $v \in X$. And then, by Theorem 3.1, problem (3.8) is null controllable. Moreover, there exist a constant $C(T) > 0$ such that

$$\int_0^T \int_\omega |h^{\varepsilon,v}|^2 dxdt \le C_T \|u_0\|^2_{L^2(\Omega)}. \tag{3.9}$$

Now, we prove that τ_ε has a fixed point $u^{\varepsilon,v}$, that is, $\tau_\varepsilon(u^{\varepsilon,v}) = u^{\varepsilon,v}$, by using Schauder's fixed point theorem. It is sufficient to prove that

(i) $\tau_\varepsilon : B_X \to B_X$,
(ii) τ_ε is a compact mapping,
(iii) τ_ε is a continuous mapping,

where

$$B_X := \{v \in X : \|v\|_X \le R\}, \ \|v\|_X := \sup_{t\in[0,T]} \|v(t)\|_{H^1_{\alpha,0}}.$$

First, we prove (i). Multiplying the first equation in (3.8) by $u^{\varepsilon,v}$ and integrating in $(0,1)$ we get

$$\frac{d}{dt}\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + 2\left(\int_0^1 x^\alpha |u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}|u^{\varepsilon,v}|^2 + \eta|u^{\varepsilon,v}|^2\right)$$
$$= -2\int_0^1 c^v(t,x)|u^{\varepsilon,v}|^2 + 2\eta\int_0^1 |u^{\varepsilon,v}|^2 + 2\int_0^1 h^{\varepsilon,v}u^{\varepsilon,v}dx. \tag{3.10}$$

Hence, by the Cauchy inequality and (2.1), we obtain

$$\frac{d}{dt}\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + 2C(\lambda,\alpha,\beta)\|u^{\varepsilon,v}\|^2_{H^1_{\alpha,0}}$$
$$\le (2\|c^v\|_{L^\infty(Q_T)} + 2\eta + 1)\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + \|h^{\varepsilon,v}\|^2_{L^2(0,1)}. \tag{3.11}$$

Using the Gronwall inequality, we deduce from (3.11) that

$$\|u^{\varepsilon,v}(t)\|^2_{L^2(0,1)} \le e^{(2\|c^v\|_{L^\infty(Q_T)}+2\eta+1)t}\|u_0\|^2_{L^2(0,1)}$$
$$+ \int_0^t e^{-(2\|c^v\|_{L^\infty(Q_T)}+2\eta+1)(s-t)}\|h^{\varepsilon,v}(s)\|^2_{L^2(0,1)}ds. \tag{3.12}$$

And then

$$\sup_{t\in[0,T]}\|u^{\varepsilon,v}\|^2_{L^2(0,1)} \le e^{(2\|c^v\|_{L^\infty(Q_T)}+2\eta+1)T}\left(\|u_0\|^2_{L^2(0,1)} + \int_0^T \|h^{\varepsilon,v}(s)\|^2_{L^2(0,1)}\right). \tag{3.13}$$

Now, multiplying the first equation in (3.8) by $u_t^{\varepsilon,v}$ and integrating in $(0,1)$, we have

$$\|u_t^{\varepsilon,v}\|^2_{L^2(0,1)} + \frac{d}{dt}\left(\int_0^1 x^\alpha|u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}|u^{\varepsilon,v}|^2 + \eta|u^{\varepsilon,v}|^2\right)$$
$$= -\int_0^1 c^v(t,x)u^{\varepsilon,v}u_t^{\varepsilon,v} + \eta\int_0^1 u^{\varepsilon,v}u_t^{\varepsilon,v} + \int_0^1 h^{\varepsilon,v}u_t^{\varepsilon,v}. \tag{3.14}$$

Using the Cauchy inequality, we have

$$\|u_t^{\varepsilon,v}\|^2_{L^2(0,1)} + \frac{d}{dt}\left(\int_0^1 x^\alpha |u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}u^2 + \eta |u^{\varepsilon,v}|^2\right)$$
$$\leq \frac{1}{2}\|u_t^{\varepsilon,v}\|^2_{L^2(0,1)} + \big(2\|c^v\|^2_{L^\infty(Q_T)} + \eta^2\big)\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + 2\|h^{\varepsilon,v}\|^2_{L^2(0,1)}. \quad (3.15)$$

