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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A SYSTEM OF FUNCTIONAL EQUATIONS AND APPLICATIONS TO PARTIAL METRIC SPACES

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Abstract. In this paper, we discuss the existence and uniqueness of solutions to the system of functional equations:

$$\begin{cases} Tx = x \\ \varphi(x) = 0 \end{cases}$$

where $T : X \to X$ is a given mapping and $\varphi : X \to [0, \infty)$ is a lower semi-continuous function on X endowed with a metric d. We apply our obtained results to derive some fixed point theorems on partial metric spaces. This answers three open problems posed by Ioan A. Rus in [Fixed point theory in partial metric spaces, Anal. Univ. de Vest, Timisoara, Seria Matematică-Informatică. 46 (2) (2008) 141-160].
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1. Introduction

In 1994, Matthews [4] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks, and showed that the Banach's contraction principle can be generalized to the partial metric context for applications in program verification. Later on, many authors studied fixed point theorems on partial metric spaces (see, for example [1, 3, 5, 6, 7, 8, 9] and references therein).

We start by recalling some basic definitions and properties of partial metric spaces (see [4, 5] for more details).

Definition 1.1 A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$, we have:
(P1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2), $x = y$; but if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, \infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [4]. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x)+\varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2 Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim\limits_{n\to+\infty} p(x, x_n)$. We may write this as $x_n \to x$;

(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim\limits_{n,m\to+\infty} p(x_n, x_m)$ exists and is finite;

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim\limits_{n,m\to+\infty} p(x_n, x_m)$.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, \infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a metric on X.

Lemma 1.3 *Let (X, p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s);*

(b) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim\limits_{n\to+\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m).$$

In [4], Matthews extended the Banach contraction principle to the setting of partial metric spaces.

Theorem 1.4 (The partial metric contraction mapping theorem) *Let (X, p) be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \le k\, d(x, y),\ \forall\, x, y \in X.$$

Then T has a unique fixed point $x^* \in X$. Moreover, we have $p(x^*, x^*) = 0$.

Remark 1.5 *Under the assumptions of Theorem 1.4, we can say that the system of functional equations:*

$$\begin{cases} Tx = x \\ p(x, x) = 0 \end{cases}$$

has a unique solution. A point $x \in X$ such that $p(x, x) = 0$ is called a total element (see [8]*).*

In [8], Ioan A. Rus presented three interesting open problems. Let (X, p) be a complete partial metric space.

Problem 1. If $T : (X, p) \to (X, p)$ is a generalized contraction, which condition satisfies T with respect to p^s?

Problem 2. The problem is to give fixed point theorems for these new classes of operators on a metric space.
Problem 3. Use the results for the above problems to give fixed point theorems in a partial metric space.

The purpose of this paper is to answer to the above problems of Ioan A. Rus. More precisely, we consider the system of functional equations:

$$\begin{cases} Tx = x \\ \varphi(x) = 0 \end{cases}$$

where (X, d) is a complete metric space, $T : X \to X$ is a given mapping and $\varphi : X \to [0, \infty)$ is a lower semi-continuous function. Under a generalized contractive condition imposed on T that involves the function φ, we establish the existence and uniqueness of solutions to the considered system. Finally, we use our obtained results to give fixed point theorems on partial metric spaces.

2. Solutions to a system of functional equations on a metric space

Let (X, d) be a metric space, $T : X \to X$ and $\varphi : X \to [0, \infty)$ are two given mappings. We consider the system of functional equations:

$$(S) : \begin{cases} Tx = x \\ \varphi(x) = 0 \end{cases}$$

In this section, we give sufficient conditions that assure the existence and uniqueness of solutions to the system (S).

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is upper semi-continuous from the right (i.e. for all $t \geq 0$, for any sequence $\{t_n\} \subset [0, \infty)$ such that $t_n \geq t$ and $t_n \to t$ as $n \to \infty$, we have $\limsup_{n\to\infty} \psi(t_n) \leq \psi(t)$);
(ii) $\psi(t) < t$ for all $t > 0$.

