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ON NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS FOR IMPULSIVE NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for impulsive nonlinear differential equations of fractional order with nonlocal integral boundary condition. Our results are based on some suitable fixed point theorems. An illustrative example is presented.
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1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied by many researchers. Fractional differential equations appear naturally in various fields of science and engineering, and constitute an important field of research. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [18, 20, 21, 22]. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 27] and the references therein.

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation. The impulsive differential equations of fractional order have also attracted a considerable attention and a variety of results can be found in the papers [4, 11, 12, 19, 24, 25, 28].

Integral boundary conditions are found to be a useful tool in the mathematical modelling of many practical situations such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [13, 16] and references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are taken to be fixed, for instance, from

0 to 1 in case the independent variable belongs to the interval $[0,1]$. It is imperative to note that the available literature on nonlocal boundary conditions is confined to the nonlocal parameters involvement in the solution or gradient of the solution of the problem. In [26], a nonlocal boundary value problem of impulsive fractional differential equations is studied to obtain the sufficient conditions for the existence of at least one solution of the problem. In [17], the author discussed the existence of solutions for a fractional nonlocal impulsive quasilinear multi-delay integro-differential systems. In [6], a nonlinear fractional boundary value problem with three-point nonlocal integral boundary conditions is addressed.

In this paper, we consider a nonlinear nonlocal impulsive fractional boundary value problem given by

$$\begin{cases} {}^{C}D^{q}x(t) = f(t, x(t)), \ 1 < q \le 2, \ t \in J', \\ \triangle x(t_k) = I_k(x(t_k)), \ \triangle x'(t_k) = I_k^*(x(t_k)), \ k = 1, 2, \cdots, p, \\ x(0) = 0, \ x(1) = \beta \int_0^{\eta} x(s)ds, \ 0 < \eta < 1, \end{cases} \tag{1.1}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative, $f \in C(J\times\mathbb{R},\mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$, $\beta \in \mathbb{R}$, $\beta \neq 2/\eta^2$, $J = [0,1]$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = 1$, $J' = J\backslash\{t_1, t_2, \cdots, t_p\}$, $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k (k = 1, 2, \cdots, p)$, respectively. $\triangle x'(t_k)$ have a similar meaning for $x'(t)$.

We prove some existence and uniqueness results for problem (1.1). The main tools of our study include a well known fixed point theorem (Theorem 3) and Banach's contraction mapping principle. This choice of fixed point theorems ensures less restrictive criteria for our existence results and can readily be verified. In fact, our approach is simple and is applicable to a variety of problems. We demonstrate it by providing an example.

2. Preliminaries

Let $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, 1]$, and we introduce the spaces:

$PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} | x \in C(J_k),\ k = 0, 1, \cdots, p,$ and $x(t_k^+)$ exist, $k = 1, 2, \cdots, p, \}$

with the norm $\|x\| = \sup_{t\in J} |x(t)|$, and

$$PC^1(J,\mathbb{R}) = \{x : J \to \mathbb{R} \mid x \in C^1(J_k),\ k = 0, 1, \cdots, p, \text{ and } x(t_k^+), \\ x'(t_k^+) \text{ exist}, \ k = 1, 2, \cdots, p, \}$$

with the norm $\|x\|_{PC^1} = \max\{\|x\|, \|x'\|\}$. Obviously, $PC(J,\mathbb{R})$ and $PC^1(J,\mathbb{R})$ are Banach spaces.

Definition 2.1. For a continuous function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α.

Definition 2.2. The Riemann-Liouville fractional integral of order α is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ \alpha>0,$$

provided the integral exists.

Definition 2.3. A function $x\in PC^1(J,\mathbb{R})$ with its Caputo derivative of order q existing on J is a solution of (1.1) if it satisfies (1.1).

The following lemma plays a pivotal role in the forthcoming analysis.