On the other hand, integrating (3.10) from t to $t+1$ with $t \leq T-1$ and using the Cauchy inequality we get

$$\int_t^{t+1}\left(\int_0^1 x^\alpha |u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}|u^{\varepsilon,v}|^2 + \eta |u^{\varepsilon,v}|^2\right)$$
$$\leq C\Big(\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + \int_t^{t+1}\|u^{\varepsilon,v}(s)\|^2_{L^2(0,1)}ds + \|c^v\|_{L^\infty(Q_T)} + \int_0^T \|h^{\varepsilon,v}\|^2_{L^2(Q_T)}\Big). \quad (3.16)$$

Therefore, by (3.13) we deduce from (3.16) that

$$\int_t^{t+1}\left(\int_0^1 x^\alpha |u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}|u^{\varepsilon,v}|^2 + \eta |u^{\varepsilon,v}|^2\right) \leq C\Big(\|c^v\|_{L^\infty(Q_T)} + \|u_0\|^2_{L^2(0,1)}$$
$$+ \int_0^T \|h^{\varepsilon,v}\|^2_{L^2(0,1)}\Big). \quad (3.17)$$

By the uniform Gronwall inequality, we obtain from (3.15) and (3.17) that

$$\int_0^1 \big(x^\alpha |u_x^{\varepsilon,v}|^2 - \frac{\lambda}{x^\beta}|u^{\varepsilon,v}|^2 + \eta |u^{\varepsilon,v}|^2\big)$$
$$\leq C\big(\|c^v\|_{L^\infty(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \int_0^T \|h^{\varepsilon,v}\|^2_{L^2(0,1)}\big). \quad (3.18)$$

So,

$$\|u^{\varepsilon,v}\|^2_{H^1_{\alpha,0}} \leq C\Big(\|c^v\|_{L^\infty(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \int_0^T \|h^{\varepsilon,v}\|^2_{L^2(0,1)}\Big). \quad (3.19)$$

Thus, we obtain (i) with $R^2 = C\Big(\|c^v\|_{L^\infty(Q_T)} + \|u_0\|^2_{L^2(0,1)} + \int_0^T \|h^{\varepsilon,v}\|^2_{L^2(0,1)}\Big)$.

We immediately obtain (ii) by using the compactness of the injection

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(A)) \hookrightarrow C([0,T];L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,0}(0,1)).$$

This compactness embeding is also useful for the proof of (iii). Indeed, let $v_k \in X$ be such that $v_k \to v$ in X, as $k \to \infty$. We want to prove that $u^{\varepsilon,v_k} \to u^{\varepsilon,v}$ in X, as $k \to \infty$. Here u^{ε,v_k} and $u^{\varepsilon,v}$ are the solutions of (3.8) associated to v_k, h^{ε,v_k} and $v, h^{\varepsilon,v}$ respectively.

Because u^{ε,v_k} and $u^{\varepsilon,v}$ are the solutions of problem (3.8) associated to v_k, h^{ε,v_k} and $v, h^{\varepsilon,v}$ respectively with the same initial datum u_0, we can see that $w^k := u^{\varepsilon,v_k} - u^{\varepsilon,v}$ satisfies

$$w_t^k - (x^\alpha w_x^k)_x - \frac{\lambda}{x^\beta}w^k = c^{v_k}w^k + \big(c^{v_k} - c^v\big)u^{\varepsilon,v} - 1_\omega(h^{\varepsilon,v_k} - h^{\varepsilon,v}). \quad (3.20)$$

Multiplying (3.20) by w^k and integrating in $[0,1]$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w^k\|^2_{L^2(0,1)} &+ \int_0^1 \left(x^\alpha |w^k_x|^2 - \frac{\lambda}{x^\beta}|w^k|^2\right)\\ &= \int_0^1 c^{v_k}|w^k|^2dx - \int_0^1 (c^{v_k}-c^v)u^{\varepsilon,v}w^k dx + \int_0^1 (h^{\varepsilon,v_k}-h^{\varepsilon,v})w^k dx. \end{aligned}\tag{3.21}$$