We have the following result.

Theorem 2.1 *Let (X, d) be a complete metric space and let $T : X \to X$, $\varphi : X \to [0, \infty)$ be two mappings such that φ is lower semi-continuous and*

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \psi\big(d(x, y) + \varphi(x) + \varphi(y)\big),\ \forall\, x, y \in X, \tag{2.1}$$

where $\psi \in \Psi$. Then T has a unique fixed point $z \in X$. Moreover, z is the unique solution to the system (S).

Proof. At first, let us prove that T admits at least one fixed point. Let $x_0 \in X$ be an arbitrary point. Consider the sequence $\{x_n\} \subset X$ defined by: $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If for some $N \in \mathbb{N}$, we have $x_N = x_{N-1}$, then x_{N-1} will be a fixed point of T. So, we can suppose that

$$d(x_{n-1}, x_n) > 0,\ \forall\, n \in \mathbb{N}. \tag{2.2}$$

Using (2.1), for all $n \in \mathbb{N}$, we get that

$$d(Tx_n, Tx_{n-1}) + \varphi(Tx_n) + \varphi(Tx_{n-1}) \leq \psi\big(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})\big).$$

This implies from the definition of $\{x_n\}$, (2.2) and the condition (ii), that for all $n \in \mathbb{N}$,

$$\begin{cases} d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n) \leq \psi\big(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})\big) \\ \psi\big(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})\big) < d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}) \end{cases} \tag{2.3}$$

It follows from (2.3) that there is $c \geq 0$ such that

$$\begin{aligned} &\lim_{n\to\infty} d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n) \\ &= \lim_{n\to\infty} \psi\big(d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n)\big) = c. \end{aligned} \tag{2.4}$$

If $c > 0$, using that ψ is upper semi-continuous from the right and condition (ii), we obtain from (2.4) that

$$c = \limsup_{n\to\infty} \psi\big(d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n)\big) \leq \psi(c) < c,$$

a contradiction. So, we have

$$\begin{aligned} &\lim_{n\to\infty} d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n) \\ &= \lim_{n\to\infty} \psi\big(d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n)\big) = 0. \end{aligned} \tag{2.5}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in (X, d).
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integer k,

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.6}$$

From (2.6), we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + d(x_{n(k)}, x_{n(k)-1}). \end{aligned}$$

Thus, for all k, we have

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) < \varepsilon + d(x_{n(k)}, x_{n(k)-1}).$$

Letting $k \to \infty$ in the above inequality and using (2.5), we obtain

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon^+. \tag{2.7}$$

From (2.5) and (2.7), we get that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)}) = \varepsilon^+. \tag{2.8}$$

Since ψ is upper semi-continuous from the right,we deduce from (2.7) that

$$\limsup_{k\to\infty} \psi\big(d(x_{n(k)}, x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)})\big) \leq \psi(\varepsilon). \tag{2.9}$$

On the other hand, for each $k \in \mathbb{N}$, from (2.1) and (2.7), we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)+1}) + \psi\big(d(x_{n(k)}, x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)})\big) + d(x_{m(k)+1}, x_{m(k)}). \end{aligned}$$

So, from (2.5) and (2.9), we get that

$$\varepsilon \leq \limsup_{k\to\infty} \psi\big(d(x_{n(k)}, x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)})\big) \leq \psi(\varepsilon),$$

a contradiction because $\psi(\varepsilon) < \varepsilon$. Consequently, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Hence, there is $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.10}$$

Next, we shall prove that

$$\varphi(z) = 0. \tag{2.11}$$

From (2.5), we have

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{2.12}$$

Since φ is lower semi-continuous, it follows from (2.10) and (2.12) that

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(x_n) = 0,$$

which implies (2.11).
Now, we show that z is a fixed point of T.
From (2.1) and (2.11), for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, Tz) \leq \psi\big(d(x_n, z) + \varphi(x_n) + \varphi(z)\big) = \psi\big(d(x_n, z) + \varphi(x_n)\big). \tag{2.13}$$

