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VISCOSITY APPROXIMATION METHOD FOR EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves and extends some recent results.

Key Words and Phrases: Equilibrium problem, fixed point, nonexpansive mapping, viscosity approximation method, variational inequality.

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1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let A be a bounded operator on C. In this paper, we assume A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in C$. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0, \quad \text{for all } v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Some authors have proposed some useful methods for solving the equilibrium problem (1.1); see [6, 8, 14, 22]. The problem (1.1) is very general in the sense that it includes, as special cases, numerous problems in physics and economics, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [1, 4, 5, 7, 12].

A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Let $F(T)$ denote the fixed points set of T. Recall that a contraction on C is a self-mapping f of C such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$, where $\alpha \in (0, 1)$ is a constant. In 2000, Mudafi [19] proved the following strong convergence theorem.

Theorem 1.1 [19] *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive self-mapping on C such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n),$$

for all $n \geq 1$, where $\varepsilon_n \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \text{and} \quad \lim_{n\to\infty} |\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}| = 0.$$

Then, the sequence $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Such a method for approximation of fixed points is called the viscosity approximation method.

Finding an optimal point in the intersection F of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g., [3, 11]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbb{R}$ over F is of wide interdisciplinary interest and practical importance see, e.g., [2, 10, 13, 27]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [15, 27]. The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by inner product of H) from any given point a in H is the simplest case of our problem.

Marino and Xu [18] considered a general iterative method for a single nonexpansive mapping. Let f be a contraction on H and $A : H \to H$ be a strongly positive bounded linear operator. Starting with arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0, \tag{1.2}$$

where $\gamma > 0$ is a constant and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.
Consequently, Marino and Xu [18] proved the sequence $\{x_n\}$, generated by (1.2) converges strongly to the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - x \rangle \leq 0, \quad x \in F(T),$$

which is the optimality condition for minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Yao et al. [25] introduced the iterative sequence:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n, \quad \text{for all } \ n \geq 0. \tag{1.3}$$

where f is a contraction on H and $A : H \to H$ is a strongly positive bounded linear operator, W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $(0,1)$. Under very mild conditions on the parameters, it was proved the sequence $\{x_n\}$, generated by (1.3), converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^* \rangle \leq 0, \quad \text{for all } \ x^* \in F,$$

which is the optimality condition for minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - h(x).$$

On the other hand, Ceng and Yao [9], inspired by Moudafi [19], Tada and Takahashi [21], Takahashi and Takahashi [22] and Yao et al. [26], introduced an iterative scheme by

$$\begin{cases} \phi(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n \rangle \geq 0, & \text{for all } x \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n W_n u_n, \\ x_{n+1} = \eta_n W_n y_n + \alpha_n f(y_n) + (1 - \eta_n - \alpha_n)x_n, \end{cases} \tag{1.4}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, f is a contraction of C into itself, $\{\alpha_n\}, \{\eta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \eta_n \leq 1$, $\{r_n\} \subset (0, \infty)$ and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$. They proved the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by (1.4), converge strongly to $x^* \in F$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)} f(x^*)$.

In this paper, motivated by Yao et al. [25] and Ceng and Yao [9], we introduce a new iterative scheme by the viscosity approximation method:

$$\begin{cases} \phi(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n \rangle \geq 0, & \text{for all } x \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n W_n u_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n, \end{cases} \tag{1.5}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, A is a strongly positive bounded linear operator on C, f is a contraction of C into itself, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$, $\{r_n\} \subset (0, \infty)$ and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, for finding a common element of the set of solutions of the equilibrium problem (1.1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves the main results of [9].

2. Main results

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and the norm $\|.\|$. Weak and strong convergence are denoted by notation $\rightharpoonup$ and $\to$, respectively. In a real Hilbert space H,

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2,$$

for all $x,y\in H$ and $\lambda\in[0,1]$. Let C be a nonempty closed convex subset of H. Then, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x-P_C(x)\|\leq\|x-y\|,\quad \text{for all } y\in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Further, for $x\in H$ and $z\in C$,

$$z=P_C(x)\ \Leftrightarrow\ \langle x-z,z-y\rangle\geq 0,\quad \text{for all } y\in C.$$

Now, we collect some lemmas which will be used in the main results.

