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AN ITERATIVE METHOD FOR A FUNCTIONAL-DIFFERENTIAL EQUATION OF SECOND ORDER WITH MIXED TYPE ARGUMENT

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Abstract. In this paper we shall study a functional differential equation of second order with mixed type argument. For this problem we give an algorithm based on the step method and the successive approximation method.
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1. Introduction

In this paper we study the following problem

$$x''(t) = f(t, x(t), x'(t), x(t-h), x(t+h)),\ t \in [-T, T], \tag{1.1}$$

$$x(t) = \varphi(t),\ t \in [-h, h]. \tag{1.2}$$

Functional-differential equations with mixed type argument have been studied by R. Driver [4], I. A. Rus and C. Iancu [17], V.A. Darzu [3], [8], R. Precup [12], D. Otrocol, V.A. Ilea and C. Revnic [11]. For this problem the above authors studied the existence and uniqueness of the solution using the step method. For other results in this field we quote here the papers of J.S. Cassell and Z. Hou [2], L.J. Grimm and H. Schmidt [5], Z. Hou and J.S. Cassell [6], V. Hutson [7], J. Mallet-Paret [10], I.A. Rus [13], L.S. Schulman [18], J. Wu and X. Zou [19].

The purpose of this paper is to elaborate an algorithm based on the step method and the successive approximation method and to apply it on the problem (1.1)-(1.2) (see [14], [15], [16], [11]).

Algorithm. At each step we have a problem like this

$$\begin{cases} x''(t) = f(t, x(t), x'(t), x(t-h), x(t+h)), \ t \in [a+h, a+2h] \\ x(t) = \theta(t), \ t \in [a, a+2h], \end{cases} \tag{1.3}$$

where $f \in C([a+h, a+2h] \times \mathbb{R}^4, \mathbb{R}), \theta \in C([a, a+2h], \mathbb{R})$ and $x : [a, a+3h] \to \mathbb{R}$.

Follows that we have

$$\theta''(t) = f(t, \theta(t), \theta'(t), \theta(t-h), x(t+h)), \ t \in [a+h, a+2h].$$

We denote $\xi := t+h, \ \xi \in [a+2h, a+3h]$. Then

$$\theta''(\xi - h) = f(\xi - h, \theta(\xi - h), \theta'(\xi - h), \theta(\xi - 2h), x(\xi)), \ \xi \in [a+2h, a+3h].$$

We denote $F(\xi, x(\xi)) := f(\xi - h, \theta(\xi - h), \theta'(\xi - h), \theta(\xi - 2h), x(\xi)) - \theta''(\xi - h)$. So

$$F(\xi, x(\xi)) = 0. \tag{1.4}$$

The purpose is to impose conditions on f such that equation (1.4) has a unique solution who can be approximated by Newton's method.

In order to study the problem (1.1)-(1.2) we need the following well known result.

Implicit function theorem. ([1]) *We suppose that $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions*

(i) $F \in C^1([a, b] \times \mathbb{R})$;

(ii) $\frac{\partial F(t,u)}{\partial u} \in \mathbb{R}^*$ *and* $\left|\frac{\partial F(t,u)}{\partial u}\right| \leq M_1, \ \forall t \in [a, b], u \in \mathbb{R}$;

(iii) *for each* $t_0 \in [a, b]$ *there exists* $u_0 \in \mathbb{R}$ *such that* $F(t_0, u_0) = 0$.

Then, there exists a unique function $x \in C^1[a, b]$ *such that* $F(t, x(t)) = 0, \forall t \in [a, b]$ *with* $x(t_0) = u_0$. *This solution can be obtained using the successive approximation method.*

In terms of f, for the problem (1.1)-(1.2), the conditions from the above theorem are:

(C_1) $f \in C^k([-T, T] \times \mathbb{R}^4, \mathbb{R}), \varphi \in C^k([-h, h], \mathbb{R}), k = [\frac{T}{h}] + 1$;

(C_2) $\frac{\partial f(t,u,v,w,z)}{\partial z} \in \mathbb{R}^*$, $\forall t \in [-T, T]$, $\forall u, v, w, z \in \mathbb{R}$;

(C_3) $\left|\frac{\partial f(t,u,v,w,z)}{\partial z}\right| \leq M_1, \forall t \in [-T, T], \ \forall u, v, w, z \in \mathbb{R}$;

(C_4) $\forall t \in [-T, T], u, v, w, z, \eta \in \mathbb{R}$, the equation $f(t, u, v, w, z) - \eta = 0$ has a unique solution.