Using the Cauchy inequality and (2.1), we infer from (3.21) that

$$\begin{aligned}\frac{d}{dt}\|w^k\|^2_{L^2(0,1)} + 2C(\alpha,\lambda,\beta)\|w^k\|^2_{H^1_{\alpha,0}} &\le (2\eta + \|c^{v_k}\|_{L^\infty(Q_T)}+2)\|w^k\|^2_{L^2(0,1)}\\ &+ \|c^{v_k}-c^v\|^2_{L^\infty(0,1)}\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + \int_0^1 |h^{\varepsilon,v_k}-h^{\varepsilon,v}|^2 dx.\end{aligned}\tag{3.22}$$

Using the Gronwall inequality, we have from (3.22)

$$\begin{aligned}&\|w^k(t)\|^2_{L^2(0,1)} \le e^{(2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)t}\|w^k(0)\|^2_{L^2(0,1)}\\ &+ \int_0^t e^{(2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)(t-s)}\Big(\|c^{v_k}(s)-c^v(s)\|^2_{L^\infty(0,1)}\|u^{\varepsilon,v}(s)\|^2_{L^2(0,1)}\Big)ds\\ &+ \int_0^t e^{(2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)(t-s)}\int_0^1 |h^{\varepsilon,v_k}(s)-h^{\varepsilon,v}(s)|^2\,dx\,ds.\end{aligned}\tag{3.23}$$

Therefore, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\|w^k(t)\|^2_{L^2(0,1)} \le e^{(2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)T}\|w^k(0)\|^2_{L^2(0,1)}\\ &+ e^{(2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)T}\Big(\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}\|u^{\varepsilon,v}\|^2_{L^2(Q_T)} + \|h^{\varepsilon,v_k}-h^{\varepsilon,v}\|^2_{L^2(Q_T)}\Big).\end{aligned}\tag{3.24}$$

Now, multiplying (3.20) by w^k_t and integrating in $[0,1]$, we get

$$\begin{aligned}\|w^k_t\|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(x^\alpha|w^k_x|^2 - \frac{\lambda}{x^\beta}|w^k|^2 + \eta|w^k|^2\right) &= \int_0^1 c^{v_k}w^k w^k_t\\ + \int_0^1 (c^{v_k}-c^v)u^{\varepsilon,v}w^k_t + \eta\int_0^1 w^k w^k_t &- \int_0^1 (h^{\varepsilon,v_k}-h^{\varepsilon,v})w^k_t.\end{aligned}\tag{3.25}$$

Using the Cauchy inequality we infer from (3.25) that

$$\begin{aligned}\|w^k_t\|^2_{L^2(0,1)} &+ \frac{1}{2}\frac{d}{dt}\int_0^1 \left(x^\alpha|w^k_x|^2 - \frac{\lambda}{x^\beta}|w^k|^2 + \eta|w^k|^2\right)\\ &\le \frac{1}{2}\|w^k_t\|^2_{L^2(Q_T)} + \left(2\|c^{v_k}\|^2_{L^\infty(Q_T)} + 2\eta^2\right)\|w^k\|^2_{L^2(0,1)}\\ &+ 2\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}\|u^{\varepsilon,v}\|^2_{L^2(0,1)} + 2\|h^{\varepsilon,v_k}-h^{\varepsilon,v}\|^2_{L^2(0,1)}.\end{aligned}\tag{3.26}$$