Lemma 2.1 [4, 12] *Let C be a nonempty closed convex subset of H and $\phi: C\times C\to\mathbb{R}$ be a bifunction satisfying the following conditions:*

(A_1) *$\phi(x,x)=0$ for all $x\in C$;*
(A_2) *ϕ is monotone, i.e., $\phi(x,y)+\phi(y,x)\leq 0$ for all $x,y\in C$;*
(A_3) *for each $x,y,z\in C$,*

$$\lim_{t\downarrow 0}\phi(tz+(1-t)x,y)\leq\phi(x,y);$$

(A_4) *for each $x\in C$, $y\mapsto\phi(x,y)$ is convex and lower semicontinuous.*

Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that

$$\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq\ 0,\quad \textit{for all } y\in C.$$

Lemma 2.2 [12] *Assume $\phi: C\times C\to\mathbb{R}$ satisfies $(A_1)-(A_4)$. For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_rx=\{z\in C:\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \textit{for all } y\in C\},$$

for all $x\in H$. Then, the following hold:
(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive, i.e., for any $x,y\in H$

$$\|T_rx-T_ry\|^2\leq\langle T_rx-T_ry,x-y\rangle;$$

(iii) $F(T_r)=EP(\phi)$;
(iv) $EP(\phi)$ is closed and convex.

Lemma 2.3 [18] *Assume A is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma}>0$ and $0<\rho\leq\|A\|^{-1}$. Then $\|I-\rho A\|\leq 1-\rho\overline{\gamma}$.*

Lemma 2.4 [23] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. Suppose $x_{n+1}=(1-\beta_n)y_n+\beta_nx_n$ for all integers $n\geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0$. Then $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

Lemma 2.5 [24] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq(1-\gamma_n)a_n+\gamma_nv_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty}\gamma_n=\infty$;

(ii) $\limsup_{n\to\infty} v_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n v_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [16] *Each Hilbert space H satisfies Opial's condition, i. e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for each $y \in H$ with $x \ne y$.

Let C be a nonempty closed convex subset of H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \ge 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{2.1}$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, \ldots, T_{n-1}, T_n$ and $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n$; see [17].

Lemma 2.7 [20] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \ge 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.8 [26] *It can be known from Lemma 2.7 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \ge k$ such that for all $n > n_0$,*

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \le \varepsilon.$$

Remark 2.9 [26] *Using Lemma 2.7, one can define mapping $W : C \to C$ as follows:*

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for all $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \ge 0\}$. Hence, it is clear from Remark 2.8 that for an arbitrary $\varepsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$,

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x\in D} \|U_{n,1}x - U_1 x\| \le \varepsilon.$$

This implies

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.10 [20] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such* *that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers in* $[0, b]$ *for some* $b \in (0, 1)$*. Then* $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$*.*

Now, we prove the following strong convergence theorem concerning the iterative scheme (1.5) for finding a common element of the set of solutions of the equilibrium problem (1.1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space.

Theorem 2.11 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\phi : C \times C \to \mathbb{R}$ *be a bifunction satisfying* $(A_1) - (A_4)$ *and* A *be a strongly positive bounded linear operator on* C *with coefficient* $\overline{\gamma} > 0$ *and* $\|A\| \leq 1$*. Let* $\{T_n\}_{n=1}^{\infty}$ *be an infinite family of nonexpansive self-mappings on* C *which satisfies* $F := \bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi) \neq \emptyset$*. Suppose* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *is a real sequence satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(iv) $0 < \liminf_{n\to\infty} r_n$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Let f *be a contraction of* C *into itself with constant* $\alpha \in (0, 1)$ *and* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ *where* γ *is some constant. Let* $x_0 \in C$*. Then, the sequences* $\{x_n\}$ *and* $\{u_n\}$*, generated iteratively by (1.5) where* $\{\lambda_n\}_{n=1}^{\infty}$ *is a sequence of positive numbers in* $[0, b]$ *for some* $b \in (0, 1)$*, converge strongly to* $x^* \in F$*, where* $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)(x^*)$*.*

Proof. Let $Q = P_F$. Then

$$\begin{aligned}\|Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y)\| \leq\ & \|(I - A + \gamma f)(x) \\ & \quad -(I - A + \gamma f)(y)\| \\ \leq\ & \|(I - A)(x) - (I - A)(y)\| \\ & \quad +\gamma\|f(x) - f(y)\| \\ \leq\ & (1 - \overline{\gamma})\|x - y\| + \gamma\alpha\|x - y\| \\ =\ & (1 - (\overline{\gamma} - \gamma\alpha))\|x - y\|,\end{aligned}$$

for all $x, y \in F$. Therefore, $Q(I - A + \gamma f)$ is a contraction of F into itself. So, there exists a unique element $x^* \in F$ such that $x^* = Q(I - A + \gamma f)(x^*) = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)(x^*)$. Note that from the conditions (i) and (ii), we may assume, without loss of generality, $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on C, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in C, \|x\| = 1\}.$$

Observe

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle =\ & (1 - \beta_n) - \alpha_n\langle Ax, x\rangle \\ \geq\ & 1 - \beta_n - \alpha_n\|A\| \\ \geq\ & 0,\end{aligned}$$

that is to say $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n A\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n A)x, x\rangle : x \in C, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in C, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned} \tag{2.2}$$