2. Main result

In this section we apply the algorithm from section 1 for the problem (1.1)-(1.2).

Let $n \in \mathbb{N}^*$ be such that $nh \leq T, \ (n+1)h > T$. In the conditions (C_1)-(C_3), the step method consists in the following:

For $t \in [h, 2h]$ we have

$$x''(t) = f(t, x(t), x'(t), x(t-h), x(t+h)),$$
$$x''(t-h) = f(t-h, x(t-h), x'(t-h), x(t-2h), x(t)),$$
$$x_0''(t-h) = f(t-h, x_0(t-h), x_0'(t-h), x_{-1}(t-2h), x(t)),$$

where $x(t) = \varphi(t) = \begin{cases} x_{-1}(t), t \in [-h, 0], \\ x_0(t), t \in [0, h]. \end{cases}$

We denote $x(t) := x_1(t),\ t \in [h, 2h]$. Let

$$F(t, x_1(t)) := f(t-h, x_0(t-h), x_0'(t-h), x_{-1}(t-2h), x_1(t)) - x_0''(t-h) = 0, \quad t \in [h, 2h].$$

$$F(t, x_1(t)) = 0,\ t \in [h, 2h].$$

From the implicit function theorem there exists a solution

$$x_1^* \in C^2[h, 2h] \Rightarrow x_1^* \in C^k[h, 2h], k = [\frac{T}{h}] + 1$$

such that

$$F(t, x_1^*(t)) = 0, \forall t \in [h, 2h].$$

The key of each step is to approximate the solution x_1^* with the method of Newton ([9]):

$$x_{1m}(t) = x_{1,m-1}(t) - G(t, x_1^*(t))F(t, x_{1,m-1}(t)),$$

where $G(t, x_1^*(t)) \neq 0$ and $x_{1,m-1}(t) - G(t, x_1^*(t))F(t, x_{1,m-1}(t))$ is a contraction.

We choose the function $G : [h, 2h] \times \mathbb{R} \to \mathbb{R}$ with $G(t, x_1^*(t)) := M \left(\frac{\partial F(h, x_1(h))}{\partial x_1}\right)^{-1}$, where $M \in (0, 1)$ is a constant. It is obvious that $G(t, x_1^*(t)) \neq 0$.

Now we consider the operator $A_1 : C[h, 2h] \to C[h, 2h]$, defined by

$$A_1(x_{1,m-1})(t) := x_{1,m-1}(t) - G(t, x_1^*(t))F(t, x_{1,m-1}(t)).$$

Proving that A_1 is a contraction we have the uniqueness of the solution x_{1m} on $[h, 2h]$. We have that $x_{1m} \overset{unif}{\to} x_1^*$ on $[h, 2h]$, so in the next step we shall use x_{1m} instead of x_1^*.

For $t \in [2h, 3h]$ we have

$$x''(t) = f(t, x(t), x'(t), x(t-h), x(t+h)),$$
$$x''(t-h) = f(t-h, x(t-h), x'(t-h), x(t-2h), x(t)),$$
$$x''_{1m}(t-h) = f(t-h, x_{1m}(t-h), x'_{1m}(t-h), x_0(t-2h), x(t)).$$

We denote $x(t) := x_2(t),\ t \in [2h, 3h]$. Let

$$F(t, x_2(t)) := f(t-h, x_{1m}(t-h), x'_{1m}(t-h), x_0(t-2h), x_2(t)) - x''_{1m}(t-h) = 0, \quad t \in [2h, 3h].$$

$$F(t, x_2(t)) = 0,\ t \in [2h, 3h].$$

Applying implicit function theorem, there exists the solution

$$x_2^* \in C^2[2h, 3h] \Rightarrow x_2^* \in C^k[2h, 3h]$$

such that

$$F(t, x_2^*(t)) = 0, \forall t \in [2h, 3h].$$

Now we approximate the solution $x_2^* \in [2h, 3h]$ with the method of Newton:

$$x_{2m}(t) = x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)),$$

where $G(t, x_2^*(t)) \neq 0$ and $x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t))$ is a contraction.

