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MULTIPLICITY OF CONCAVE AND MONOTONE POSITIVE SOLUTIONS FOR NONLINEAR FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. By constructing an equivalent operator and using fixed point theorem, multiple positive solutions for some fourth-order multi-point boundary value problems with nonlinearity depending on all order derivatives are obtained. The associated Green's functions are also given.
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1. Introduction

In this paper, we are interested in the positive solution for fourth-order m-point boundary value problems

$$\begin{cases} x^{(4)}(t) + f(t,\ x(t),\ x'(t),\ x''(t),\ x'''(t)) = 0,\ t \in [0,1] & (1.1) \\ x'''(0) = 0,\ x''(0) = 0,\ x'(0) = 0,\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) & (1.2) \end{cases}$$

and

$$\begin{cases} x^{(4)}(t) + f(t,\ x(t),\ x'(t),\ x''(t),\ x'''(t)) = 0,\ t \in [0,1] & (1.1) \\ x'''(1) = 0,\ x''(1) = 0,\ x'(1) = 0,\ x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) & (1.3) \end{cases}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \beta_i < 1,\ i = 1, 2, \cdots, m-2,\ \sum_{i=1}^{m-2} \beta_i < 1$ and $f \in C([0,1] \times R^4, [0, +\infty))$.

It is well known that boundary value problems for nonlinear differential equations arise in a large number problems in physics, biology and chemistry. For example, the deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations. Owing to its

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importance in application, the existence of positive solutions for nonlinear second-order or high-order boundary value problems have been studied by many authors. We refer to recent contributions of Ma[1, 2, 3], He and Ge [4], Guo and Ge [5], R. I. Avery et al.[6, 7], J. Henderson [8], P.W. Eloe and J. Henderson [9], Yang et al. [10], J.R.L. Webb and G. Infante [11, 12], R.P. Agarwal and D. O'Regan [13]. For survey of known results and additional references, we refer the reader to the monographs by Agarwal [14] and Agarwal et al [15].

Equation (1.1), often referred to as the beam equation, has been studied under a variety of two point boundary conditions, see in [16-24]. But few work is concerned with the positive solutions with the m-point boundary conditions. Furthermore, for nonlinear fourth-order equations, only the situation that the nonlinear term does not depend on the first, second and third order derivatives are considered, see in [16-23]. Few paper deals with the situation that lower order derivatives are involved in the nonlinear term explicitly. In fact, the derivatives are of great importance in problems in some cases. For example, in the linear elastic beam equation (Euler-Bernoulli equation)

$$(EIu''(t))'' = f(t),\ t \in (0, L),$$

where $u(t)$ is the deformation function, L is the length of the beam, $f(t)$ is the load density, E is the Young's modulus of elasticity and I is the moment of inertia of the cross-section of the beam. In this problem, the physical meaning of the derivatives of the function $u(t)$ is as follows: $u^{(4)}(t)$ is the load density stiffness, $u'''(t)$ is the shear force stiffness, $u''(t)$ is the bending moment stiffness and the $u'(t)$ is the slope. If the payload depends on the shear force stiffness, bending moment stiffness or the slope, the derivatives of the unknown function are involved in the nonlinear term explicitly.

The goal of the present paper is to study the fourth-order multi-point boundary value problems (1.1, 1.2) and (1.1, 1.3) which all order derivatives are involved in the nonlinear term explicitly. In this sense, the problem studied in this paper are more general than before. In order to overcome the difficulty of the derivatives that appear, our main technique is to transfer the problem to a equivalent operator equation by constructing the associate Green's function and apply a fixed point theorem due to Avery and Peterson [25]. In this paper, multiple concave and monotone positive solutions for problem (1.1, 1.2) and (1.1, 1.3) are established. The results presented extends the study of boundary value problems for fourth-order nonlinear ordinary differential equations.

This paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to the existence of at least three concave and decreasing positive solutions for problem (1.1, 1.2). In section 4, we prove that there exist at least three concave and increasing positive solutions for problem (1.1, 1.3).

2. Preliminaries and Lemmas

In this section, some preliminaries and lemmas used later are presented.

Definition 2.1 Let E be real Banach space. A nonempty convex closed set $P \subset E$ is called a cone provided that

(1) $au \in P$, for all $u \in P,\ a \geq 0$;
(2) $u, -u \in P$ implies $u = 0$.
Definition 2.2 The map α is said to be a nonnegative continuous convex functional on cone P of a real Banach space E provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y),$$

for all $x,\ y \in P$ *and* $t \in [0, 1]$.
Definition 2.3 The map β is said to be a nonnegative continuous concave functional on cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y),$$

for all $x,\ y \in P$ and $t \in [0, 1]$.