Let $p \in F$. From the definition of T_r, we know $u_n = T_{r_n} x_n$. It follows that

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\|,$$

and hence

$$\begin{aligned} \|y_n - p\| = & \ \|(1-\gamma_n)(x_n - p) + \gamma_n (W_n u_n - p)\| \\ \le & \ (1-\gamma_n)\|x_n - p\| + \gamma_n \|W_n u_n - p\| \\ \le & \ (1-\gamma_n)\|x_n - p\| + \gamma_n \|u_n - p\| \\ \le & \ (1-\gamma_n)\|x_n - p\| + \gamma_n \|x_n - p\| \\ = & \ \|x_n - p\|. \end{aligned}$$

First, we claim $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, from (1.5), (2.1) and (2.2), we obtain

$$\begin{aligned} \|x_{n+1} - p\| = & \ \|\alpha_n(\gamma f(y_n) - Ap) + \beta_n (x_n - p) \\ & \ +((1-\beta_n)I - \alpha_n A)(W_n y_n - p)\| \\ \le & \ (1 - \beta_n - \alpha_n \overline{\gamma})\|y_n - p\| + \beta_n \|x_n - p\| + \alpha_n \|\gamma f(y_n) - Ap\| \\ \le & \ (1 - \alpha_n \overline{\gamma})\|x_n - p\| + \alpha_n \gamma \|f(y_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| \\ \le & \ (1 - \alpha_n(\overline{\gamma} - \alpha\gamma))\|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|. \end{aligned} \tag{2.3}$$

It follows from (2.3) that

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|\}, \quad n \ge 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{y_n\}$, $\{f(y_n)\}$, $\{W_n u_n\}$ and $\{W_n y_n\}$. Define

$$x_{n+1} = (1-\beta_n) z_n + \beta_n x_n, \quad n \ge 0.$$

Then

$$\begin{aligned} z_{n+1} - z_n = & \ \tfrac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ = & \ \tfrac{\alpha_{n+1}\gamma f(y_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}} \\ & \ - \tfrac{\alpha_n \gamma f(y_n) + ((1-\beta_n)I - \alpha_n A)W_n y_n}{1-\beta_n} \\ = & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(y_{n+1}) - \tfrac{\alpha_n}{1-\beta_n}\gamma f(y_n) + W_{n+1}y_{n+1} \\ & \ -W_n y_n + \tfrac{\alpha_n}{1-\beta_n} A W_n y_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} A W_{n+1} y_{n+1} \\ = & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(y_{n+1}) - AW_{n+1}y_{n+1}] + \tfrac{\alpha_n}{1-\beta_n}[AW_n y_n - \gamma f(y_n)] \\ & \ +W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_{n+1}y_n\| \leq\ & \|y_{n+1} - y_n\| \\ =\ & \|(1-\gamma_{n+1})x_{n+1} + \gamma_{n+1}W_{n+1}u_{n+1} - (1-\gamma_n)x_n \\ & -\gamma_n W_n u_n\| \\ \leq\ & (1-\gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| \\ & +\gamma_{n+1}\|W_{n+1}u_{n+1} - W_n u_n\| \\ & +|\gamma_{n+1} - \gamma_n|\|W_n u_n\| \\ \leq\ & (1-\gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| \\ & +\gamma_{n+1}(\|W_{n+1}u_{n+1} - W_{n+1}u_n\| + \\ & +\|W_{n+1}u_n - W_n u_n\|) + |\gamma_{n+1} - \gamma_n|\|W_n u_n\|.\end{aligned} \tag{2.5}$$

From (2.1), Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_{n+1}u_n - W_n u_n\| =\ & \|\lambda_1 T_1 U_{n+1,2}u_n - \lambda_1 T_1 U_{n,2}u_n\| \\ \leq\ & \lambda_1\|U_{n+1,2}u_n - U_{n,2}u_n\| \\ =\ & \lambda_1\|\lambda_2 T_2 U_{n+1,3}u_n - \lambda_2 T_2 U_{n,3}u_n\| \\ \leq\ & \lambda_1\lambda_2\|U_{n+1,3}u_n - U_{n,3}u_n\| \\ \leq\ & \dots \\ \leq\ & \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \\ \leq\ & M\textstyle\prod_{i=1}^n \lambda_i,\end{aligned} \tag{2.6}$$

for all $n \geq 1$ and similarly,

$$\|W_{n+1}y_n - W_n y_n\| \leq \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \leq M\prod_{i=1}^n \lambda_i, \tag{2.7}$$

for some constant $M \geq 0$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we obtain

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C, \tag{2.8}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \text{for all } y \in C. \tag{2.9}$$