We choose $G : [2h, 3h] \times \mathbb{R} \to \mathbb{R}$ with $G(t, x_2^*(t)) := M\left(\frac{\partial F(2h, x_2(2h))}{\partial x_2}\right)^{-1}$, where $M \in (0, 1)$ is a constant. Then we have $G(t, x_2^*(t)) \neq 0$.

Let us consider the operator $A_2 : C[2h, 3h] \to C[2h, 3h]$, defined by

$$A_2(x_{2,m-1})(t) := x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)).$$

In the same way as in the previous step we prove that A_2 is a contraction. Follows that $x_{2m} \overset{unif}{\to} x_2^*$ on $[2h, 3h]$, so in the next step we shall use x_{2m} instead of x_2^*.

By induction, for $t \in [nh, T]$ we have

$$\begin{aligned}
x''(t) &= f(t, x(t), x'(t), x(t-h), x(t+h)),\\
x''(t-h) &= f(t-h, x(t-h), x'(t-h), x(t-2h), x(t)),\\
x''_{n-1,m}(t-h) &= f(t-h, x_{n-1,m}(t-h), x'_{n-1,m}(t-h), x_{n-2,m}(t-2h), x(t)).
\end{aligned}$$

We denote $x(t) := x_n(t),\ t \in [nh, T]$. Let

$$\begin{aligned}
F(t, x_n(t)) :=& f(t-h, x'_{n-1,m}(t-h), x_{n-1,m}(t-h), x_{n-2,m}(t-2h), x(t))\\
&- x''_{n-1,m}(t-h) = 0,\\
F(t, x_n(t)) =& 0,\ t \in [nh, T].
\end{aligned}$$

Applying implicit function theorem, there exists the solution $x_n^* \in C^2[nh, T] \Rightarrow x_n^* \in C^k[nh, T]$ such that

$$F(t, x_n^*(t)) = 0, \forall t \in [nh, T].$$

We approximate the solution $x_n^* \in [nh, T]$ with the method of Newton:

$$x_{nm}(t) = x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t)),$$

where $G(t, x_n^*(t)) \neq 0$ and $x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t))$ is a contraction.

The function chosen here is $G : [nh, T] \times \mathbb{R} \to \mathbb{R},\ G(t, x_n^*(t)) := M\left(\frac{\partial F(nh, x_n(nh))}{\partial x_n}\right)^{-1}$, where $M \in (0, 1)$ is a constant. Then $G(t, x_n^*(t)) \neq 0$.

Let the operator $A_n : C[nh, T] \to C[nh, T]$ defined by

$$A_n(x_{n,m-1})(t) := x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t)).$$

It is trivial to prove that A_n is a contraction. Then we have that $x_{nm} \overset{unif}{\to} x_n^*$ on $[nh, T]$.

So, the following convergence takes place

$$\widetilde{x} = \begin{cases} x_{-1}, & t \in [-h, 0]\\ x_0, & t \in [0, h]\\ x_{1m}, & t \in [h, 2h]\\ \vdots & \\ x_{nm}, & t \in [nh, T] \end{cases} \to x^* = \begin{cases} x_{-1}, & t \in [-h, 0]\\ x_0, & t \in [0, h]\\ x_1^*, & t \in [h, 2h]\\ \vdots & \\ x_n^*, & t \in [nh, T]. \end{cases}$$

In what follows we present the step method for the solution determined with the above algorithm.

$$(p_0)\ x(t) = \varphi(t) = \begin{cases} x_{-1}(t), t \in [-h, 0], \\ x_0(t), t \in [0, h]; \end{cases}$$

$$(p_1)\ x_{1m}(t) = x_{1,m-1}(t) - G(t, x_1^*(t))F(t, x_{1,m-1}(t)), t \in [h, 2h];$$

$$(p_2)\ x_{2m}(t) = x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)), t \in [2h, 3h];$$

$$(p_3)\ x_{3m}(t) = x_{3,m-1}(t) - G(t, x_3^*(t))F(t, x_{3,m-1}(t)), t \in [3h, 4h];$$

$$\vdots$$

$$(p_n)\ x_{nm}(t) = x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t)), t \in [nh, T].$$