Lemma 2.1. *Let $\beta\neq 2/\eta^2, 0<\eta<1, \eta\in J_m$, where m is a nonnegative integer, $0\le m\le p$. For a given $y\in C[0,1]$, a function x is a solution of the integral boundary value problem*

$$\begin{cases} {}^C D^q x(t)=y(t),\ 1<q\le 2,\ t\in J', \\ \triangle x(t_k)=I_k(x(t_k)),\ \triangle x'(t_k)=I_k^*(x(t_k)),\ k=1,2,\cdots,p, \\ x(0)=0,\ x(1)=\beta\int_0^{\eta}x(s)ds, \end{cases} \tag{2.1}$$

if and only if x is a solution of the impulsive fractional integral equation

$$x(t)=\begin{cases} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds+\frac{2\beta t}{2-\beta\eta^2}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}y(s)ds \\ -\frac{2t}{2-\beta\eta^2}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds+M(t),\ t\in J_0; \\ \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds+\frac{2\beta t}{2-\beta\eta^2}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}y(s)ds \\ -\frac{2t}{2-\beta\eta^2}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds \\ +\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big] \\ +\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big] \\ +\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]+M(t),\ t\in J_k, \\ k=1,2,\cdots,p. \end{cases} \tag{2.2}$$

where

$$M(t)=\frac{-2t}{2-\beta\eta^2}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$

$$+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{\beta(t_{j+1}-s)^q}{\Gamma(q+1)}y(s)ds$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j}\beta(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}\beta(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{i=1}^{m}\beta(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$
$$-\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]\Big\}.$$

Proof. Let x be a solution of (2.1). Then, for $t\in J_0$, there exist constants $c_1,c_2\in\mathbb{R}$ such that

$$x(t)=I^qy(t)-c_1-c_2t=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s)ds-c_1-c_2t, \tag{2.3}$$
$$x'(t)=\frac{1}{\Gamma(q-1)}\int_0^t(t-s)^{q-2}y(s)ds-c_2.$$

For $t\in J_1$, there exist constants $d_1,d_2\in\mathbb{R}$, such that

$$x(t)=\frac{1}{\Gamma(q)}\int_{t_1}^t(t-s)^{q-1}y(s)ds-d_1-d_2(t-t_1),$$
$$x'(t)=\frac{1}{\Gamma(q-1)}\int_{t_1}^t(t-s)^{q-2}y(s)ds-d_2.$$

Then we have

$$x(t_1^-)=\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)ds-c_1-c_2t_1,\ x(t_1^+)=-d_1,$$
$$x'(t_1^-)=\frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)ds-c_2,\ x'(t_1^+)=-d_2,$$

In view of the impulse conditions

$$\triangle x(t_1)=x(t_1^+)-x(t_1^-)=I_1(x(t_1))$$

and

$$\triangle x'(t_1) = x'(t_1^+) - x'(t_1^-) = I_1^*(x(t_1)),$$

we have that

$$-d_1 = \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} y(s) ds - c_1 - c_2 t_1 + I_1(x(t_1)),$$

$$-d_2 = \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} y(s) ds - c_2 + I_1^*(x(t_1)).$$

Consequently,

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} y(s) ds + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} y(s) ds \\ & + \frac{t - t_1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} y(s) ds \\ & + I_1(x(t_1)) + (t - t_1) I_1^*(x(t_1)) - c_1 - c_2 t, \ t \in J_1. \end{aligned}$$

By a similar process, we get

$$\begin{aligned} x(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} y(s) ds + I_i(x(t_i)) \Big] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} y(s) ds + I_i^*(x(t_i)) \Big] \\ & + \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} y(s) ds + I_i^*(x(t_i)) \Big] - c_1 - c_2 t, \ t \in J_k, \ k = 1, 2, \cdots, p. \end{aligned} \tag{2.4}$$

By the condition $x(0) = 0$, we get $c_1 = 0$. For a given $\eta \in J_m, \ 0 \le m \le p$, we have

$$\begin{aligned} \int_0^{\eta} x(s) ds = & \Big(\int_0^{t_1} + \int_{t_1}^{t_2} + \cdots + \int_{t_{m-1}}^{t_m} + \int_{t_m}^{\eta} \Big) x(s) ds \\ = & \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \Big(\int_{t_j}^{s} \frac{(s-r)^{q-1}}{\Gamma(q)} y(r) dr \Big) ds \\ & + \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \Big(\sum_{i=1}^{j} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{q-1}}{\Gamma(q)} y(r) dr + I_i(x(t_i)) \Big] \Big) ds \\ & + \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \Big(\sum_{i=1}^{j-1} (t_j - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{q-2}}{\Gamma(q-1)} y(r) dr + I_i^*(x(t_i)) \Big] \Big) ds \\ & + \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \Big(\sum_{i=1}^{j} (s - t_j) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - r)^{q-2}}{\Gamma(q-1)} y(r) dr + I_i^*(x(t_i)) \Big] \Big) ds \\ & + \int_{t_m}^{\eta} \Big(\int_{t_m}^{s} \frac{(s-r)^{q-1}}{\Gamma(q)} y(r) dr \Big) ds \end{aligned}$$