Putting $y = u_{n+1}$ in (2.8) and $y = u_n$ in (2.9), we have

$$\phi(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$

and

$$\phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A_2), we get

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0,$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, we may assume there exists a real number r such that $0 < r < r_n$ for all $n \geq 0$. Therefore

$$\begin{aligned}\|u_{n+1} - u_n\|^2 \leq & \quad \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \tfrac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ \leq & \quad \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}.\end{aligned}$$

So

$$\begin{aligned}\|u_{n+1} - u_n\| \leq & \quad \|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\| \\ \leq & \quad \|x_{n+1} - x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L,\end{aligned} \tag{2.10}$$

where $L = \sup\{\|u_n - x_n\| : n \geq 0\}$. Substituting (2.6) and (2.10) in (2.5), we have

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_{n+1}y_n\| \leq & \quad (1 - \gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| \\ & \quad +\gamma_{n+1}(\|x_{n+1} - x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L) \\ & \quad +\gamma_{n+1}M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} - \gamma_n|\|W_n u_n\| \\ \leq & \quad \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L \\ & \quad +M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} - \gamma_n|\|W_n u_n\|.\end{aligned} \tag{2.11}$$

By combining (2.4), (2.7) and (2.11), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq & \quad \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\ & \quad +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(y_n)\|) \\ & \quad +\|W_{n+1}y_{n+1} - W_{n+1}y_n\| \\ & \quad +\|W_{n+1}y_n - W_n y_n\| - \|x_{n+1} - x_n\| \\ \leq & \quad \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\ & \quad +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(y_n)\|) \\ & \quad +[\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n\| \\ & \quad +\tfrac{1}{r}|r_n - r_{n+1}|L + M\textstyle\prod_{i=1}^n \lambda_i + |\gamma_{n+1} \\ & \quad -\gamma_n|\|W_n u_n\|] + M\textstyle\prod_{i=1}^n \lambda_i - \|x_{n+1} - x_n\| \\ \leq & \quad \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(y_{n+1})\| + \|AW_{n+1}y_{n+1}\|) \\ & \quad +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n y_n\| + \|\gamma f(y_n)\|) \\ & \quad +|\gamma_{n+1} - \gamma_n|\|x_n\| + \tfrac{1}{r}|r_n - r_{n+1}|L \\ & \quad +|\gamma_{n+1} - \gamma_n|\|W_n u_n\| + 2M\textstyle\prod_{i=1}^n \lambda_i.\end{aligned} \tag{2.12}$$

Thus it follows from (2.12) and condition $(i)-(iv)$ that (noting that $0<\lambda_i\leq b<1$, for all $i\geq 1$)

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence by Lemma 2.4, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0.$$

From (2.10) and $\lim_{n\to\infty}|r_{n+1}-r_n|=0$, we get

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Since $x_{n+1}=\alpha_n\gamma f(y_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n$, we obtain

$$\|x_n-W_ny_n\|\leq\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(y_n)-AW_ny_n\|+\beta_n\|x_n-W_ny_n\|.$$

That is,

$$\|x_n-W_ny_n\|\leq\frac{1}{1-\beta_n}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(y_n)-AW_ny_n\|.$$

It follows that

$$\lim_{n\to\infty}\|x_n-W_ny_n\|=0. \tag{2.13}$$

Let $p\in F$. Since T_r is firmly nonexpansive, we have

$$\begin{aligned}\|u_n-p\|^2 =& \quad \|T_{r_n}x_n-T_{r_n}p\|^2\leq\langle T_{r_n}x_n-T_{r_n}p,x_n-p\rangle\\ =& \quad \langle u_n-p,x_n-p\rangle=\tfrac{1}{2}(\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2),\end{aligned}$$