Let γ, θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then for positive numbers $a,\ b,\ c$ and d, we define the following convex sets:
$P(\gamma,\ d) = \{x \in P \mid \gamma(x) < d\}$,
$P(\gamma,\ \alpha,\ b,\ d) = \{x \in P \mid b \leq \alpha(x),\ \gamma(x) \leq d\}$,
$P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) = \{x \in P \mid b \leq \alpha(x),\ \theta(x) \leq c,\ \gamma(x) \leq d\}$
and a closed set

$$R(\gamma,\ \psi,\ a,\ d) = \{x \in P \mid a \leq \psi(x),\ \gamma(x) \leq d\}.$$

The following Avery and Peterson fixed point theorem will be used to prove our main results.
Lemma 2.1 *Let P be a cone in Banach space E. Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P satisfying:*

$$\psi(\lambda x) \leq \lambda\psi(x),\ for\ 0 \leq \lambda \leq 1, \tag{2.1}$$

such that for some positive numbers l and d,

$$\alpha(x) \leq \psi(x),\ \|x\| \leq l\gamma(x) \tag{2.2}$$

for all $x \in \overline{P(\gamma,\ d)}$. Suppose $T : \overline{P(\gamma,\ d)} \to \overline{P(\gamma,\ d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a < b$ such that
(S_1) $\{x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) | \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$;
(S_2) $\alpha(Tx) > b$ for $x \in P(\gamma,\ \alpha,\ b,\ d)$ with $\theta(Tx) > c$;
(S_3) $0 \notin R(\gamma,\ \psi,\ a,\ d)$ and $\psi(Tx) < a$ for $x \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x) = a$.
Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma,\ d)}$ such that:

$$\gamma(x_i) \leq d,\ i = 1,\ 2,\ 3;\ b < \alpha(x_1); a < \psi(x_2),\ \alpha(x_2) < b;\ \psi(x_3) < a. \tag{2.3}$$

3. Positive solutions for problem (1.1, 1.2)

Firstly we consider the properties of the solution for following fourth-order m-point boundary value problem

$$x^{(4)}(t) = y(t),\ t \in [0, 1] \tag{3.1}$$

$$x'''(0) = 0,\ x''(0) = 0,\ x'(0) = 0,\quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \tag{3.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \beta_i < 1,\ i = 1, 2, \cdots, m-2$, and $\sum_{i=1}^{m-2} \beta_i < 1$.

Lemma 3.1 *Denote* $\xi_0 = 0,\ \xi_{m-1} = 1,\ \beta_0 = \beta_{m-1} = 0,\ y(t) \in C[0,1]$*, problem* $(3.1), (3.2)$ *has the unique solution*

$$x(t) = \int_0^1 G(t,s) y(s) ds,$$

where for $i = 1,\ 2,\ \cdots,\ m-1$,

$$G(t,s) = \begin{cases} -\dfrac{1}{6}s^3 + \dfrac{\dfrac{1}{6} - \dfrac{1}{2}s + \dfrac{1}{2}s^2 - \dfrac{1}{6}\sum\limits_{k=0}^{i-1}\beta_k s^3 + \sum\limits_{k=i}^{m-2}\beta_k\xi_k(-\dfrac{1}{6}\xi_k^2 + \dfrac{1}{2}\xi_k s - \dfrac{1}{2}s^2)}{1 - \sum\limits_{k=0}^{m-1}\beta_k}, \\ \qquad t \le s,\ \xi_{i-1} \le s \le \xi_i \\ -\dfrac{1}{6}t^3 + \dfrac{1}{2}st^2 - \dfrac{1}{2}s^2 t \\ + \dfrac{\dfrac{1}{6} - \dfrac{1}{2}s + \dfrac{1}{2}s^2 - \dfrac{1}{6}\sum\limits_{k=0}^{i-1}\beta_k s^3 + \sum\limits_{k=i}^{m-2}\beta_k\xi_k(-\dfrac{1}{6}\xi_k^2 + \dfrac{1}{2}\xi_k s - \dfrac{1}{2}s^2)}{1 - \sum\limits_{k=0}^{m-1}\beta_k}, \\ \qquad t \ge s,\ \xi_{i-1} \le s \le \xi_i \end{cases}$$

Proof. Let $G(t,s)$ be the Green's function of problem $-x^{(4)}(t) = 0$ with boundary condition (3.2). We can suppose

$$G(t,s) = \begin{cases} a_3 t^3 + a_2 t^2 + a_1 t + a_0, & t \le s,\ \xi_{i-1} \le s \le \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1 \\ b_3 t^3 + b_2 t^2 + b_1 t + b_0, & t \ge s,\ \xi_{i-1} \le s \le \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1 \end{cases}$$

where $a_i,\ b_i, i = 0,\ 1,\ 2,\ 3$ are unknown coefficients. Considering the properties of Green's function and boundary condition (3.2), we have

$$\begin{cases} a_3 s^3 + a_2 s^2 + a_1 s + a_0 = b_3 s^3 + b_2 s^2 + b_1 s + b_0 \\ 3a_3 s^2 + 2a_2 s + a_1 = 3b_3 s^2 + 2b_2 s + b_1 \\ 6a_3 s + 2a_2 = 6b_3 s + 2b_2 \\ 6a_3 - 6b_3 = 1 \\ a_3 = 0, \\ a_2 = 0, \\ a_1 = 0 \\ b_3 + b_2 + b_1 + b_0 = \sum\limits_{k=0}^{i=1}\beta_k(a_3\xi_k^3 + a_2\xi_k^2 + a_1\xi_k + a_0) + \sum\limits_{k=i}^{m-2}\beta_k(b_3\xi_k^3 + b_2\xi_k^2 + b_1\xi_k + b_0) \end{cases}$$