$$+\int_{t_m}^{\eta}\Big(\sum_{i=1}^{m}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-1}}{\Gamma(q)}y(r)dr+I_i(x(t_i))\Big]\Big)ds$$

$$+\int_{t_m}^{\eta}\Big(\sum_{i=1}^{m-1}(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]\Big)ds$$

$$+\int_{t_m}^{\eta}\Big(\sum_{i=1}^{m}(s-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]\Big)ds-c_2\int_0^{\eta}sds$$

$$=\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-r)^q}{\Gamma(q+1)}y(r)dr$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-1}}{\Gamma(q)}y(r)dr+I_i(x(t_i))\Big]$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]$$

$$+\frac{1}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]$$

$$+\int_{t_m}^{\eta}\frac{(\eta-r)^q}{\Gamma(q+1)}y(r)dr+\sum_{i=1}^{m}(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-1}}{\Gamma(q)}y(r)dr+I_i(x(t_i))\Big]$$

$$+\sum_{i=1}^{m-1}(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]$$

$$+\frac{1}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-r)^{q-2}}{\Gamma(q-1)}y(r)dr+I_i^*(x(t_i))\Big]-\frac{\eta^2}{2}c_2.$$

Using the condition $x(1)=\beta\int_0^{\eta}x(s)ds$, we find that

$$-c_2=\frac{-2}{2-\beta\eta^2}\Big\{\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)ds+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$

$$+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$

$$-\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{\beta(t_{j+1}-s)^q}{\Gamma(q+1)}y(s)ds$$

$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j}\beta(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}\beta(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\int_{t_m}^{\eta}\frac{\beta(\eta-s)^q}{\Gamma(q+1)}y(s)ds-\sum_{i=1}^{m}\beta(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}y(s)ds+I_i(x(t_i))\Big]$$
$$-\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}y(s)ds+I_i^*(x(t_i))\Big]\Big\}.$$

Substituting the value of c_1, c_2 in (2.3) and (2.4), we obtain (2.2). Conversely, assume that x is a solution of the impulsive fractional integral equation (2.2), then by a direct computation, it follows that the solution given by (2.2) satisfies (2.1). This completes the proof.

3. Main results

Let $\beta \neq 2/\eta^2, 0<\eta<1, \eta\in J_m$, where m is a nonnegative integer, $0\le m\le p$. Define an operator $T: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ as

$$Tx(t)=\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\frac{2\beta t}{2-\beta\eta^2}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}f(s,x(s))ds$$
$$-\frac{2t}{2-\beta\eta^2}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds$$
$$+\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$+\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\frac{2t}{2-\beta\eta^2}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{\beta(t_{j+1}-s)^q}{\Gamma(q+1)}f(s,x(s))ds$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j}\beta(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}\beta(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{i=1}^{m}\beta(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$
$$-\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]\Big\}. \quad (3.1)$$

Observe that the problem (1.1) has a solution if and only if the operator T has a fixed point.

Lemma 3.1. The operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by (3.1) is completely continuous.

Proof. It is obvious that T is continuous in view of continuity of f, I_k and I_k^*. Let $\Omega \subset PC(J,\mathbb{R})$ be bounded. Then, there exist a function $L_1(t) \in C(J,\mathbb{R}^+)$ and positive constants $L_2, L_3 > 0$ such that $|f(t,x(t))| \le L_1(t), |I_k(x)| \le L_2$ and $|I_k^*(x)| \le L_3$, $\forall x \in \Omega$. Thus, $\forall x \in \Omega$, we have

$$|Tx(t)| \le \int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds$$
$$+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds$$
$$+\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{|\beta|(t_{j+1}-s)^{q}}{\Gamma(q+1)}|f(s,x(s))|ds$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j}|\beta|(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}|\beta|(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\frac{|\beta|}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{i=1}^{m}|\beta|(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{i=1}^{m-1}|\beta|(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\frac{|\beta|}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\Big\}$$