and hence

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|x_n-u_n\|^2.$$

Therefore

$$
\begin{aligned}
\|x_{n+1}-p\|^2 = {} & \|\alpha_n\gamma f(y_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n-p\|^2 \\
= {} & \|(1-\beta_n)(W_n y_n-p)+\beta_n(x_n-p)+\alpha_n\gamma f(y_n)-\alpha_n AW_n y_n\|^2 \\
= {} & \alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2+\|\beta_n(x_n-p)+(1-\beta_n)(W_n y_n-p)\|^2 \\
& +2\alpha_n\langle\beta_n(x_n-p)+(1-\beta_n)(W_n y_n-p),\gamma f(y_n)-AW_n y_n\rangle \\
\le {} & \alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n)\|W_n y_n-p\|^2 \\
& +2\alpha_n(1-\beta_n)\langle W_n y_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
& +2\alpha_n\beta_n\langle x_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
& \le (1-\beta_n)\|y_n-p\|^2+\beta_n\|x_n-p\|^2+\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n(1-\beta_n)\langle W_n y_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
& +2\alpha_n\beta_n\langle x_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
= {} & (1-\beta_n)\|(1-\gamma_n)(x_n-p)+\gamma_n(W_n u_n-p)\|^2+\beta_n\|x_n-p\|^2 \\
& +\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n(1-\beta_n)\langle W_n y_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
& +2\alpha_n\beta_n\langle x_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
\le {} & (1-\beta_n)(1-\gamma_n)\|x_n-p\|^2+(1-\beta_n)\gamma_n\|u_n-p\|^2+\beta_n\|x_n-p\|^2 \\
& +\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2+2\alpha_n(1-\beta_n)\langle W_n y_n-p,\gamma f(y_n) \\
& -AW_n y_n\rangle+2\alpha_n\beta_n\langle x_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
\le {} & (1-\beta_n)(1-\gamma_n)\|x_n-p\|^2+(1-\beta_n)\gamma_n(\|x_n-p\|^2-\|x_n-u_n\|^2) \\
& +\beta_n\|x_n-p\|^2+\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n(1-\beta_n)\langle W_n y_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
& +2\alpha_n\beta_n\langle x_n-p,\gamma f(y_n)-AW_n y_n\rangle \\
\le {} & \|x_n-p\|^2+\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n(1-\beta_n)\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\| \\
& +2\alpha_n\beta_n\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\|-(1-\beta_n)\gamma_n\|x_n-u_n\|^2 \\
= {} & \|x_n-p\|^2+\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\|-(1-\beta_n)\gamma_n\|x_n-u_n\|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(1-\beta_n)\gamma_n\|x_n-u_n\|^2 \le {} & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\| \\
= {} & (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|) \\
& +\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\| \\
\le {} & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
& +\alpha_n^2\|\gamma f(y_n)-AW_n y_n\|^2 \\
& +2\alpha_n\|x_n-p\|\|\gamma f(y_n)-AW_n y_n\|.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$ and $\liminf_{n\to\infty}\gamma_n>0$, it is easy to see $\liminf_{n\to\infty}(1-\beta_n)\gamma_n>0$. So

$$
\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{2.14}
$$

Observe

$$\begin{aligned}\|y_n - u_n\| \leq & \ \|y_n - x_n\| + \|x_n - u_n\| \\ \leq & \ \gamma_n\|W_n u_n - x_n\| + \|x_n - u_n\| \\ \leq & \ \gamma_n\|W_n u_n - W_n y_n + W_n y_n - x_n\| + \|x_n - u_n\| \\ \leq & \ \gamma_n[\|y_n - u_n\| + \|W_n y_n - x_n\|] + \|x_n - u_n\|,\end{aligned}$$

and hence

$$(1-\gamma_n)\|y_n - u_n\| \leq \|W_n y_n - x_n\| + \|x_n - u_n\|.$$

So, from (2.13), (2.14) and $\limsup_{n\to\infty}\gamma_n < 1$, we get

$$\lim_{n\to\infty}\|y_n - u_n\| = 0 \tag{2.15}$$

and hence $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since

$$\begin{aligned}\|W_n u_n - u_n\| \leq & \ \|W_n u_n - W_n y_n\| + \|W_n y_n - x_n\| + \|x_n - u_n\| \\ \leq & \ \|y_n - u_n\| + \|W_n y_n - x_n\| + \|x_n - u_n\|,\end{aligned}$$

we have $\lim_{n\to\infty}\|W_n u_n - u_n\| = 0$. On the other hand, observe

$$\|Wu_n - u_n\| \leq \|W_n u_n - Wu_n\| + \|W_n u_n - u_n\|. \tag{2.16}$$

It follows from (2.16) and Remark 2.9 that

$$\lim_{n\to\infty}\|Wu_n - u_n\| = 0.$$

Next, we claim

$$\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, x_n - x^*\rangle \leq 0,$$

where $x^* = P_{F(W)\bigcap EP(\phi)}(I - A + \gamma f)x^*$. First, we can choose a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\lim_{j\to\infty}\langle\gamma f(x^*) - Ax^*, u_{n_j} - x^*\rangle = \limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, u_n - x^*\rangle.$$

Since $\{u_{n_j}\}$ is bounded, there exists a subsequence of $\{u_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume $u_{n_j} \rightharpoonup w$. From $\|Wu_n - u_n\| \to 0$, $Wu_{n_j} \rightharpoonup w$. Now, we show $w \in EP(\phi)$. By $u_n = T_{r_n}x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C.$$

From (A_2), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n),$$

and hence

$$\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle \geq \phi(y, u_{n_j}).$$

Since $\frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup w$, from (A_4), we have

$$\phi(y, w) \leq 0, \quad \text{for all } y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\phi(y_t, w) \leq 0$. Therefore, from (A_1) and (A_4), we obtain

