Fixed Point Theory, 14(2013), No. 2, 401-412
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

POSITIVE SOLUTION FOR SECOND ORDER MULTI-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

YANG LIU*,** AND ZHANG WEIGUO*

*College of Science, University of Shanghai for Science and Technology
Shanghai, 200093, PR China

**Department of Mathematics, Hefei Normal University
Hefei, Anhui Province, 230061, PR China

Abstract. By using Leggett-Williams norm-type theorem due to O'Regan and Zima, we establish the existence of positive solution for a class of second-order m-point boundary value problem at resonance.
Key Words and Phrases: M-point boundary value problem, resonance, positive solution, cone, fixed point.
2010 Mathematics Subject Classification: 34B10, 34B15, 47H10.

1. Introduction

In the paper we study the existence of positive solution for the second-order m-point boundary value problem

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \ t \in (0, 1) \\ x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \end{cases} \tag{1.1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 \le \alpha_i \le 1$, $\beta_i \ge 0$, $i = 1, 2, \cdots, m-2$, $\sum_{i=1}^{m-2} \beta_i \xi_i < 1$, under the resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \neq 1$.

The multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics, for example, in heat flow problems. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Then Gupta [3,4,5] considered three point boundary value problems for nonlinear ordinary differential equations. Since then, by using various methods, such as Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory and different fixed point theorems, the more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [6-13] and references along this line.

For problem (1.1) under the case $f(t,x)=a(t)f(x)$, Ma [14] established the existence results of positive solutions by using Krasnosel'skii-Guo fixed point theorem on cone expansion and compression. When the first order derivative is considered in the nonlinear term, Yang, Liu and Jia [15] obtained the triple positive solutions by using a fixed point theorem due to Avery and Peterson [16]. For problem (1.1) with the one-dimensional p-Laplacian, Su [17] and Wang [18,19], Ma [20], Ji [21] established the existence results of positive solutions. All these results were established under the non-resonant conditions

$$\alpha_i \geq 0,\ \beta_i \geq 0,\ i=1,\ 2,\ \cdots,\ m-2,\ 0<\sum_{i=1}^{m-2}\alpha_i<1,\ 0<\sum_{i=1}^{m-2}\beta_i<1.$$

When the condition $\sum_{i=1}^{m-2}\beta_i=1$ is considered, the problem studied is called boundary value problem at resonance, that is , the associated linear operator $Lx=-x''$ is non-invertible. In the resonance case for $\alpha_i=0,\ i=1,\ 2,\ \cdots,\ m-2$ and $\sum_{i=1}^{m-2}\beta_i=1$, Feng and Webb [22], Ma [23] obtained the existence of solutions by using Mawhin continuous theorem [24]. Liu [25] considered second order m-point boundary value problem

$$\begin{cases} x''(t)=f(t,x(t),x'(t))+e(t),\ t\in(0,1) \\ x'(0)=\sum_{i=1}^{m-2}\alpha_i x'(\xi_i),\ x(1)=\sum_{j=1}^{n-2}\beta_j x(\eta_j). \end{cases}$$

Each of the following resonant conditions

$$(1)\ \sum_{i=1}^{m-2}\alpha_i=\sum_{j=1}^{n-2}\beta_j=1;\ (2)\ \sum_{i=1}^{m-2}\alpha_i=1,\ \sum_{j=1}^{n-2}\beta_j\eta_j=1;(3)\ \sum_{i=1}^{m-2}\beta_i=1,\sum_{j=1}^{n-2}\beta_j\eta_j=1$$

is considered. By using Mawhin continuous theorem, he established the existence results of solution for this resonant problem.

Though a lot of attention has been devoted to the study of positive solution of boundary value problem with non-resonant conditions and solution of boundary value problem with resorant conditions, only few papers deal with positive solution to boundary value problems at resonance. Bai and Fang [26] established the existence of positive solutions of the following second-order differential equation

$$\begin{cases} (p(t)x'(t))'=f(t,x(t),x'(t)),\ t\in(0,1) \\ x'(0)=0,\ x(1)=x(\eta) \end{cases}$$

by using a fixed point index theorem for semi-linear A-proper maps due to Cremins [27]. Infante and Zima [28] obtained the existence of positive solution for problem

$$\begin{cases} x''(t)+f(t,x(t))=0,\ t\in(0,1) \\ x'(0)=0,\ x(1)=\sum_{i=1}^{m-2}\alpha_i x(\eta_i) \end{cases}$$

with resonance condition $\sum_{i=1}^{m-2} \alpha_i = 1$. The result was based on the Leggett-Williams norm-type theorem due to O'Regan and Zima [29].