A straightforward calculation shows that

$$b_3 = -\frac{1}{6}, a_3 = a_2 = a_1 = 0,\ b_2 = \frac{s}{2},\ b_1 = -\frac{s^2}{2}$$

$$a_0 = -\frac{1}{6}s^3 + \frac{\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k\xi_k(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2)}{1 - \sum_{k=0}^{m-1}\beta_k},$$

$$b_0 = \frac{\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k\xi_k(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2)}{1 - \sum_{k=0}^{m-1}\beta_k}$$

These give the explicit expression of the Green's function and the proof of Lemma 3.1 is completed.

Lemma 3.2 *One can see that* $G(t,s) \geq 0,\ t,\ s \in [0,1]$.

Proof. For $\xi_{i-1} \leq s \leq \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1,$

$$\frac{\partial G(t,\ s)}{\partial t} = \begin{cases} 0, & t \leq s,\ \xi_{i-1} \leq s \leq \xi_i \\ -\frac{1}{2}(t-s)^2, & t \geq s,\ \xi_{i-1} \leq s \leq \xi_i \end{cases}$$

Then $\frac{\partial G(t,\ s)}{\partial t} \leq 0,\ 0 \leq t,\ s \leq 1$, which induces that $G(t,\ s)$ is decreasing on t.

By a simple computation, we see

$$G(1,s) = -\frac{1}{6} + \frac{1}{2}s - \frac{1}{2}s^2 + \frac{\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k\xi_k(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2)}{1 - \sum_{k=0}^{m-1}\beta_k}$$

$$= \frac{\frac{1}{6}\sum_{k=0}^{m-1}\beta_k(1-s)^3 + \frac{1}{6}\sum_{k=i}^{m-2}\frac{1}{2}\beta_k(s-\xi_k)^3}{1 - \sum_{k=0}^{m-1}\beta_k} \geq 0.$$

This ensures that $G(t,s) \geq 0,\ t,\ s \in [0,1]$.

Lemma 3.3 *If* $x(t) \in C^3[0,1]$ *and*

$$x'''(0) = 0,\ x''(0) = 0,\ x'(0) = 0,\quad x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i).$$

Furthermore $x^{(4)}(t) \leq 0$ *and there exist* t_0 *such that* $x^{(4)}(t_0) < 0$, *then* $x(t)$ *has the following properties*

(1) $\min_{0\le t\le 1}|x(t)|\ge\delta\max_{0\le t\le 1}|x(t)|,$

(2) $\max_{0\le t\le 1}|x(t)|\le\gamma\max_{0\le t\le 1}|x'(t)|,$

(3) $\max_{0\le t\le 1}|x'(t)|\le\max_{0\le t\le 1}|x''(t)|,\ \max_{0\le t\le 1}|x''(t)|\le\max_{0\le t\le 1}|x'''(t)|$

where $\delta=\sum_{i=1}^{m-2}\beta_i(1-\xi_i)/(1-\sum_{i=1}^{m-2}\beta_i\xi_i),\gamma=(1-\sum_{i=1}^{m-2}\beta_i\xi_i)/(1-\sum_{i=1}^{m-2}\beta_i)$ *are positive constants.*

Proof. Since $x^{(4)}(t)\le 0,\ t\in[0,1]$, then $x'''(t)$ is decreasing on [0,1]. Considering $x'''(0)=0$, we have $x'''(t)\le 0,\ t\in[0,1]$. Thus $x''(t)$ is decreasing on [0,1]. Considering this together with the boundary condition $x''(0)=0$, we conclude that $x''(t)\le 0$. Then $x(t)$ is concave on [0,1]. Taking into account that $x'(0)=0$, we get that

$$\max_{0\le t\le 1}x(t)=x(0),\ \min_{0\le t\le 1}x(t)=x(1).$$

(1) From the concavity of $x(t)$, we have

$$\xi_i(x(1)-x(0))\le x(\xi_i)-x(0).$$

Multiplying both sides with β_i and considering the boundary condition, we have

$$(1-\sum_{i=1}^{m-2}\beta_i\xi_i)x(1)\ge\sum_{i=1}^{m-2}\beta_i(1-\xi_i)x(0). \tag{3.3}$$

Thus

$$\min_{0\le t\le 1}|x(t)|\ge\delta\max_{0\le t\le 1}|x(t)|.$$

(2) Considering the mean-value theorem together with the concavity of $x(t)$, we have

$$x(\xi_i)-x(1)=(1-\xi_i)|x'(\eta_i)|<(1-\xi_i)|x'(1)|,\ \eta_i\in(\xi_i,1). \tag{3.4}$$