$$\leq\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}L_1(s)ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^{q}}{\Gamma(q+1)}L_1(s)ds$$

$$+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}L_1(s)ds+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]$$

$$+\sum_{i=1}^{p-1}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]$$

$$+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]$$

$$+\sum_{i=1}^{p-1}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]$$

$$+\sum_{j=0}^{p-1}|\beta|\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{q}}{\Gamma(q+1)}L_1(s)ds+\sum_{j=1}^{p-1}\sum_{i=1}^{j}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]$$

$$+\sum_{j=1}^{p-1}\sum_{i=1}^{j-1}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]$$

$$+\sum_{j=1}^{p-1}\sum_{i=1}^{j}\frac{|\beta|}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]$$

$$+\sum_{i=1}^{p}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]$$

$$+\sum_{i=1}^{p-1}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]$$

$$+\sum_{i=1}^{p}\frac{|\beta|\eta^2}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\Big\}$$

$$\le I^qL_1(t)+\frac{2|\beta|}{|2-\beta\eta^2|}I^{q+1}L_1(\eta)+\frac{2}{|2-\beta\eta^2|}I^qL_1(1)+p[I^qL_1(1)+L_2]$$

$$+(p-1)[I^{q-1}L_1(1)+L_3]+p[I^{q-1}L_1(1)+L_3]$$

$$+\frac{2}{|2-\beta\eta^2|}\Big\{p[I^qL_1(1)+L_2]+(p-1)[I^{q-1}L_1(1)+L_3]$$

$$+p[I^{q-1}L_1(1)+L_3]+p|\beta|I^{q+1}L_1(1)+\frac{p(p-1)|\beta|}{2}[I^qL_1(1)+L_2]$$

$$+\frac{(p-1)(p-2)|\beta|}{2}[I^{q-1}L_1(1)+L_3]$$

$$+\frac{p(p-1)|\beta|}{4}[I^{q-1}L_1(1)+L_3]+p|\beta|\eta[I^qL_1(1)+L_2]$$

$$+(p-1)|\beta|\eta[I^{q-1}L_1(1)+L_3]+\frac{p|\beta|\eta^2}{2}[I^{q-1}L_1(1)+L_3]\Big\}$$

$$\leq \frac{2|\beta|}{|2-\beta\eta^2|}\Big[I^{q+1}L_1(\eta)+pI^{q+1}L_1(1)\Big]+\Big(1+\frac{2}{|2-\beta\eta^2|}\Big)I^qL_1(1)$$

$$+\Big[p+\frac{2p(1+|\beta|\eta)+p(p-1)|\beta|}{|2-\beta\eta^2|}\Big]\Big[I^qL_1(1)+L_2\Big]$$

$$+\Big\{2p-1+\frac{2}{|2-\beta\eta^2|}\Big[2p-1+\frac{(p-1)(3p-4)|\beta|}{4}+(p-1+\frac{p\eta}{2})|\beta|\eta\Big]\Big\}\Big[I^{q-1}L_1(1)+L_3\Big]=L, \tag{3.2}$$

which implies that $\|Tx\|\leq L$. On the other hand, for any $t\in J_k, 0\leq k\leq p$, we have

$$|(Tx)'(t)|\leq\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds$$

$$+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds$$

$$+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{|\beta|(t_{j+1}-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j}|\beta|(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}|\beta|(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\frac{|\beta|}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$+\sum_{i=1}^{m}|\beta|(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$+\sum_{i=1}^{m-1}|\beta|(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$