To the best of our knowledge, positive solution of problem (1.1) with resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$ has not been considered before. The main purpose of this paper is to fill this gap. In this paper we will give sufficient conditions to ensure the existence of positive solution for problem (1.1) with resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$. Our method is based on the Leggett-Williams norm-type theorem.

2. Some background definitions and results

For the convenience of the reader,we present here the necessary definitions and a new fixed point theorem due to O'Regan and Zima. Let X, Y be real Banach spaces. A nonempty convex closed set $C \subset X$ is said to be a cone provided that

(i) $ax \in C$,for all $x \in C, a \geq 0$;

(ii) $x, -x \in C$ implies $x = 0$.

Note that every cone $C \subset X$ induces an ordering in X given by $x \leq y$ if $y - x \in C$.

$L : domL \subset X \to Y$ is called a Fredholm operator with index zero if ImL is closed and dim Ker L=codim ImL$< \infty$. This implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$ and $KerQ = ImL$. Moreover, since dim Im Q=codim Im L, there exists an isomorphism $J : ImQ \to KerL$. Denote by L_P the restriction of L to $KerP \cap domL$ to ImL and its inverse by K_P, so $K_P : ImL \to KerP \cap domL$ and the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma x = x$ for all $x \in C$ and

$$\Psi := P + JQN + K_P(I - Q)N, \ \Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.1 ([29]) *Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that $L : domL \subset X \to Y$ is a Fredholm operator of index zero and*

(C1) $QN : X \to Y$ is continuous and bounded, $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X,

(C2) $Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap domL$ and $\lambda \in (0, 1)$,

(C3) γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,

(C4) $d_B([I - (P + JQN)\gamma]|_{kerL}, KerL \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,

(C5) There exists $u_0 \in C\backslash\{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x\}$ for some $\mu > 0$ and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,

(C6) $(P + JQN)\gamma(\partial\Omega_2) \subset C$,

(C7) $\Psi_\gamma(\overline{\Omega}_2\backslash\Omega_1) \subset C$,
then the equation $Lx = Nx$ *has a solution in the set* $C \cap (\overline{\Omega}_2\backslash\Omega_1)$.

3. Main result

Consider the Banach spaces $X = Y = C[0,1]$ endowed with the norm $\|x\| = \max_{0\le t\le 1} |x(t)|$. Define the linear operator $L : domL \subset X \to Y, Lx = -x''(t),\ t \in [0,1]$, where

$$domL = \{x \in X|x'' \in C[0,1],\ x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i),\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)\}.$$

Define the operator $N : X \to Y$ by

$$(Nx)(t) = f(t, x(t)), t \in [0,1].$$

It is obvious that

$$KerL = \{x \in domL : x(t) \equiv c\ , t \in [0,1]\}.$$

Denote $\xi_0 = 0,\ \xi_{m-1} = 1,\ \alpha_0 = \alpha_{m-1} = \beta_0 = \beta_{m-1} = 0$ and the function $G(s),\ s \in [0,1]$ as follow:

$$G(s) = 1-s+\frac{\sum_{i=0}^{m-1}\beta_i\xi_i - 1}{\sum_{i=0}^{m-1}\alpha_i - 1}\sum_{i=k}^{m-1}\alpha_i - \sum_{i=k}^{m-1}\beta_i(\xi_i - s),\ \xi_{k-1} \le s \le \xi_k,\ k = 1,\ 2,\ \cdots,\ m-1$$

Note that $G(s) \ge 0,\ s \in [0,1]$.
Denote the function $U(t,s)$ as follow:

$$U(t,s) = \begin{cases} \dfrac{s^2}{2} + \dfrac{\sum_{k=0}^{i-1}\beta_k(s-\xi_k)(\frac{1}{2}-t)}{1-\sum_{i=0}^{m-1}\beta_i\xi_i} + \dfrac{3t^2+5}{6\int_0^1 G(s)ds}G(s), 0 \le t \le s \le 1 \\[2ex] \dfrac{s^2}{2} + t - s + \dfrac{\sum_{k=0}^{i-1}\beta_k(s-\xi_k)(\frac{1}{2}-t)}{1-\sum_{i=0}^{m-1}\beta_i\xi_i} + \dfrac{3t^2+5}{6\int_0^1 G(s)ds}G(s),\ 0 \le s \le t \le 1 \end{cases}$$

for $\xi_{i-1} \le s \le \xi_i, i = 1,\ 2,\ \cdots,\ m-1$ and positive number

$$\kappa := min\{1, \min_{s\in[0,1]} \frac{\int_0^1 G(s)ds}{G(s)}, \min_{t,s\in[0,1]} \frac{1}{U(t,s)}\}.$$