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, w) \leq t\phi(y_t, y),$$

and so $\phi(y_t, y) \geq 0$. From (A_3), we have

$$\phi(w, y) \geq 0, \quad \text{for all } y \in C,$$

and hence $w \in EP(\phi)$. Next, we show $w \in F(W)$. Assume $w \notin F(W)$. Since $u_{n_j} \rightharpoonup w$ and $Ww \neq w$, from Lemma 2.6, we obtain

$$\begin{aligned} \liminf_{j\to\infty} \|u_{n_j} - w\| &< \liminf_{j\to\infty} \|u_{n_j} - Ww\| \\ &\leq \liminf_{j\to\infty} (\|u_{n_j} - Wu_{n_j}\| + \|Wu_{n_j} - Ww\|) \\ &\leq \liminf_{j\to\infty} \|u_{n_j} - w\|. \end{aligned}$$

This is a contradiction. So, $w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, $w \in F$. Since $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)x^*$, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle &= \lim_{j\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n_j} - x^* \rangle \\ &= \lim_{j\to\infty} \langle \gamma f(x^*) - Ax^*, u_{n_j} - x^* \rangle \\ &= \langle \gamma f(x^*) - Ax^*, w - x^* \rangle \leq 0. \end{aligned} \tag{2.17}$$

From (2.13), we obtain

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, W_n y_n - x^* \rangle = \limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0 \tag{2.18}$$

Finally, we prove $\{x_n\}$ converges strongly to $x^* = P_{F(W) \bigcap EP(\phi)}(I - A + \gamma f)x^*$. Indeed, from (1.5), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 = &\ \|\alpha_n(\gamma f(y_n) - Ax^*) + \beta_n(x_n - x^*) \\ &\quad + ((1-\beta_n)I - \alpha_n A)(W_n y_n - x^*)\|^2 \\ = &\ \alpha_n^2 \|\gamma f(y_n) - Ax^*\|^2 + \|\beta_n(x_n - x^*) \\ &\quad + ((1-\beta_n)I - \alpha_n A)(W_n y_n - x^*)\|^2 \\ &\quad + 2\beta_n\alpha_n \langle x_n - x^*, \gamma f(y_n) - Ax^* \rangle \\ &\quad + 2\alpha_n \langle ((1-\beta_n)I - \alpha_n A)(W_n y_n - x^*), \gamma f(y_n) - Ax^* \rangle \\ \leq &\ ((1 - \beta_n - \alpha_n\overline{\gamma})\|W_n y_n - x^*\| + \beta_n\|x_n - x^*\|)^2 \\ &\quad + \alpha_n^2 \|\gamma f(y_n) - Ax^*\|^2 + 2\beta_n\alpha_n\gamma \langle x_n - x^*, f(y_n) - f(x^*) \rangle \\ &\quad + 2\beta_n\alpha_n \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle \\ &\quad + 2(1-\beta_n)\gamma\alpha_n \langle W_n y_n - x^*, f(y_n) - f(x^*) \rangle \\ &\quad + 2(1-\beta_n)\alpha_n \langle W_n y_n - x^*, \gamma f(x^*) - Ax^* \rangle \\ &\quad - 2\alpha_n^2 \langle A(W_n y_n - x^*), \gamma f(y_n) - Ax^* \rangle, \end{aligned}$$

Which implies

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq\ & [(1-\alpha_n\overline{\gamma})^2+2\alpha\beta_n\alpha_n\gamma+2\alpha(1-\beta_n)\alpha_n\gamma]\|x_n-x^*\|^2\\ & +2\beta_n\alpha_n\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle\\ & +\alpha_n^2\|\gamma f(y_n)-Ax^*\|^2\\ & +2(1-\beta_n)\alpha_n\langle W_ny_n-x^*,\gamma f(x^*)-Ax^*\rangle\\ & -2\alpha_n^2\langle A(W_ny_n-x^*),\gamma f(y_n)-Ax^*\rangle\\ \leq\ & [1-2\alpha_n(\overline{\gamma}-\alpha\gamma)]\|x_n-x^*\|^2+\alpha_n^2\overline{\gamma}^2\|x_n-x^*\|^2\\ & +2\beta_n\alpha_n\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle+\alpha_n^2\|\gamma f(y_n)-Ax^*\|^2\\ & +2(1-\beta_n)\alpha_n\langle W_ny_n-x^*,\gamma f(x^*)-Ax^*\rangle\\ & +2\alpha_n^2\|\gamma f(y_n)-Ax^*\|\|A(W_ny_n-x^*)\|\\ =\ & [1-2\alpha_n(\overline{\gamma}-\alpha\gamma)]\|x_n-x^*\|^2+\alpha_n\{\alpha_n(\overline{\gamma}^2\|x_n-x^*\|^2\\ & +\|\gamma f(y_n)-Ax^*\|^2+2\|\gamma f(y_n)-Ax^*\|\|A(W_ny_n-x^*)\|)\\ & +2\beta_n\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle\\ & +2(1-\beta_n)\langle W_ny_n-x^*,\gamma f(x^*)-Ax^*\rangle\}.\end{aligned}$$

