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BEST PROXIMITY POINT THEOREMS FOR NON-SELF MAPPINGS

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Abstract. Let us consider a pair (A, B) of nonempty subsets of a metric space X and a mapping $T : A \to B$. In this article, we introduced a notion called $P-$property and used it to prove sufficient conditions for the existence of a point $x_0 \in A$, called best proximity point, satisfying $d(x_0, Tx_0) = \text{dist}(A, B) := \inf\{d(a, b) : a \in A,\, b \in B\}$.
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1. Introduction

Let (X, d) be a metric space and A be a nonempty subset of X. A map $T : A \to X$ is said to be a contraction mapping if there is a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq k\, d(x, y)$, for all x, y in A. The well-known Banach contraction principle states that if A is a complete subset of a metric space X and $T : A \to A$ is a contraction self map map then, the fixed point equation $Tx = x$ has a unique solution in A. Due to the wide applications of this principle, there are enormous extensions and generalization of Banach contraction principle are available in the literature. For interesting and important extensions, generalizations of contraction principle, one may refer [1, 2]. The following generalization of Banach contraction principle was proved in [3].

Theorem 1.1. [3] *Let A and B be nonempty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a mapping satisfies*

(1) *$T(A) \subseteq B$ and $T(B) \subseteq A$,*
(2) *$d(Tx, Ty) \leq k\, d(x, y)$, for all $x \in A$ and $y \in B$, where $k \in (0, 1)$.*

Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.

The interesting feature of the above theorem is that the continuity of T is no longer needed. Note that the conditions in the above theorem forces us to conclude that $A \cap B$ is nonempty and the mapping T restricted to $A \cap B$ is a contraction mapping. In [4], the authors generalized Theorem 1.1 which does not forces the set $A \cap B$ to be empty. Let us define the notion called *cyclic contraction* which was introduced in [4].

Definition 1.2. [4] *Let A and B be nonempty subsets of a metric space X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic contraction mapping if there exists $k \in (0, 1)$ and satisfies*

(1) *$T(A) \subseteq B$ and $T(B) \subseteq A$,*
(2) *$d(Tx, Ty) \leq k\, d(x, y) + (1 - k)\,\mathrm{dist}(A, B)$, for all $x \in A$ and $y \in B$.*

Note that if $\mathrm{dist}(A, B) = 0$, then the notion of cyclic contraction reduces to a mapping which satisfy the assumptions of Theorem 1.1. Interestingly, consider a mapping $T : A \to B$ and if $A \cap B$ is an empty set, then $d(x, T(x)) > 0$ and $d(x, Tx) \geq \mathrm{dist}(A, B)$, for all $x \in A$. This motivates us to define the following notion called *best proximity points*.

Definition 1.3. [4] *Let A, B be nonempty subsets of a metric space X and $T : A \to B$ be a given mapping. A point $x_0 \in A$ is said to be a best proximity point of T if $d(x_0, Tx_0) = \mathrm{dist}(A, B)$.*

In [4], the authors proved the following best proximity point theorem.

Theorem 1.4. [4] *Let A, B be nonempty closed convex subsets of a uniformly convex Banach space and $T : A \cup B \to A \cup B$ be a cyclic contraction mapping. Then there exists a unique $x_0 \in A$ such that $\|x_0 - Tx_0\| = \mathrm{dist}(A, B)$.*

Some generalized versions of Theorem 1.4 can be found in [5, 6, 7, 8, 9]. Recently, in [10], Basha introduced a class of non-self mappings called *proximal contraction* and proved an another generalization of Banach contraction principle.

Definition 1.5. [10] *Let A and B be nonempty subsets of a metric space X. A mapping $T : A \to B$ is said to be a proximal contraction if there exists $\alpha \in [0, 1)$ such that*

$$d(u, Tx) + d(Tx, Ty) + d(Ty, v) \leq \alpha\, d(x, y)$$

whenever x and y are distinct elements in A satisfying the condition that $d(u, Tx) = \mathrm{dist}(A, B)$ and $d(v, Ty) = \mathrm{dist}(A, B)$ for some $u, v \in A$.