Multiplying both sides with β_i and considering the boundary condition, we have

$$(1-\sum_{i=1}^{m-2}\beta_i)x(1)\le\sum_{i=1}^{m-2}\beta_i(1-\xi_i)|x'(1)|. \tag{3.5}$$

Comparing (3.3) and (3.5) yields that

$$x(0)\le(1-\sum_{i=1}^{m-2}\beta_i\xi_i)/(1-\sum_{i=1}^{m-2}\beta_i)|x'(1)|=\gamma\max_{0\le t\le 1}|x'(t)|.$$

(3) Since $x'(t)=x'(0)+\int_0^t x''(s)ds$ and $x'(0)=0$, we get

$$|x'(t)|=|\int_0^t x''(s)ds|\le\int_0^1|x''(s)|ds.$$

Since $x''(t) = x''(0) + \int_0^t x'''(s)ds$ and $x''(0) = 0$, we get

$$|x''(t)| = |\int_t^1 x''(s)ds| \le \int_0^1 |x'''(s)|ds.$$

Consequently

$$\max_{0\le t\le 1} |x'(t)| \le \max_{0\le t\le 1} |x''(t)|,\ \max_{0\le t\le 1} |x''(t)| \le \max_{0\le t\le 1} |x'''(t)|.$$

These give the proof of Lemma 3.3.
Remark. Lemma 3.3 ensures that

$$\max\{\max_{0\le t\le 1} |x(t)|,\ \max_{0\le t\le 1} |x'(t)|,\ \max_{0\le t\le 1} |x''(t)|,\ \max_{0\le t\le 1} |x'''(t)|\} \le \gamma \max_{0\le t\le 1} |x'''(t)|.$$

Let Banach space $E = C^3[0,\ 1]$ be endowed with the norm

$$\|x\| = \max\{\max_{0\le t\le 1} |x(t)|,\ \max_{0\le t\le 1} |x'(t)|,\ \max_{0\le t\le 1} |x''(t)|, \max_{0\le t\le 1} |x'''(t)|\},\ x \in E.$$

Define the cone $P \subset E$ by

$$P = \{x \in E \mid x(t) \ge 0,\ x'''(0) = 0,\ x''(0) = 0,\ x'(0) = 0,$$

$$x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i),\ x(t) \text{ is concave on } [0,1]\}.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals $\gamma,\ \theta$ and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(x) = \max_{0\le t\le 1} |x'''(t)|,\ \theta(x) = \psi(x) = \max_{0\le t\le 1} |x(t)|,\ \alpha(x) = \min_{0\le t\le 1} |x(t)|.$$

By Lemma 3.3, the functionals defined above satisfy

$$\delta\theta(x) \le \alpha(x) \le \theta(x) = \psi(x),\ \|x\| \le \gamma\gamma(x). \tag{3.6}$$

Denote

$$m = \int_0^1 G(1,s)ds,\ N = \int_0^1 G(0,s)ds,\ \lambda = \min\{m,\ \delta\gamma\}.$$

Assume that there exist constants $0 < a,\ b,\ d$ with $a < b < \lambda d$ such that

$A_1)$ $f(t,\ u,\ v,\ w,\ p) \le d,\ (t,\ u,\ v,\ w,\ p) \in [0,1]\times[0,\ \gamma d]\times[-d,\ 0]\times[-d,\ 0]\times[-d,\ 0]$,

$A_2)$ $f(t,\ u,\ v,\ w,\ p) > b/m,\ (t,\ u,\ v,\ w,\ p) \in [0,\ 1]\times[b,\ b/\delta]\times[-d,\ 0]\times[-d,\ 0]\times[-d,\ 0]$,

$A_3)$ $f(t,\ u,\ v,\ w,\ p) < a/N,\ (t,\ u,\ v,\ w,\ p) \in [0,\ 1]\times[0,\ a]\times[-d,\ 0]\times[-d,\ 0]\times[-d,\ 0]$.

Theorem 3.1 *Under assumptions $A_1) - A_3)$, problem (1.1, 1.2) has at least three positive solutions $x_1,\ x_2,\ x_3$ satisfying*

$$\max_{0\le t\le 1} |x_i'''(t)| \le d,\ i = 1,\ 2,\ 3;\ b < \min_{0\le t\le 1} |x_1(t)|;$$

$$a < \max_{0\le t\le 1} |x_2(t)|, \min_{0\le t\le 1} |x_2(t)| < b;\ \max_{0\le t\le 1} |x_3(t)| \le a.$$

Proof. Problem (1.1, 1.2) has a solution $x = x(t)$ if and only if $x(t)$ is a fixed point of operator T

$$T(x(t)) = \int_0^1 G(t,s)f(s,\ x(s),\ x'(s),\ x''(s),\ x'''(s))ds.$$

A straightforward calculation shows that

$$(Tx)'''(t) = -\int_0^t f(s,\ x,\ x',\ x'',\ x''')ds$$

For $x \in \overline{P(\gamma,\ d)}$, considering Lemma 3.3 and assumption A_1), we have

$$f(t,\ x(t),\ x'(t),\ x''(t),\ x'''(t)) \le d.$$

Thus

$$\gamma(Tx) = |(Tx)'''(1)| = |-\int_0^1 f(s,\ x,\ x',\ x'',\ x''')ds| = \int_0^1 |f(s,\ x,\ x',\ x'',\ x''')|ds \le d.$$

Hence $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.
An application of the Arzela-Ascoli theorem yields that T is a completely continuous operator.
The fact that the constant function $x(t) = b/\delta \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$ and $\alpha(b/\delta) > b$ implies that

$$\{x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)|\alpha(x) > b\} \ne \emptyset.$$

Thus condition (S1) of Lemma 2.1 is fulfilled.