$$
\begin{aligned}
&+\frac{|\beta|}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\Big\}\\
&\le \int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^{q}}{\Gamma(q+1)}L_1(s)ds\\
&+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}L_1(s)ds\\
&+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]\\
&+\sum_{i=1}^{p-1}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\\
&+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\\
&+\sum_{j=0}^{p-1}|\beta|\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^{q}}{\Gamma(q+1)}L_1(s)ds\\
&+\sum_{j=1}^{p-1}\sum_{i=1}^{j}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]\\
&+\sum_{j=1}^{p-1}\sum_{i=1}^{j-1}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\\
&+\sum_{j=1}^{p-1}\sum_{i=1}^{j}\frac{|\beta|}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\\
&+\sum_{i=1}^{p}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}L_1(s)ds+L_2\Big]\\
&+\sum_{i=1}^{p-1}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\\
&+\sum_{i=1}^{p}\frac{|\beta|\eta^2}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}L_1(s)ds+L_3\Big]\Big\}\\
&\le \frac{2|\beta|}{|2-\beta\eta^2|}\Big[I^{q+1}L_1(\eta)+pI^{q+1}L_1(1)\Big]+\frac{2}{|2-\beta\eta^2|}I^{q}L_1(1)\\
&+\frac{2p(1+|\beta|\eta)+p(p-1)|\beta|}{|2-\beta\eta^2|}\Big[I^{q}L_1(1)+L_2\Big]+I^{q-1}L_1(1)\\
&+\frac{2}{|2-\beta\eta^2|}\Big[2p-1+\frac{(p-1)(3p-4)|\beta|}{4}+(p-1+\frac{p\eta}{2})|\beta|\eta\Big]\Big[I^{q-1}L_1(1)+L_3\Big]=\overline{L}.
\end{aligned}
$$

Hence, for $t_1, t_2 \in J_k,\ t_1 < t_2,\ 0 \le k \le p$, we have

$$|(Tx)(t_2) - (Tx)(t_1)| \le \int_{t_1}^{t_2} |(Tx)'(s)|ds \le \overline{L}(t_2 - t_1).$$

This implies that T is equicontinuous on all $J_k, k = 0, 1, 2, \cdots, p$. Thus, by the Arzela-Ascoli Theorem, the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. This completes the proof.

We need the following known fixed point theorem [23] to prove the existence of solution for (1.1).

Theorem 3.2. *Let E be a Banach space. Assume that $T : E \to E$ be a completely continuous operator and the set $V = \{x \in E \mid x = \mu Tx, 0 < \mu < 1\}$ be bounded. Then T has a fixed point in E.*

Now, we are in a position to prove the main results of this paper.

Theorem 3.3. *Assume that*
(H_1) there exist a function $L_1(t) \in C(J, \mathbb{R}^+)$ and positive constants L_i $(i = 2, 3)$ such that $|f(t, x)| \le L_1(t),\ |I_k(x)| \le L_2,\ |I_k^*(x)| \le L_3,\ for\ t \in J,\ x \in \mathbb{R}$ and $k = 1, 2, \cdots, p$. Then the problem (1.1) has at least one solution.

Proof. Let us consider the set

$$V = \{x \in PC(J, \mathbb{R}) \mid x = \mu Tx, 0 < \mu < 1\},$$

where the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is defined by (3.1). We just need to show that the set V is bounded as it has already been proved that the operator T is completely continuous in the lemma 3.1. Let $x \in V$, then $x = \mu Tx,\ 0 < \mu < 1$. For any $t \in J$, we have

$$\begin{aligned} x(t) = &\int_{t_k}^{t} \frac{\mu(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + \frac{2\mu\beta t}{2-\beta\eta^2} \int_{t_m}^{\eta} \frac{(\eta-s)^q}{\Gamma(q+1)} f(s, x(s))ds \\ &- \frac{2\mu t}{2-\beta\eta^2} \int_{t_p}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\ &+ \sum_{i=1}^{k} \mu \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + I_i(x(t_i)) \Big] \\ &+ \sum_{i=1}^{k-1} (t_k - t_i)\mu \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds + I_i^*(x(t_i)) \Big] \\ &+ \sum_{i=1}^{k} (t - t_k)\mu \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds + I_i^*(x(t_i)) \Big] \\ &- \frac{2\mu t}{2-\beta\eta^2} \Big\{ \sum_{i=1}^{p} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + I_i(x(t_i)) \Big] \end{aligned}$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\mu\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$+\sum_{i=1}^{p}(1-t_p)\mu\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{\mu\beta(t_{j+1}-s)^{q}}{\Gamma(q+1)}f(s,x(s))ds$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j}\beta\mu(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$
$$-\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}\beta\mu(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta\mu}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\sum_{i=1}^{m}\beta\mu(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+I_i(x(t_i))\Big]$$
$$-\sum_{i=1}^{m-1}\beta\mu(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]$$
$$-\frac{\beta\mu}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds+I_i^*(x(t_i))\Big]\Big\}. \tag{3.3}$$