Thus we have the following theorem

Theorem 2.1. *a) In the conditions $(C_1)-(C_3)$ we obtain that the problem (1.1)-(1.2) has in $C^2[-T,T]$ (which is in fact in $C^k[-T,T]$) a unique solution*

$$x^*(t) = \begin{cases} \varphi(t), & t \in [-h, h] \\ x_1^*(t), & t \in [h, 2h] \\ \vdots & \\ x_n^*(t), & t \in [nh, T]. \end{cases}$$

b) We suppose that the conditions $(C_1) - (C_3)$ and

(C_4) there exists $L_f > 0$ such that

$$|f(t, u, v, w, z_1) - f(t, u, v, w, z_2)| \le L_f\,|z_1 - z_2|\,, \forall t \in [-T, T], u, v, w, z_1, z_2 \in \mathbb{R};$$

are satisfied. Then the sequence defined by

$$(p_0)\ x(t) = \varphi(t) = \begin{cases} x_{-1}(t), t \in [-h, 0], \\ x_0(t), t \in [0, h]; \end{cases}$$

$$(p_1)\ x_{1m}(t) = x_{1,m-1}(t) - G(t, x_1^*(t))F(t, x_{1,m-1}(t)), t \in [h, 2h];$$

$$(p_2)\ x_{2m}(t) = x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)), t \in [2h, 3h];$$

$$(p_3)\ x_{3m}(t) = x_{3,m-1}(t) - G(t, x_3^*(t))F(t, x_{3,m-1}(t)), t \in [3h, 4h];$$

$$\vdots$$

$$(p_n)\ x_{nm}(t) = x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t)), t \in [nh, T].$$

is convergent and $\lim\limits_{m\to\infty} x_{im} = x_i^*,\ i = \overline{1,n}$.

3. Numerical example

In this section we give an example to test the numerical method presented above. We consider the following functional-differential problem with mixed type argument:

$$x''(t) = 3x(t) + x'(t) - x(t-h) - 2x(t+h) - (2h-2)/3,\ t \in [-7; 7],\ h = 1, \tag{3.1}$$

$$x(t) = (1 - 2t)/3,\ t \in [-1; 1].$$

We divide the working interval $[-1;7]$ by the points $P_n = nh$, $n = \overline{-1,7}$. We develop the solution for the step of time $s = 0.1$ so we obtain $N = 10$ points on each subinterval $I_n = [P_{n-1}, P_n]$. From implicit function theorem, on each I_n, there exists a solution $x_n(t)$ and this solution is approximated by Newton's method.

The algorithm from the section 2 is implemented using Matlab in the following way:
Step 0: We construct the vector t formed by $2N+1$ points of the interval $[-h;h]$ at each step s. Further, we initialize the known solution for this interval with $\varphi(t) = (1-2t)/3$ and its derivative with $\varphi'(t) = \frac{-2}{3}$, $\varphi''(t) = 0$.
Step k: We concatenate to the initial vector t the rest of the points till T, constructing the interval $[nh, T]$, $n = \overline{2,7}$. For this interval we get the solution applying Newton's method. For starting this method, we initialize the value of the first solution with that computed to the last knot at the previous step.
Stoping test: We evaluate the difference in norm between two consecutive computed values $x_n^{(k)}$ and $x_n^{(k+1)}$ and the iterations stop when it is less than a chosen value (in our case 10^{-6}). The last values of the solution are retained in the solution vector and are plotted along to the exact solution of the equation (3.1). These solutions are presented in Fig.1. We can see from Fig.1 that for this example, the equation is (3.1), our algorithm works perfectly. The exact solution $x(t) = (1-2t)/3$ is designed graphically by line and the numerical solution $x = x_n(t)$ by circles. We observe that the numerical solution is overlapping the exact solution.

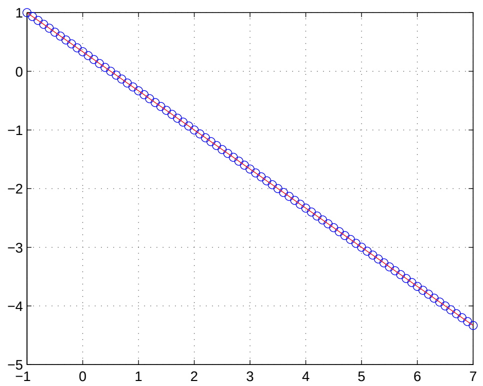

FIGURE 1. Exact and numerical solution for equation (3.1)

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