For $x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$, it is easy to check that $b \le x(t) \le b/\delta$ and $|x'''(t)| < d$. From assumption A_2), we see

$$f(t,\ x,\ x',\ x'',\ x''') > b/m.$$

Hence, by definition of α and the cone P, we can get

$$\alpha(Tx) = (Tx)(1) = \int_0^1 G(1,s)f(s,\ x,\ x',\ x'',\ x''')ds \ge \frac{b}{m}\int_0^1 G(1,s)ds > \frac{b}{m}m = b,$$

which ensures that $\alpha(Tx) > b,\ \forall x \in P(\gamma,\ \theta,\ \alpha,\ b,\ b/\delta,\ d)$. Second, from (3.4) and $b < \lambda d$, we have

$$\alpha(Tx) \ge \delta\theta(Tx) > \delta \times \frac{b}{\delta} = b$$

for all $x \in P(\gamma,\ \alpha,\ b,\ d)$ with $\theta(Tx) > \dfrac{b}{\delta}$. Thus, condition (S_2) of Lemma 2.1 holds.

Finally we show that (S_3) also holds. For $\psi(0) = 0 < a$, we see $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x) = a$, then by the assumption of A_3),

$$\psi(Tx) = \max_{0\le t\le 1} |(Tx)(t)| = \int_0^1 G(0,\ s)f(s,\ x,\ x',\ x'',\ x''')ds < \frac{a}{N}\int_0^1 G(0,s)ds = a,$$

which ensures that condition (S_3) of Lemma 2.1 is fulfilled. Thus, an application of Lemma 2.1 implies that the fourth-order m-point boundary value problem (1.1, 1.2) has at least three concave and decreasing positive solutions $x_1,\ x_2,\ x_3$ with the properties that

$$\max_{0\le t\le 1}|x_i'''(t)|\le d,\ i=1,\ 2,\ 3;\ b<\min_{0\le t\le 1}|x_1(t)|;$$

$$a<\max_{0\le t\le 1}|x_2(t)|,\ \min_{0\le t\le 1}|x_2(t)|<b;\ \max_{0\le t\le 1}|x_3(t)|\le a.$$

4. Positive solutions for problem (1.1, 1.3)

Lemma 4.1. *Denote* $\xi_0=0,\ \xi_{m-1}=1,\ \beta_0=\beta_{m-1}=0$, *the Green's function of problem*

$$-x^{(4)}(t)=0, \tag{4.1}$$

$$x'''(1)=0,\ x''(1)=0,\ x'(1)=0,\quad x(0)=\sum_{i=1}^{m-2}\beta_i x(\xi_i), \tag{4.2}$$

is given as follow: for $i=1,\ 2,\ \cdots,\ m-1$

$$H(t,s)=\begin{cases}\dfrac{1}{6}t^3-\dfrac{1}{2}st^2+\dfrac{1}{2}s^2t+\dfrac{\sum\limits_{k=0}^{i-1}\beta_k(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2 s+\frac{1}{2}\xi_k s^2)+\frac{1}{6}\sum\limits_{k=i}^{m-2}\beta_k s^3}{1-\sum\limits_{k=0}^{m-1}\beta_k}, \\ \xi_{i-1}\le s\le \xi_i \\ \\ \dfrac{1}{6}s^3+\dfrac{\sum\limits_{k=0}^{i-1}\beta_k(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2 s+\frac{1}{2}\xi_k s^2)+\frac{1}{6}\sum\limits_{k=i}^{m-2}\beta_k s^3}{1-\sum\limits_{k=0}^{m-1}\beta_k}, \\ \xi_{i-1}\le s\le \xi_i\end{cases}$$

Proof. Suppose

$$G(t,s)=\begin{cases}a_3t^3+a_2t^2+a_1t+a_0 & t\le s,\ \xi_{i-1}\le s\le\xi_i,\ i=1,\ 2,\ \cdots,\ m-1\\ b_3t^3+b_2t^2+b_1t+b_0 & t\ge s,\ \xi_{i-1}\le s\le\xi_i,\ i=1,\ 2,\ \cdots,\ m-1\end{cases}$$