Using the assumption (H_1) in (3.3), we obtain

$$|x(t)|=\mu|Tx(t)|\le\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^{q}}{\Gamma(q+1)}|f(s,x(s))|ds$$
$$+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds$$
$$+\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$
$$+\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]$$
$$+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]$$

$$
\begin{aligned}
&+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\\
&+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\\
&+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{|\beta|(t_{j+1}-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds\\
&+\sum_{j=1}^{m-1}\sum_{i=1}^{j}|\beta|(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]\\
&+\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}|\beta|(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\\
&+\frac{|\beta|}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\\
&+\sum_{i=1}^{m}|\beta|(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|I_i(x(t_i))|\Big]\\
&+\sum_{i=1}^{m-1}|\beta|(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\\
&+\frac{|\beta|}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|I_i^*(x(t_i))|\Big]\Big\}\\
&\le\frac{2|\beta|}{|2-\beta\eta^2|}\Big[I^{q+1}L_1(\eta)+pI^{q+1}L_1(1)\Big]+\Big(1+\frac{2}{|2-\beta\eta^2|}\Big)I^qL_1(1)\\
&+\Big[p+\frac{2p(1+|\beta|\eta)+p(p-1)|\beta|}{|2-\beta\eta^2|}\Big]\Big[I^qL_1(1)+L_2\Big]\\
&+\Big\{2p-1+\frac{2}{|2-\beta\eta^2|}\Big[2p-1+\frac{(p-1)(3p-4)|\beta|}{4}\\
&+(p-1+\frac{p\eta}{2})|\beta|\eta\Big]\Big\}\Big[I^{q-1}L_1(1)+L_3\Big]=M,
\end{aligned}
$$

which implies that $\|x\|\le M$ for any $t\in J$. So, the set V is bounded. Thus, by the conclusion of Theorem 3, the operator T has at least one fixed point, which implies that (1.1) has at least one solution. This completes the proof.

Theorem 3.4. *Assume that*

(H_2) *there exist a function $K_1\in C(J,\mathbb{R}^+)$ and nonnegative constants $K_i(i=2,3)$ such that $|f(t,x)-f(t,y)|\le K_1(t)|x-y|$, $|I_k(x)-I_k(y)|\le K_2|x-y|$, $|I_k^*(x)-I_k^*(y)|\le K_3|x-y|$, for $t\in J$, $x,y\in\mathbb{R}$ and $k=1,2,\cdots,p$. Then problem* (1.1) *has a unique solution if $\mathcal{H}<1$, where*

$$\mathcal{H}=\frac{2|\beta|}{|2-\beta\eta^2|}\Big[I^{q+1}K_1(\eta)+pI^{q+1}K_1(1)\Big]+\Big(1+\frac{2}{|2-\beta\eta^2|}\Big)I^qK_1(1)$$

$$+\lambda_1\Big[I^qK_1(1)+K_2\Big]+\lambda_2\Big[I^{q-1}K_1(1)+K_3\Big], \tag{3.4}$$

$$\lambda_1=p+\frac{2p(1+|\beta|\eta)+p(p-1)|\beta|}{|2-\beta\eta^2|},$$

$$\lambda_2=2p-1+\frac{2}{|2-\beta\eta^2|}\Big[2p-1+\frac{(p-1)(3p-4)|\beta|}{4}+(p-1+\frac{p\eta}{2})|\beta|\eta\Big].$$