Integrating (3.22) from t to $t+1$, $t \le T-1$, we get

$$\begin{aligned} C(\lambda,\alpha,\beta)\int_t^{t+1}\int_0^1 &\left(x^\alpha|w_x^k|^2-\frac{\lambda}{x^\beta}|w^k|^2+\eta|w^k|^2\right)\\ &\le (2\eta+\|c^{v_k}\|_{L^\infty(Q_T)}+2)\int_t^{t+1}\|w^k\|^2_{L^2(0,1)} \\ &+\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}\int_t^{t+1}\|u^{\varepsilon,v}\|^2_{L^2(0,1)}+\int_t^{t+1}\int_0^1|h^{\varepsilon,v_k}-h^{\varepsilon,v}|^2. \end{aligned} \tag{3.27}$$

Using (3.24) then (3.27) becomes

$$\begin{aligned} \int_t^{t+1}\int_0^1\left(x^\alpha|w_x^k|^2-\frac{\lambda}{x^\beta}|w^k|^2+\eta|w^k|^2\right)&\le C(\eta,T)\Big[e^{\|c^{v_k}\|_{L^\infty(Q_T)}T}\|w^k(0)\|^2_{L^2(0,1)}\\ &+\Big(\|c^{v_k}\|_{L^\infty(Q_T)}e^{\|c^{v_k}\|_{L^\infty(Q_T)}T}+1\Big)\times\\ \times\Big(\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}&\|u^{\varepsilon,v}\|^2_{L^2(Q_T)}+\|h^{\varepsilon,v_k}-h^{\varepsilon,v}\|^2_{L^2(Q_T)}\Big)\Big]. \end{aligned} \tag{3.28}$$

Now, using (3.26) and (3.28) and by the uniform Gronwall inequality, we have

$$\begin{aligned} \int_0^1\left(x^\alpha|w_x^k|^2-\frac{\lambda}{x^\beta}|w^k|^2+\eta|w^k|^2\right)&\le C(\eta,T)\Big[e^{\|c^{v_k}\|_{L^\infty(Q_T)}T}\|w^k(0)\|^2_{L^2(0,1)}\\ +\Big(\|c^{v_k}\|_{L^\infty(Q_T)}e^{\|c^{v_k}\|_{L^\infty(Q_T)T}}+1\Big)&\Big(\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}\|u^{\varepsilon,v}\|^2_{L^2(Q_T)}\\ &+\|h^{\varepsilon,v_k}-h^{\varepsilon,v}\|^2_{L^2(Q_T)}\Big)\Big]. \end{aligned} \tag{3.29}$$

From (3.24) and (3.29), we obtain that

$$\begin{aligned} \|w^k\|^2_X\le &C(\eta,T)\Big[e^{\|c^{v_k}\|_{L^\infty(Q_T)}T}\|w^k(0)\|^2_{L^2(0,1)}\\ &+e^{\|c^{v_k}\|_{L^\infty(Q_T)T}}\Big(\|c^{v_k}-c^v\|^2_{L^\infty(Q_T)}\|u^{\varepsilon,v}\|^2_{L^2(Q_T)}\\ &+\|h^{\varepsilon,v_k}-h^{\varepsilon,v}\|^2_{L^2(Q_T)}\Big)\Big]. \end{aligned} \tag{3.30}$$

The fact that $v_k \to v$ in X as $k\to\infty$ implies that

$$c^{v_k}\to c^v \text{ in } L^\infty(Q_T) \text{ as } k\to\infty, \tag{3.31}$$

$$h^{\varepsilon,v_k}\to h^{\varepsilon,v} \text{ in } L^2(Q_T) \text{ as } k\to\infty. \tag{3.32}$$

Hence,

$$w^k\to 0 \text{ as } k\to\infty \text{ in } X,$$

and the proof is complete. □

Using the above approximate null controllability result, we now prove the null controllability of problem (1.5) when the initial data $u_0\in H^1_{\alpha,0}(0,1)$.