Considering the definition and properties of Green's function together with the boundary condition (4.2), we have

$$\begin{cases}a_3s^3+a_2s^2+a_1s+a_0=b_3s^3+b_2s^2+b_1s+b_0\\ 3a_3s^2+2a_2s+a_1=3b_3s^2+2b_2s+b_1\\ 6a_3s+2a_2=6b_3s+2b_2\\ 6a_3-6b_3=1\\ b_3=0,6b_3+2b_2=0,3b_3+2b_2+b_1=0,\\ a_0=\sum\limits_{k=0}^{i=1}\beta_k(a_3\xi_k^3+a_2\xi_k^2+a_1\xi_k+a_0)+\sum\limits_{k=i}^{m-2}\beta_k(b_3\xi_k^3+b_2\xi_k^2+b_1\xi_k+b_0)\end{cases}$$

Hence

$$a_3=\frac{1}{6},\ a_2=-\frac{1}{2}s,\ a_1=\frac{1}{2}s^2,\ b_3=b_2=b_1=0,$$

$$a_0=\frac{\sum_{k=0}^{i-1}\beta_k(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2)+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3}{1-\sum_{k=0}^{m-1}\beta_k},$$

$$b_0=\frac{1}{6}s^3+\frac{\sum_{k=0}^{i-1}\beta_k(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2)+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3}{1-\sum_{k=0}^{m-1}\beta_k}$$

The proof of Lemma 4.1 is completed.

Lemma 4.2 *One can see that* $H(t,s)\geq 0,\ t,\ s\in[0,1]$.

Proof. For $\xi_{i-1}\leq s\leq\xi_i,\ i=1,\ 2,\ \cdots,\ m-1$,

$$\frac{\partial H(t,\ s)}{\partial t}=\begin{cases}\frac{1}{2}(s-t)^2, & t\leq s,\ \xi_{i-1}\leq s\leq\xi_i\\ 0, & t\geq s,\ \xi_{i-1}\leq s\leq\xi_i\end{cases}$$

Then $\dfrac{\partial H(t,\ s)}{\partial t}\geq 0,\ 0\leq t,\ s\leq 1$, which implies that $H(t,\ s)$ is increasing on t. The fact that

$$H(0,s)=\frac{\sum_{k=0}^{i-1}\beta_k(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2)+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3}{1-\sum_{k=0}^{m-1}\beta_k}\geq 0$$

ensures that $H(t,s)\geq 0,\ t,\ s\in[0,1]$.

Lemma 4.3 *If* $x(t)\in C^3[0,1]$,

$$x'''(1)=0,\ x''(1)=0,\ x'(1)=0,\quad x(0)=\sum_{i=1}^{m-2}\beta_ix(\xi_i),$$

and $x^{(4)}(t)\leq 0$, *there exist* t_0 *such that* $x^{(4)}(t_0)<0$, *then*

(1) $\min\limits_{0\leq t\leq 1}|x(t)|\geq\delta_1\max\limits_{0\leq t\leq 1}|x(t)|$,

(2) $\max\limits_{0\leq t\leq 1}|x(t)|\leq\gamma_1\max\limits_{0\leq t\leq 1}|x'(t)|$,

(3) $\max\limits_{0\leq t\leq 1}|x'(t)|\leq\max\limits_{0\leq t\leq 1}|x''(t)|,\ \max\limits_{0\leq t\leq 1}|x''(t)|\leq\max\limits_{0\leq t\leq 1}|x'''(t)|$

where $\delta_1 = \sum_{i=1}^{m-2} \beta_i\xi_i/(1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i)), \gamma_1 = (1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i))/(1-\sum_{i=1}^{m-2}\beta_i)$ *are positive constants.*

Proof. It follows from the same methods as Lemma 3.3 that $x(t)$ is concave on [0,1]. Taking into account that $x'(1) = 0$, one see that $x(t)$ is increasing on [0,1] and

$$\max_{0\le t\le 1} x(t) = x(1), \ \min_{0\le t\le 1} x(t) = x(0).$$

(1) From the concavity of $x(t)$, we have

$$\xi_i(x(1) - x(0)) \le x(\xi_i) - x(0).$$

Multiplying both sides with β_i and considering the boundary condition, we have

$$\sum_{i=1}^{m-2} \beta_i\xi_i x(1) \le (1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i))x(0). \tag{4.3}$$

Thus

$$\min_{0\le t\le 1}|x(t)| \ge \delta_1 \max_{0\le t\le 1}|x(t)|.$$

(2) Considering the mean-value theorem, we get

$$x(\xi_i) - x(0) = \xi_i x'(\eta_i), \ \eta \in (\xi_i, 1).$$

From the concavity of x similarly with above we know

$$(1-\sum_{i=1}^{m-2}\beta_i)x(0) < \sum_{i=1}^{m-2}\beta_i\xi_i x'(0). \tag{4.4}$$

Considering (4.3) together with (4.4) we have $x(1) \le \gamma_1|x'(0)| = \gamma_1 \max_{0\le t\le 1}|x'(t)|$.