In [10], Basha obtained necessary conditions for the existence of a best proximity point of a proximal contraction mapping $T : A \to B$. In this article, we introduce a notion called $P-$property which is used to prove an extended version of Banach contraction principle. Also, we proved that the restriction of a proximal contraction mapping $T : A \to B$ to A_0 is nothing but the usual contraction mapping, provided the pair (A, B) has $P-$property and $T(A_0) \subseteq B_0$.

2. Preliminaries

Let A, B be two nonempty subsets of a metric space X and let us fix the following notations for our further use in this article.

$$A_0 = \{x \in A : d(x, y) = \mathrm{dist}(A, B) \text{ for some } y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = \mathrm{dist}(A, B) \text{ for some } x \in A\}$$

In [11], the authors discussed sufficient conditions which guarantees the nonemptiness of A_0 and B_0. Also, in [12], the authors proved that A_0 is contained in the boundary of A. It is easy to verify that A_0, B_0 are convex subsets of A and B respectively, if given A and B are convex subsets of a normed linear space. Let us define the non-self contraction map as follows.

Definition 2.1. *Let A, B be two nonempty subsets of a metric space X. A map $T : A \to B$ is said to be an (A, B) non-self $k-$contraction map if there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq k\, d(x, y), \text{ for all } x, y \in A.$$

It is worth to mention that, since T is not a self-map, for any x in A, we can not define $T(Tx)$. That is, for a fixed $x_0 \in A$, it is not possible to define the iterated sequence $x_n = Tx_{n-1}$, for each $n \in \mathbb{N}$, in a usual way. Now, let us introduce a notion called $P-$property, which we will play an important role in our main results.

Definition 2.2. *Let (A, B) be a pair of nonempty subsets of a metric space X. Then the pair (A, B) is said to have P-property if and only if*

$$\left.\begin{array}{l} d(x_1, y_1) = \operatorname{dist}(A, B) \\ d(x_2, y_2) = \operatorname{dist}(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is easy to see that, for any nonempty subset A of X, the pair (A, A) has P-property. Also, Theorem 3.5 shows that the pair (A, B) of nonempty closed convex subsets of a real Hilbert space H has $P-$property. Before proving Theorem 3.5, let us recall the notion of metric projection. Let C be a nonempty subset of a normed linear space N. Then the metric projection operator $P_C : N \to 2^C$ is defined as $P_C(x) = \{y \in C \ : \ \|x - y\| = \operatorname{dist}(x, C)\}$, for each $x \in N$, where 2^C denotes the set of all subsets of C. It is well known the fact that the metric projection operator P_C on a strictly convex Banach space N is a single valued map from N to C, where C is a nonempty weakly compact convex subset of N. The following lemma, which has been proved in [13], play an important role in proving Theorem 3.5.

Lemma 2.3. [13] *Let (A, B) be a pair of nonempty closed convex subsets of a real Hilbert space and let $x, y \in A_0$. Then $\|x - P_B(y)\| = \|y - P_B(x)\|$.*

3. Main results

Theorem 3.1. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) with $A_0 \neq \emptyset$. Let $T : A \to B$ be an (A, B) non-self $k-$contraction map such that $T(A_0) \subseteq B_0$. Assume that (A, B) satisfies P-property. Then there exists a unique element x^* in A such that*

$$d(x^*, Tx^*) = \operatorname{dist}(A, B).$$

Further, for each fixed x_0 in A_0, there exists a sequence $\{x_n\}$ such that for each $n \in \mathbb{N}$, $d(x_n, Tx_{n-1}) = \operatorname{dist}(A, B)$ and $\{x_n\}$ converges to the best proximity point x^* of the map T.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = \operatorname{dist}(A, B)$. Again, since $Tx_1 \in B_0$, there exists an element $x_2 \in A_0$

such that $d(x_2, Tx_1) = \text{dist}(A, B)$. This process can be continued. Having chosen x_n in A_0, it is possible to find x_{n+1} in A_0 such that

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B) \tag{3.1}$$

because of the fact that $T(A_0) \subseteq B_0$. Now, let us claim that $\{x_n\}$ is a Cauchy sequence. Since (A, B) satisfies $P-$property, for any $n \in \mathbb{N}$, we have

$$\left.\begin{array}{r} d(x_{n+1}, Tx_n) = \text{dist}(A, B), \\ d(x_n, Tx_{n-1}) = \text{dist}(A, B) \end{array}\right\} \Rightarrow d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}).$$