Since $\{x_n\},\{f(y_n)\}$ and $\{W_ny_n\}$ are bounded, we can take a constant $M_1\geq 0$ such that

$$\overline{\gamma}^2\|x_n-x^*\|^2+\|\gamma f(y_n)-Ax^*\|^2+2\|\gamma f(y_n)-Ax^*\|\|A(W_ny_n-x^*)\|\leq M_1,$$

for all $n\geq 0$. So

$$\|x_{n+1}-x^*\|^2\leq[1-2\alpha_n(\overline{\gamma}-\alpha\gamma)]\|x_n-x^*\|^2+\alpha_n\xi_n, \tag{2.19}$$

where

$$\xi_n=2\beta_n\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle+2(1-\beta_n)\langle W_ny_n-x^*,\gamma f(x^*)-Ax^*\rangle+\alpha_nM_1.$$

By (i), (2.17) and (2.18), we get $\limsup_{n\to\infty}\xi_n\leq 0$. Now applying Lemma 2.5 to (2.19) concludes that $x_n\to x^*$ as $n\to\infty$. This completes the proof.

As direct consequences of Theorem 2.11, we obtain two corollaries.

Corollary 2.12 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi: C\times C\to\mathbb{R}$ be a bifunction satisfying $(A_1)-(A_4)$ such that $EP(\phi)\neq\emptyset$ and A be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma}>0$ and $\|A\|\leq 1$. Suppose $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{r_n\}\subset(0,\infty)$ is a real sequence satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(iii) $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$ *and* $\lim_{n\to\infty}|\gamma_{n+1}-\gamma_n|=0$;

(iv) $0<\liminf_{n\to\infty}r_n$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n|=0$.

Let f be a contraction of C into itself and given $x_0\in C$ arbitrarily. Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases}\phi(u_n,x)+\frac{1}{r_n}\langle x-u_n,u_n-x_n\rangle\geq 0, \quad \text{for all } x\in C,\\ x_{n+1}=\alpha_n\gamma f((1-\gamma_n)x_n+\gamma_nu_n)+\beta_nx_n\\ \qquad\quad +((1-\beta_n)I-\alpha_nA)((1-\gamma_n)x_n+\gamma_nu_n),\end{cases}$$

converges strongly to $x^*\in EP(\phi)$, where $x^*=P_{EP(\phi)}(I-A+\gamma f)(x^*)$.

Proof. Put $T_ix=x$ for all $i=1,2,\ldots$ and for all $x\in C$ in (2.1). Then, $W_nx=x$ for all $x\in C$. The conclusion follows immediately from Theorem 2.11.

Corollary 2.13 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C which satisfies $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and A be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Let f be a contraction of C into itself and given $x_0 \in C$ arbitrarily. Then, the sequences $\{x_n\}$ generated iteratively by

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n W_n P_C x_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n, \end{cases}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$, converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n)}(I - A + \gamma f)(x^*)$.

Proof. Put $\phi(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 0$. Then, $u_n = P_C x_n$. From (1.5),

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n W_n P_C x_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n. \end{cases}$$

So, the conclusion follows immediately from Theorem 2.11.

Remark 2.14 Let $\eta_n = 1 - \alpha_n - \beta_n$, $A = I$ (identity map) with constant $\overline{\gamma} = 1$ and $\gamma = 1$ in Theorem 2.11, then Theorem 2.11 is a generalization of [9, Theorem 3.1]. Also, Corollaries 2.12 and 2.13 are generalizations of [9, Corollary 3.1] and [9, Corollary 3.2], respectively.

3. Numerical Test

In this section, we give an example to illustrate the scheme (1.5) given in Theorem 2.11.

Example 3.1 *Let $C = [-1,1] \subset H = \mathbb{R}$ and define $\phi(x,y) = -5x^2 + xy + 4y^2$. First, we verify that ϕ satisfies the conditions $(A_1) - (A_4)$ as follows:*
(A_1) $\phi(x,x) = -5x^2 + x^2 + 4x^2 = 0$, *for all $x \in [-1,1]$;*
(A_2) $\phi(x,y) + \phi(y,x) = -(x-y)^2 \leq 0$, *for all $x, y \in [-1,1]$;*
(A_3) *For all $x, y, z \in [-1,1]$,*

$$\limsup_{t\to 0^+} \phi(tz + (1-t)x, y) = -5x^2 + xy + 4y^2 \leq \phi(x,y).$$

(A_4) *For all $x \in [-1,1]$, $\Phi(y) = \phi(x,y) = -5x^2 + xy + 4y^2$ is a lower semicontinuous and convex function.*