(3) For $x'(t) = x'(1) - \int_t^1 x''(s)ds, \ x''(t) = x''(1) - \int_t^1 x'''(s)ds$ and $x'(1) = 0, \ x''(1) = 0$, we get

$$|x'(t)| = |\int_t^1 x''(s)ds| \le \int_0^1 |x''(s)|ds, \ |x''(t)| = |\int_t^1 x'''(s)ds| \le \int_0^1 |x'''(s)|ds.$$

Thus

$$\max_{0\le t\le 1}|x'(t)| \le \max_{0\le t\le 1}|x''(t)|, \ \max_{0\le t\le 1}|x''(t)| \le \max_{0\le t\le 1}|x'''(t)|.$$

Remark. Then we see that

$$\max\{\max_{0\le t\le 1}|x(t)|, \ \max_{0\le t\le 1}|x'(t)|, \ \max_{0\le t\le 1}|x''(t)|, \ \max_{0\le t\le 1}|x'''(t)|\} \le \gamma_1 \max_{0\le t\le 1}|x'''(t)|.$$

Denote

$$m_1 = \int_0^1 H(0,s)ds, \ N_1 = \int_0^1 H(1,s)ds, \ \lambda_1 = \min\{m_1, \ \delta_1\gamma_1\}.$$

Assume that there exist constants $0 < a, \ b, \ d$ with $a < b < \lambda_1 d$ such that

$A_4)$ $f(t, \ u, \ v, \ w, \ p) \le d, \ (t, \ u, \ v, \ w, \ p) \in [0,1]\times[0, \ \gamma_1 d]\times[0, \ d]\times[-d, \ 0]\times[-d, \ 0]$,

$A_5)$ $f(t, \ u, \ v, \ w, \ p) > b/m_1, \ (t, \ u, \ v, \ w, \ p) \in [0, \ 1]\times[b, \ b/\delta_1]\times[0, \ d]\times[-d, \ 0]\times[-d, \ 0]$,

$A_6)$ $f(t, \ u, \ v, \ w, \ p) < a/N_1, \ (t, \ u, \ v, \ w, \ p) \in [0, \ 1]\times[0, \ a]\times[0, \ d]\times[-d, \ 0]\times[-d, \ 0]$.

Theorem 4.1 *Under assumptions $A_4) - A_6)$, problem (1.1, 1.3) has at least three concave and increasing positive solutions $x_1,\ x_2,\ x_3$ with the properties that*

$$\max_{0\le t\le 1}|x_i'''(t)|\le d,\ i=1,\ 2,\ 3;\ b<\min_{0\le t\le 1}|x_1(t)|;$$

$$a<\max_{0\le t\le 1}|x_2(t)|,\ \min_{0\le t\le 1}|x_2(t)|<b;\ \max_{0\le t\le 1}|x_3(t)|\le a.$$

The proof is similar with Theorem 3.1 and is omitted here.

5. Example

In this section, we present an example to illustrate the main results. Consider the third-order four-point boundary value problem

$$\begin{cases} x^{(4)}(t)+f(t,\ x(t),\ x'(t),\ x''(t),\ x'''(t))=0,\ t\in[0,1], \\ x'''(0)=0,\ x''(0)=0,\ x'(0)=0,\ x(1)=\dfrac{1}{2}x(\dfrac{1}{2}), \end{cases} \tag{4.1}$$

where

$$f(t,\ u,\ v,\ w,\ p)=\begin{cases} \dfrac{1}{60}e^t+\dfrac{2u^5}{10\pi}+\dfrac{1}{60}\left(\dfrac{p}{1800}\right)^4, & 0\le u\le 6, \\ \dfrac{1}{60}e^t+\dfrac{15552}{10\pi}+\dfrac{1}{60}\left(\dfrac{p}{1800}\right)^4, & u\ge 6, \end{cases}$$

By a simple computation, the function $G(t,s)$ is given by

$$G(t,s)=\begin{cases} -\dfrac{1}{6}s^3+\dfrac{3}{4}s^2-\dfrac{7}{8}s+\dfrac{5}{16}, & 0\le s\le\dfrac{1}{2},\ t\le s, \\ -\dfrac{1}{6}t^3+\dfrac{1}{2}st^2-\dfrac{1}{2}s^2t+\dfrac{3}{4}s^2-\dfrac{7}{8}s+\dfrac{5}{16}, & 0\le s\le\dfrac{1}{2},\ t\ge s, \\ -\dfrac{1}{3}s^3+\dfrac{1}{3}-s+s^2, & \dfrac{1}{2}\le s\le 1,\ t\le s, \\ -\dfrac{1}{6}t^3+\dfrac{1}{2}st^2-\dfrac{1}{2}s^2t+\dfrac{1}{3}-s+s^2-\dfrac{1}{6}s^3, & \dfrac{1}{2}\le s\le 1,\ t\ge s, \end{cases}$$

Choosing $a=1,\ b=5,\ d=1800$, we note that

$$\gamma=\frac{3}{2},\delta=\frac{1}{3},\ m=\int_0^1 G(1,\ s)ds=\frac{5}{128},\ N=\int_0^1 G(0,\ s)ds=\frac{31}{384}.$$

We can check that $f(t,\ u,\ v,\ w,\ p)$ satisfies that
$f(t,\ u,\ v,\ w,\ p)\le 1800,$

$$(t,\ u,\ v,\ w)\in[0,\ 1]\times[0,\ 2700]\times[-1800,\ 0]\times[-1800,\ 0]\times[-1800,\ 0];$$