Therefore,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq k\, d(x_n, x_{n-1}).$$

Consequently,

$$d(x_n, x_{n+p}) \leq \left(\frac{k^n}{1-k}\right) d(x_1, x_0).$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Since X is complete and A is a closed subset of X, the sequence $\{x_n\}$ converges to some element x^* in A. Since T is a continuous map on A, we have $Tx_n \to Tx^*$. The continuity of the metric implies that $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$. But, (3.1) implies that $\{d(x_{n+1}, Tx_n)\}$ is a constant sequence which converges to $\text{dist}(A, B)$. Therefore,

$$d(x^*, Tx^*) = \text{dist}(A, B),$$

and thus $x^* \in A$ is a best proximity point for T. Suppose there are $x, y \in A$ with $x \neq y$ satisfying

$$d(x, Tx) = \text{dist}(A, B) \quad \text{and}$$
$$d(y, Ty) = \text{dist}(A, B).$$

Since (A, B) satisfies P-property and T is a (A, B) non-self $k-$contraction map,

$$d(x, y) = d(Tx, Ty) \leq k\, d(x, y) < d(x, y),$$

we arrived at a contradiction. Hence T has a unique best proximity point. □

Since (A, A) satisfies $P-$property, for any subset A of X, we can deduce the following Banach's contraction principle as a corollary to Theorem 3.1.

Corollary 3.2. *Let A be a nonempty closed subset of a complete metric space X and $T : A \to A$ be a contraction map. Then T has a unique fixed point x^* in A. Further, for each fixed x_0 in A, the iterated sequence $\{x_n\}$ defined as $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$, converges to the fixed point x^*.*

The following example illustrate Theorem 3.1.

Example 3.3. *Consider a pair (A, B) of nonempty closed subsets of $\mathbb{R}^2$ where $A = \{(1, t) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$ and $B = \{(2, t) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$. Let $T : A \to B$ be a mapping defined by $T(1, t) = (2, \frac{t}{2})$. Then T is an (A, B) non-self $k-$contraction map and $(1, 0)$ is the unique best proximity point for the map T in A.*

The following example shows that the condition $P-$property in Theorem 3.1 can not be relaxed to ensure the existence of a best proximity point for a non-self $k-$contraction mapping.

Example 3.4. *Fix $R > 0$ and $r = \frac{R}{4}$. Let $X = \mathbb{C}$, the set of all complex numbers. Consider $A = \{R\,e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and $B = \{r\,e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Then A, B are nonempty closed subsets of $\mathbb{C}$ with $A_0 = A$ and $B_0 = B$. It is worth to note that (A, B) does not have $P-$property. Let $T : A \to B$ be a mapping defined by $T(R\,e^{i\theta}) = r\,e^{i(\theta+\pi)}$. Then T is a contraction mapping having no best proximity point.*

Using the Lemma 2.3, let us prove the following result.

Theorem 3.5. *Any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H has the $P-$property.*

Proof. Suppose x_1, $x_2 \in A_0$ and y_1, $y_2 \in B_0$ such that $\|x_1 - y_1\| = \text{dist}(A, B)$ and $\|x_2 - y_2\| = \text{dist}(A, B)$. The uniqueness of metric projection operator on a Hilbert space implies that $y_1 = P_B(x_1)$ and $y_2 = P_B(x_2)$. By Lemma 2.3, we have $\|x_1 - y_2\| = \|x_2 - y_1\|$. Therefore, by Pythagoras theorem,

$$\begin{aligned} \|x_1 - y_2\|^2 &= \|x_1 - x_2\|^2 + \|x_2 - y_2\|^2 \\ &= \|x_1 - x_2\|^2 + \text{dist}(A, B)^2 \qquad (3.2) \\ \|x_2 - y_1\|^2 &= \|y_1 - y_2\|^2 + \|x_2 - y_2\|^2 \\ &= \|y_1 - y_2\|^2 + \text{dist}(A, B)^2 \qquad (3.3) \end{aligned}$$

The above equality follows from the fact that $x_1 - x_2 \perp x_2 - y_2$ and $y_1 - y_2 \perp y_2 - x_2$. From (3.2) and (3.3), we conclude that $\|x_1 - x_2\| = \|y_1 - y_2\|$. Hence, any pair (A, B) of nonempty closed convex subsets a real Hilbert space has the $P-$property. $\square$

The following theorem ensures the existence of a best proximity point for a proximal contraction mappings.