Theorem 3.4. *Let $T>0$ and $u_0\in H^1_{\alpha,0}(0,1)$ be given. Under assumptions* (1.6), (1.7) *and* (1.8)*, problem* (1.5) *is null controllable, that is, there exists $h\in L^2(\omega\times(0,T))$ such that*

$$u(T,x)=0 \text{ for every } x\in(0,1), \tag{3.33}$$

and

$$\int_0^T\int_\omega h^2dxdt \le C_T\int_0^1 u_0^2dx,\ \textit{for some positive constant } C_T. \tag{3.34}$$

Proof. By Theorem 3.3, problem (1.5) is approximately null controllable. Thus, for all $\varepsilon > 0$, there exists $h^\varepsilon \in L^2(\omega\times(0,T))$ such that (3.6) and (3.7) hold. Using (3.7), we have that h^εconverges weakly to h^0 in $L^2(\omega\times(0,T))$ as $\varepsilon\to 0$ and, by the semicontinuity of the norm, it results

$$\int_0^T\int_\omega |h_0|^2dxdt \le \liminf_{\varepsilon\to0}\int_0^T\int_\omega |h^\varepsilon|^2dxdt \le C_T\int_0^1 u_0^2dxdt. \tag{3.35}$$

Now, for all $t\in[0,T]$, proceeding as in Theorem 3.3, one can prove that

$$u^{h^\varepsilon}(t,\cdot)\to u^{h^0}(t,\cdot) \text{ strongly in } X \text{ as } \varepsilon\to 0. \tag{3.36}$$

And then, using (1.6)-(1.7), we can prove u^{h^0} is the solution of (1.5) with $h = h^0$. Moreover, by (3.36) and (3.6), we deduce that

$$u^{h^0}(T,x) = 0 \text{ for all } x\in(0,1).$$

□

We are now ready to prove the main result of the paper.

Theorem 3.5. *Let $T > 0$ and $u_0\in L^2(0,1)$ be given. Under assumptions* (1.6), (1.7) *and* (1.8)*, problem* (1.5) *is null controllable, that is, there exists $h\in L^2(\omega\times(0,T))$ such that*

$$u(T,x) = 0 \textit{ for every } x\in(0,1), \tag{3.37}$$

and

$$\int_0^T\int_\omega h^2dxdt \le C_T\int_0^1 u_0^2dx,\ \textit{for some positive constant } C_T. \tag{3.38}$$

Proof. In the first step, we consider the problem

$$\begin{cases} v_t - (x^\alpha v_x)_x - \dfrac{\lambda}{x^\beta}v + f(t,x,v) = 0, & (t,x)\in Q_{T/2},\\ v(t,1) = v(t,0) = 0, & t\in(0,T/2),\\ v(0,x) = u_0, & x\in(0,1). \end{cases} \tag{3.39}$$

By Theorem 2.4, problem (3.39) has a solution $v\in L^2(0,T;H^1_{\alpha,0}(0,1))$, therefore, there exists $t_0\in(0,T/2)$ such that $v(t_0,x) =: u_1(x)\in H^1_{\alpha,0}(0,1)$.

In the second step, we consider the problem

$$\begin{cases} w_t - (x^\alpha w_x)_x - \dfrac{\lambda}{x^\beta}w + f(w) = 1_\omega h_1, & (t,x)\in(0,1)\times(t_0,T),\\ w(t,1) = w(t,0) = 0, & t\in(t_0,T),\\ w(t_0,x) = u_1, & x\in(0,1). \end{cases} \tag{3.40}$$

By Theorem 3.4, problem (3.40) is null controllable, that is, there exists a control $h_1\in L^2((t_0,T)\times(0,1))$ such that

$$w(T,x) = 0, \quad \text{for all } x\in(0,1),$$

and

$$\int_{t_0}^{T}\int_{\omega} h_1^2 dxdt \le C_{t_0,T}\int_0^1 u_1^2 dx \text{ for some positive constant } C_{t_0,T}.$$

In the third step, to finish the proof, we define u and h by

$$u := \begin{cases} v(t) & \text{for all } t \in [0,t_0], \\ w(t) & \text{for all } t \in [t_0,T], \end{cases} \text{ and } h := \begin{cases} 0 & \text{for all } t \in [0,t_0], \\ h_1 & \text{for all } t \in [t_0,T]. \end{cases}$$

Then u is a solution of (1.5) and satisfies $u(T,x) = 0$ for all $x \in (0,1)$, h satisfies (3.38). □

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