Proof. For $x,y\in PC(J,\mathbb{R})$, we have

$$\begin{aligned}
|(Tx)(t)-(Ty)(t)| \le & \int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\
&+\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds+|I_i(x(t_i))-I_i(y(t_i))|\Big]\\
&+\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&\qquad+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&\qquad+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\
&\qquad+|I_i(x(t_i))-I_i(y(t_i))|\Big]\\
&+\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&\qquad+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\sum_{i=1}^{p}(1-t_p)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\frac{|\beta|(t_{j+1}-s)^q}{\Gamma(q+1)}|f(s,x(s))-f(s,y(s))|ds\\
&+\sum_{j=1}^{m-1}\sum_{i=1}^{j}|\beta|(t_{j+1}-t_j)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&+|I_i(x(t_i))-I_i(y(t_i))|\Big]\\
&+\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}|\beta|(t_{j+1}-t_j)(t_j-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\frac{|\beta|}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(t_{j+1}-t_j)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\sum_{i=1}^{m}|\beta|(\eta-t_m)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\
&+|I_i(x(t_i))-I_i(y(t_i))|\Big]\\
&+\sum_{i=1}^{m-1}|\beta|(\eta-t_m)(t_m-t_i)\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\\
&+\frac{|\beta|}{2}\sum_{i=1}^{m}(\eta-t_m)^2\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|I_i^*(x(t_i))-I_i^*(y(t_i))|\Big]\Big\}\\
&\le\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}K_1(s)\|x-y\|ds+\frac{2|\beta|}{|2-\beta\eta^2|}\int_{t_m}^{\eta}\frac{(\eta-s)^{q}}{\Gamma(q+1)}K_1(s)\|x-y\|ds\\
&+\frac{2}{|2-\beta\eta^2|}\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}K_1(s)\|x-y\|ds\\
&+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}K_1(s)ds+K_2\Big]\|x-y\|\\
&+\sum_{i=1}^{p-1}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|\\
&+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|\\
&+\frac{2}{|2-\beta\eta^2|}\Big\{\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}K_1(s)ds+K_2\Big]\|x-y\|\\
&+\sum_{i=1}^{p-1}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|
\end{aligned}
$$

$$+\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|$$
$$+\sum_{j=0}^{p-1}|\beta|\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-s)^q}{\Gamma(q+1)}K_1(s)ds\|x-y\|$$
$$+\sum_{j=1}^{p-1}\sum_{i=1}^{j}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}K_1(s)ds+K_2\Big]\|x-y\|$$
$$+\sum_{j=1}^{p-1}\sum_{i=1}^{j-1}|\beta|\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|$$
$$+\sum_{j=1}^{p-1}\sum_{i=1}^{j}\frac{|\beta|}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|$$
$$+\sum_{i=1}^{p}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}K_1(s)ds+K_2\Big]\|x-y\|$$
$$+\sum_{i=1}^{p-1}|\beta|\eta\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|$$
$$+\sum_{i=1}^{p}\frac{|\beta|\eta^2}{2}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}K_1(s)ds+K_3\Big]\|x-y\|\Big\}$$
$$\le\Big\{\frac{2|\beta|}{|2-\beta\eta^2|}\Big[I^{q+1}K_1(\eta)+pI^{q+1}K_1(1)\Big]+\Big(1+\frac{2}{|2-\beta\eta^2|}\Big)I^qK_1(1)$$
$$+\lambda_1\Big[I^qK_1(1)+K_2\Big]+\lambda_2\Big[I^{q-1}K_1(1)+K_3\Big]\Big\}\|x-y\|.$$

Consequently, we have $\|Tx-Ty\|\le\mathcal{H}\|x-y\|$, where $\mathcal{H}$ is given by (3.4). As $\mathcal{H}<1$, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof.

Remark 3.5. *The existence results for a nonlocal integral boundary value problems for impulsive nonlinear second-order differential equations follow by taking* $q=2$ *in the results of this paper. In the limit* $\eta\to1$, *our results correspond to the ones with the usual integral boundary condition* $x(1)=\beta\int_0^1 x(s)ds$.

Example 3.6. Consider the following nonlocal integral boundary value problem for impulsive nonlinear fractional differential equations

$$\begin{cases} {}^CD^qx(t)=\dfrac{x^6(t)+\sin^{\frac{2}{3}}(3x(t)+1)+5t^2}{3x^6(t)+e^{\cos x(t)}},\ 0<t<1,\ t\neq t_1,\\ \triangle x(t_1)=e^{-x^2(t_1)}+2\sin x(t_1),\ \triangle x'(t_1)=2\cos(3+5x^3(t_1)),\\ x(0)=0,\ x(1)=3\int_0^{\frac{2}{5}}x(s)ds, \end{cases} \tag{3.5}$$

where $1<q\le2,\ \beta=3,\ \eta=\dfrac{2}{5}$ and $p=1$.

In this case, $L_1(t) = (1 + 5t^2)e,\ L_2 = 3,\ L_3 = 2$, and the conditions of Theorem 3 can readily be verified. Therefore, the conclusion of Theorem 3 applies to the impulsive fractional integral boundary value problem (3.5).

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