$f(t,\ u,\ v,\ w,\ p)\ge 128,$

$$(t,\ u,\ v,\ w)\in[0,\ 1]\times[5,\ 15]\times[-1800,\ 0]\times[-1800,\ 0]\times[-1800,\ 0];$$

$f(t,\ u,\ v,\ w,\ p)\le\dfrac{384}{31},$

$$(t,\ u,\ v,\ w)\in[0,1]\times[0,1]\times[-1800,\ 0]\times[-1800,\ 0]\times[-1800,\ 0].$$

Then all assumptions of Theorem 3.1 are satisfied. Thus, problem (5.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\max_{0\le t\le 1} |x_i'(t)| \le 1800,\ i = 1,\ 2,\ 3;\ \min_{0\le t\le 1} x_1(t) > 5;$$

$$\max_{0\le t\le 1} x_2(t) > 1,\ \min_{0\le t\le 1} x_2(t) < 5;\ \max_{0\le t\le 1} x_3(t) < 1.$$

Remark. We can notice that problem (4.1) is a fourth-order three-point Bvps and the nonlinear term is involved in the third-order derivative explicitly. Earlier results for positive solutions, to author's best knowledge, are not applicable to this problem.

References

[1] R. Ma, *Positive solutions of a nonlinear three-point boundary valve problem*, Electron. J. Diff. Eqs., **34**(1999), 1-8.

[2] R. Ma, N. Cataneda, *Existence of solution for nonlinear m-point boundary value problem*, J. Math. Anal. Appl., **256**(2001), 556-567.

[3] R. Ma, H. Wang, *Positive solutions of nonlinear three-point boundary value problems*, J. Math. Anal. Appl., **279**(2003), 1216-1227.

[4] X. He and W. Ge, *Triple solutions for second-order three-point boundary value problems*, J. Math. Anal. Appl., **268**(2002), 256-265.

[5] Y. Guo and W. Ge, *Positive solutions for three-point boundary-value problems with dependence on the first order derivative*, J. Math. Anal. Appl, **290**(2004), 291-301.

[6] R.I. Avery et al., *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations*, Comput. Math. Appl., **42**(2001), 695-704.

[7] R.I. Avery and J. Henderson, *Three symmetric positive solutions for a second order boundary-value problem*, Appl. Math. Lett., **13**(2000), 1-7.

[8] J. Henderson, *Double solutions of three-point boundary-value problems for second-order differential equations*, Electron. J. Diff. Eqs. **115**(2004), 1-7.

[9] P.W. Eloe and J. Henderson, *Positive solutions for $(n-1, n)$ conjugate boundary value problems*, Nonlinear Anal., **28**(1997), 1669-1680.

[10] L. Yang et. al., *Multiplicity results for second-order m-point boundary value problem*, J. Math. Anal. Appl., **324**(2006), 532-542.

[11] J.R.L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London. Math. Soc., **74**(2006), 673-693.

[12] J.R.L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems involving integral conditions*, Nonlinear Differ. Equ. Appl., **15**(2008), 45-67.

[13] R.P. Agarwal and D. O'Regan, *A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem,* Appl. Math. Comput., **161**(2005), 433-439.

[14] R.P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer Academic, Dordrecht, 1998.

[15] R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publ., Dordrecht, 1998.

[16] Q. Yao, *Positive solutions for eigenvalue problems of fourth-order elastic beam equation*, Appl. Math. Lett., **17**(2004), 237-243.

[17] Y. Li, *On the existence of positive solutions for the bending elastic beam equations*, Appl. Math. Comput., **189**(2007), 821-827.

[18] Y. Yang and J. Zhang, *Nontrival solutions for some fourth-order boundary value problems with parametes*, Nonlinear Anal., **70**(2009), 3966-3977.

[19] Y. Yang, *Triple positive solutions of a class of fourth-order two-point boundary value problems*, Appl. Math. Lett., (in press).

[20] X. Yang and K. Lo, *Existence of a positive solution of a fourth-order boundary value problems*, Nonlinear Anal., **69**(2008), 2267-2273.

[21] G. Hang and Z. Xu, *Multiple solutions of some fourth-order beam equations*, Nonlinear Anal., **68**(2008), 3646-3656.

[22] Y. Liu, *Multiple positive solutions of nonlinear singular boundary value problem for fourth-order equations*, Appl. Math. Lett., **17**(2004), 747-757.

[23] Z. Bai and H. Wang, *On positive solutions of some fourth-order beam equations*, J. Math. Anal. Appl., **270**(2002), 357-368.

[24] R.R. Kaufmann and N. Kosmatov, *Elastic beam equation with high order derivatives*, Nonlinear Anal, Real World Appl., **8**(2007), 811-821.

[25] R.I. Avery, A.C. Peterson, *Three positive fixed points of nonlinear operators on an ordered Banach space,* Comput. Math. Appl., **208**(2001), 313-322.

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