On the other hand, since $0 \leq \psi(t) < t$ for all $t > 0$, we have $\lim_{t\to 0^+} \psi(t) = 0$. Then, it follows from (2.10) and (2.12) that

$$\lim_{n\to\infty} \psi\big(d(x_n, z) + \varphi(x_n)\big) = \lim_{t\to 0^+} \psi(t) = 0.$$

Using the above equality and letting $n \to \infty$ in (2.13), we obtain that

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = 0. \tag{2.14}$$

From the uniqueness of the limit, it follows from (2.10) and (2.14) that $z = Tz$, that is, z is a fixed point of T.
Finally, let $u \in X$ be such that $Tu = u$. Applying (2.1) with $x = z$ and $y = u$, we get that

$$d(z, u) + \varphi(z) + \varphi(u) = d(Tz, Tu) + \varphi(Tz) + \varphi(Tu) \leq \psi\big(d(z, u) + \varphi(z) + \varphi(u)),$$

which holds only if $d(z, u) = 0$, i.e., $u = z$. This concludes the proof. □

Taking $\varphi \equiv 0$ in Theorem 2.1, we obtain immediately the the celebrated Boyd and Wong fixed point theorem [2].

3. An homotopy result

We have the following homotopy result.

Theorem 3.1 *Let (X, d) be a complete metric space, U be an open subset of X and V be a closed subset of X with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(1) *$x \neq H(x, \lambda)$ for every $x \in V \backslash U$ and $\lambda \in [0, 1]$;*

(2) *There exist a lower semi-continuous function* $\varphi : X \to [0,\infty)$ *and* $L \in (0,1)$ *such that for all* $x, y \in V$ *and* $\lambda \in [0,1]$,

$$d(H(x,\lambda), H(y,\lambda)) + \varphi(H(x,\lambda)) + \varphi(H(y,\lambda)) \leq L\big(d(x,y) + \varphi(x) + \varphi(y)\big);$$

(3) *There exists a continuous function* $\eta : [0,1] \to \mathbb{R}$ *such that for all* $x \in V$ *and* $\lambda, \mu \in [0,1]$,

$$d(H(x,\lambda), H(x,\mu)) + \varphi(H(x,\lambda)) + \varphi(H(y,\lambda)) \leq |\eta(\lambda) - \eta(\mu)|.$$

Then, $H(\cdot,0)$ *has a fixed point if and only if* $H(\cdot,1)$ *has a fixed point.*

Proof. Suppose that $H(\cdot,0)$ has a fixed point. Consider the set

$$Q := \{t \in [0,1] \,|\, x = H(x,t) \text{ for some } x \in U\}.$$

Since $H(\cdot,0)$ has a fixed point and (1) holds, we have $0 \in Q$, so Q is a nonempty set. We will show that Q is both closed and open in [0, 1], and so by the connectedness of $[0,1]$, we are finished since $Q = [0,1]$.

First, let us prove that Q is open in $[0,1]$. Let $t_0 \in Q$ and $x_0 \in U$ with $x_0 = H(x_0,t_0)$. From condition (2), since $L \in (0,1)$, clearly we have $\varphi(x_0) = 0$. Moreover, we can show that for all $t \in [0,1]$, if $x \in U$ is a fixed point of $H(\cdot,t)$, then $\varphi(x) = 0$. Since U is open in (X,d), there exists $r > 0$ such that $B(x_0,r) \subseteq U$, where

$$B(x_0,r) := \{z \in X \,|\, d(x_0,z) < r\}.$$

Consider the set

$$\Lambda(x_0,\varphi) := \{z \in X \,|\, d(x_0,z) + \varphi(z) < r\}.$$

Clearly, $\Lambda(x_0,\varphi)$ is nonempty, since $x_0 \in \Lambda(x_0,\varphi)$, and $\Lambda(x_0,\varphi) \subseteq B(x_0,r) \subseteq U$. Let $\varepsilon = (1-L)r$. Since η is continuous on t_0, there exists $\alpha(\varepsilon) > 0$ such that $|\eta(t)-\eta(t_0)| < \varepsilon$ for all $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$. Let $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$, for $x \in \overline{\Lambda(x_0,\varphi)}$ (the closure of $\Lambda(x_0,\varphi)$), we have