From Lemma 2.2, T_r is single-valued, for all $r > 0$. Now, we deduce a formula for $T_r(x)$. For any $y \in [-1,1]$ and $r > 0$, we have

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Leftrightarrow 4ry^2 + ((r+1)z - x)y + xz - (5r+1)z^2 \geq 0.$$

Set $G(y) = 4ry^2 + ((r+1)z - x)y + xz - (5r+1)z^2$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4r, b = (r+1)z - x$ and $c = xz - (5r+1)z^2$. So its *discriminate* $\Delta = b^2 - 4ac$ *is*

$$\begin{aligned}\Delta = & \quad [(r+1)z - x]^2 - 16r(xz - (5r+1)z^2) \\ = & \quad (r+1)^2 z^2 - 2(r+1)xz + x^2 - 16rxz + (80r^2 + 16r)z^2 \\ = & \quad [(9r+1)z - x]^2.\end{aligned}$$

Since $G(y) \geq 0$ *for all* $y \in C$, *this is true if and only if* $\Delta \leq 0$. *That is,* $[(9r+1)z - x]^2 \leq 0$. *Therefore,* $z = \frac{x}{9r+1}$, *which yields* $T_r(x) = \frac{x}{9r+1}$. *So, from Lemma 2.2, we get* $EP(\phi) = \{0\}$. *Let* $\alpha_n = \frac{1}{n}, \beta_n = \frac{n}{3n+1}, \lambda_n = \beta \in (0,1), \gamma_n = \frac{1}{2}, r_n = 1, T_n = I$, *for all* $n \in \mathbb{N}$, $Ax = x$ *with coefficient* $\overline{\gamma} = 1$, $f(x) = \frac{1}{2}x$ *and* $\gamma = \frac{1}{2}$. *Hence,* $F = \bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi) = \{0\}$. *Also,* $W_n = I$. *Indeed, from (2.1), we have*

$$\begin{aligned}W_1 = U_{1,1} = & \quad \lambda_1 T_1 U_{1,2} + (1-\lambda_1)I = \lambda_1 T_1 + (1-\lambda_1)I, \\ W_2 = U_{2,1} = & \quad \lambda_1 T_1 U_{2,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{2,3} + (1-\lambda_2)I) \\ & \quad +(1-\lambda_1)I \\ = & \quad \lambda_1\lambda_2 T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I, \\ W_3 = U_{3,1} = & \quad \lambda_1 T_1 U_{3,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{3,3} + (1-\lambda_2)I) \\ & \quad +(1-\lambda_1)I \\ = & \quad \lambda_1\lambda_2 T_1 T_2 U_{3,3} + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I \\ = & \quad \lambda_1\lambda_2 T_1 T_2(\lambda_3 T_3 U_{3,4} + (1-\lambda_3)I) + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I \\ = & \quad \lambda_1\lambda_2\lambda_3 T_1 T_2 T_3 + \lambda_1\lambda_2(1-\lambda_3)T_1 T_2 + \lambda_1(1-\lambda_2)T_1 \\ & \quad +(1-\lambda_1)I.\end{aligned}$$

By computing in this way by (2.1), we obtain

$$\begin{aligned}W_n = U_{n,1} = & \quad \lambda_1\lambda_2 \ldots \lambda_n T_1 T_2 \ldots T_n + \lambda_1\lambda_2 \ldots \lambda_{n-1}(1-\lambda_n)T_1 T_2 \ldots T_{n-1} \\ & \quad +\lambda_1\lambda_2 \ldots \lambda_{n-2}(1-\lambda_{n-1})T_1 T_2 \ldots T_{n-2} \\ & \quad +\lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I.\end{aligned}$$

Since $T_n = I, \lambda_n = \beta$, *for all* $n \in \mathbb{N}$, *we get*

$$W_n = (\beta^n + \beta^{n-1}(1-\beta) + \ldots + \beta(1-\beta) + (1-\beta))I = I.$$

Then, from Lemma 2.5, the sequences $\{x_n\}$ *and* $\{u_n\}$, *generated iteratively by*

$$\begin{cases} u_n = T_{r_n} x_n = \frac{1}{10} x_n, \\ y_n = \frac{1}{2} x_n + \frac{1}{2} W_n u_n = \frac{11}{20} x_n, \\ x_{n+1} = \frac{168n^2 - 55n - 33}{80n(3n+1)} x_n, \end{cases} \tag{3.1}$$

converges strongly to $0 \in F$, *where* $0 = P_F(\frac{3}{4}I)(0)$.

If $\gamma = 1$, *then* $x_{n+1} = \frac{84n^2 - 11n - 11}{40n(3n+1)} x_n$ *in scheme (3.1). So, it is to see, the rate of convergence of Theorem 2.11 is faster than the rate of convergence of* [9, Theorem 3.1].

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