Theorem 3.6. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty and $T : A \to B$ be a proximal contraction mapping such that $T(A_0) \subseteq B_0$. Suppose (A, B) has the $P-$property. Then T has a unique best proximity point.*

Proof. Given $T : A \to B$ is a proximal contraction mapping. Now let us prove that the map T restricted to A_0 is a $(A_0,\, B_0)$ non-self $k-$contraction mapping into B_0.

Let $x, y \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_0, y_0 \in A_0$ such that $d(x_0, Tx) = \text{dist}(A, B)$ and $d(y_0, Ty) = \text{dist}(A, B)$. Thus the $P-$property of (A, B) concludes that $d(x_0, y_0) = d(Tx, Ty)$. Therefore,

$$\begin{aligned} d(Tx, Ty) &= d(x_0, y_0) \\ &\leq d(x_0, Tx) + d(Tx, Ty) + d(Ty, y_0) \leq \alpha\, d(x, y). \end{aligned}$$

Thus, $T : A_0 \to B_0$ is a contraction mapping. Note that A_0 and B_0 are nonempty closed subsets of A and B respectively. Since $\text{dist}(A_0, B_0) = \text{dist}(A, B)$, $(A_0)_0 =$

$A_0 \neq \emptyset$ and (A_0, B_0) has the $P-$property. Hence the conclusion follows from Theorem 3.1. □

The following theorem provides sufficient conditions to ensure the existence of a best proximity point for a nonexpansive non-self mapping.

Theorem 3.7. *Let (A, B) be a pair of nonempty compact convex subsets of a normed linear space X and $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Assume that (A, B) satisfies P-property. Then there exists an element x^* in A such that $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.*

Proof. Since A, B are compact sets, it is easy to see that A_0 is nonempty, and hence B_0 so. Fix $y_0 \in B_0$. For each fixed $n \in \mathbb{N}$, define a mapping $T_n : A \to B$ by

$$T_n(x) = \frac{1}{n}\, y_0 + \left(1 - \frac{1}{n}\right) Tx, \quad \text{for all } x \in A.$$

Since B is a convex set, the mapping T_n is well defined for all $n \in \mathbb{N}$. Also by the non-expansivity of T, it is easy to verify that each $T_n : A \to B$ is a (A, B) non-self $\frac{1}{n}-$contraction mapping. Since B_0 is convex and $T(A_0) \subseteq B_0$, we conclude that $T_n(A_0) \subseteq B_0$, for all $n \in \mathbb{N}$. Hence by Theorem 3.1, for each $n \in \mathbb{N}$, there exists $x_n \in A$ such that $\|x_n - T_n(x_n)\| = \operatorname{dist}(A, B)$. Since A is compact, with out loss of generality, let us assume that $x_n \to x_0$, for some $x_0 \in A$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} \operatorname{dist}(A, B) &\leq \|x_0 - Tx_0\| \\ &\leq \|x_0 - x_n\| + \|x_n - T_n(x_n)\| + \|T_n(x_n) - Tx_0\| \\ &\leq \|x_0 - x_n\| + \operatorname{dist}(A, B) + \frac{1}{n}\|y_0 - Tx\| + \left(1 - \frac{1}{n}\right)\|Tx_n - Tx_0\| \end{aligned}$$

Since $Tx_n \to Tx_0$ as $n \to \infty$, the above inequalities shows that $x_0 \in A$ satisfies $\|x_0 - Tx_0\| = \operatorname{dist}(A, B)$, and hence T has a best proximity point. □

Let us illustrate the above theorem with the following example.

Example 3.8. *Let $A := \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq -1,\ 0 \leq y \leq 1\}$ and $B := \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\ 0 \leq y \leq 1\}$. Then $\operatorname{dist}(A, B) = 2$ and the pair (A, B) satisfies the $P-$property. Let $T : A \to B$ be a mapping defined as $T(x, y) = (-x, y)$. It is easy to verify that T is a nonexpansive mapping. Hence by Theorem 3.7, T has at least one best proximity point in A. Note that any $(-1, y) \in A$ is a best proximity point of T.*

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