$$\begin{aligned}
d(H(x,t),x_0) + \varphi(H(x,t)) &= d(H(x,t),H(x_0,t_0)) + \varphi(H(x,t)) \\
&\leq d(H(x,t),H(x,t_0)) + \varphi(H(x,t)) + d(H(x,t_0),H(x_0,t_0)) \\
&\leq |\eta(t) - \eta(t_0)| + L\big(d(x,x_0) + \varphi(x) + \varphi(x_0)\big) \\
&= |\eta(t) - \eta(t_0)| + L\big(d(x,x_0) + \varphi(x)\big) \\
&< \varepsilon + Lr = r.
\end{aligned}$$

Thus, for all $t \in (t_0-\alpha(\varepsilon), t_0+\alpha(\varepsilon)) \cap [0,1]$, the mapping $H(\cdot,t) : \overline{\Lambda(x_0,\varphi)} \to \overline{\Lambda(x_0,\varphi)}$ is well defined. Now, from condition (2), applying Theorem 2.1, we obtain that $H(\cdot,t)$ has a fixed point in V, for all $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$. But this fixed point must be in U since (1) holds. Hence, $(t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1] \subseteq Q$ and therefore Q is open in $[0,1]$.

Next, we show that Q is closed in $[0,1]$. To see this, let $\{t_n\}$ be a sequence in Q with $t_n \to t^* \in [0,1]$ as $n \to \infty$. We have to prove that $t^* \in Q$. By the definition of Q, for all $n \in \mathbb{N}$, there exists $x_n \in U$ with $x_n = H(x_n,t_n)$ and $\varphi(x_n) = 0$. On the other hand, for all $m, n \in \mathbb{N}$, we have

$$\begin{aligned}d(x_n, x_m) &= d(H(x_n, t_n), H(x_m, t_m))\\ &\leq d(H(x_n, t_n), H(x_n, t_m)) + d(H(x_n, t_m), H(x_m, t_m))\\ &\leq |\eta(t_n) - \eta(t_m)| + Ld(x_n, x_m).\end{aligned}$$

Thus, for all $m, n \in \mathbb{N}$, we have

$$d(x_n, x_m) \leq \frac{|\eta(t_n) - \eta(t_m)|}{1 - L}.$$

Letting $n, m \to \infty$ and using the continuity of η, we obtain that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). So, there is $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$ and $\varphi(z) = 0$ (since φ is lower semi-continuous). On the other hand, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}d(x_n, H(z, t^*)) &= d(H(x_n, t_n), H(z, t^*))\\ &\leq d(H(x_n, t_n), H(x_n, t^*)) + d(H(x_n, t^*), H(z, t^*))\\ &\leq |\eta(t_n) - \eta(t^*)| + Ld(x_n, z).\end{aligned}$$

Letting $n \to \infty$, we get that $\lim_{n\to\infty} d(x_n, H(z, t^*)) = 0$. By the uniqueness of the limit, we obtain that $z = H(z, t^*)$, which implies from condition (1) that $z \in U$ and $t^* \in Q$. Thus Q is closed in $[0, 1]$.
For the reverse implication, we use the same strategy. □

4. Applications to partial metric spaces

In this section, from our previous obtained results on metric spaces, we will show that we can deduce easily various fixed point theorems on partial metric spaces including Matthews fixed point theorem.

We have the following partial metric version of Boyd and Wong fixed point theorem.

Corollary 4.1 *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping such that*

$$p(Tx, Ty) \leq \psi\big(p(x, y)\big),\ \forall\, x, y \in X, \tag{4.1}$$

where $\psi \in \Psi$. Then T has a unique fixed point $z \in X$. Moreover, we have $p(z, z) = 0$.

Proof. From (1.1), for all $x, y \in X$, we have

$$p(x, y) = \frac{p^s(x, y) + p(x, x) + p(y, y)}{2}. \tag{4.2}$$

Note that since (X, p) is complete, from Lemma 1.3, (X, p^s) is a complete metric space.

We denote by q^s the metric on X defined by

$$q^s(x, y) = \frac{p^s(x, y)}{2},\ \forall\, x, y \in X.$$

Clearly, (X, q^s) is also a complete metric space. Let $\varphi : X \to [0, \infty)$ be the function defined by

$$\varphi(x) = \frac{p(x, x)}{2},\ \forall\, x \in X.$$

We shall prove that φ is a continuous function in (X, q^s).
Indeed, let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} q^s(x_n, x) = 0$, which is equivalent to $\lim_{n\to\infty} p^s(x_n, x) = 0$. From Lemma 1.1 (b), we get that $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$, which implies that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$. Thus, we proved the continuity of φ in X with respect to the topology of q^s.
Now, taking the new expression of p given by (4.2) in (4.1), we obtain that

$$q^s(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \psi\big(q^s(x, y) + \varphi(x) + \varphi(y)\big),\ \forall\, x, y \in X.$$

Now, the desired result follows immediately from Theorem 2.1. □.

Taking in Corollary 4.1, $\psi(t) = kt$ with $k \in (0, 1)$, we obtain Matthews fixed point theorem.

Finally, we end this paper with the following homotopy result on partial metric spaces.

Corollary 4.2 *Let (X, p) be a complete partial metric space, U be an open subset of (X, p) and V be a closed subset of (X, p) with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(1) *$x \neq H(x, \lambda)$ for every $x \in V \backslash U$ and $\lambda \in [0, 1]$;*
(2) *There exists $L \in (0, 1)$ such that for all $x, y \in V$ and $\lambda \in [0, 1]$,*

$$p(H(x, \lambda), H(y, \lambda)) \leq Lp(x, y);$$

(3) *There exists a continuous function $\eta : [0, 1] \to \mathbb{R}$ such that for all $x \in V$ and $\lambda, \mu \in [0, 1]$,*

$$p(H(x, \lambda), H(x, \mu)) \leq |\eta(\lambda) - \eta(\mu)|.$$

Then, $H(\cdot, 0)$ has a fixed point if and only if $H(\cdot, 1)$ has a fixed point.

Proof. We shall prove that V is closed in (X, p^s).
Let $\{x_n\}$ be a sequence in V such that $\lim_{n\to\infty} p^s(x_n, x) = 0$, where $x \in X$. We have to prove that $x \in V$. From Lemma 1.3 (b), we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$, which implies, since V is closed in (X, p), that $x \in V$. Thus we proved that V is closed in X with respect to the topology of p^s. Since U is open in (X, p), we deduce that U is also open in the metric space (X, p^s).
Now, define the function $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = p(x, x),\ \forall\, x \in X.$$

We proved that φ is a continuous function in (X, p^s) (see the proof of Corollary 4.1). Using the equality (4.2), we obtain from condition (2) that for all $x, y \in V$ and $\lambda \in [0, 1]$,

$$p^s(H(x, \lambda), H(y, \lambda)) + \varphi(H(x, \lambda)) + \varphi(H(y, \lambda)) \leq L\big(p(x, y) + \varphi(x) + \varphi(y)\big).$$

Using the equality (4.2), we obtain from condition (3) that for all $x \in V$ and $\lambda, \mu \in [0, 1]$,

$$p^s(H(x, \lambda), H(x, \mu)) + \varphi(H(x, \lambda)) + \varphi(H(x, \mu)) \leq |2\eta(\lambda) - 2\eta(\mu)|.$$

Now, the desired result follows immediately from Theorem 3.1. □

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