Theorem 3.1 *Assume that there exists $R \in (0, \infty)$ such that $f : [0,1] \times [0, R] \to R$ is continuous and*
$(H1)$ $f(t,x) > -\kappa x$, *for all* $(t,x) \in [0,1] \times [0,R]$,
$(H2)$ $f(t,R) < 0$, *for all* $t \in [0,1]$,
$(H3)$ *there exists* $r \in (0,R)$, $t_0 \in [0,1]$, $a \in (0,1]$, $M \in (0,1)$ *and continuous functions* $g : [0,1] \to [0,+\infty)$, $h : (0,r] \to [0,+\infty)$ *such that* $f(t,x) \geq g(t)h(x)$ *for* $[t,x] \in [0,1] \times (0,r]$ *and* $h(x)/x^a$ *is non-increasing on* $(0,r]$ *with*

$$\frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)ds \geq \frac{1-M}{M^a}.$$

Then problem (1.1) with resonant condition $\sum_{i=1}^{m-2}\beta_i = 1$ *has at least one positive solution.*

Proof. Firstly we claim that

$$ImL = \{y \in Y | \int_0^1 G(s)y(s)ds = 0\}.$$

Indeed, for each $y \in \{y \in Y | \int_0^1 G(s)y(s)ds = 0\}$, we take

$$x(t) = -\int_0^t (t-s)y(s)ds + \frac{\sum_{i=0}^{m-1}\int_0^{\xi_i} y(s)ds}{\sum_{i=0}^{m-1}\alpha_i - 1}\ t.$$

It is easy to check that $-x''(t) = y(t)$, $x'(0) = \sum_{i=1}^{m-2}\alpha_i x'(\xi_i)$, $x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i)$, which means $x(t) \in domL$. Thus

$$\{y \in Y | \int_0^1 G(s)y(s)ds = 0\} \subset ImL.$$

On the other hand, for each $y(t) \in ImL$, there exists $x(t) \in domL$,

$$x''(t) = -y(t),\ x'(0) = \sum_{i=1}^{m-2}\alpha_i x'(\xi_i),\ x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i).$$

Integrating both sides on [0,t], we have

$$x(t) = -\int_0^t (t-s)y(s)ds + x'(0)t + x(0).$$

Considering the boundary condition $x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i),\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)$ and condition $\sum_{i=0}^{m-1} \beta_i = 1$, we conclude that

$$\int_0^1 (1-s)y(s)ds - \frac{\sum_{i=0}^{m-1} \beta_i \xi_i - 1}{\sum_{i=0}^{m-1} \alpha_i - 1} \sum_{i=0}^{m-1} \alpha_i \int_0^{\xi_i} y(s)ds - \sum_{i=0}^{m-1} \beta_i \int_0^{\xi_i} (\xi_i - s)y(s)ds = 0,$$

which equivalents to the conclusion that $\int_0^1 G(s)y(s)ds = 0$. So we have

$$ImL \subset \{y \in Y | \int_0^1 G(s)y(s)ds = 0\}.$$

Thus,

$$ImL = \{y \in Y | \int_0^1 G(s)y(s)ds = 0\}.$$

Clearly, dim KerL=1 and ImL is closed. Next we see $Y = Y_1 \bigoplus ImL$,where

$$Y_1 = \{y_1 | y_1 = \frac{1}{\int_0^1 G(s)ds} \int_0^1 G(s)y(s)ds, y \in Y\}.$$

In fact, for each $y(t) \in Y$, we have

$$\int_0^1 G(s)[y(s) - y_1]ds = 0.$$

This shows that $y - y_1 \in ImL$. Since $Y_1 \cap ImL = \{0\}$, we have $Y = Y_1 \bigoplus ImL$. Thus L is a Fredholm operator with index zero.

Then define the projections $P : X \to X, Q : Y \to Y$ by

$$Px = \int_0^1 x(s)ds,$$

$$Qy = \frac{1}{\int_0^1 G(s)ds} \int_0^1 G(s)y(s)ds.$$

Clearly, $ImP = KerL, KerQ = ImL$ and $KerP = \{x \in X : \int_0^1 x(s)ds = 0\}$. Note that for $y \in ImL$, the inverse K_P of L_P is given by

$$(K_P)y = \int_0^t k(t,s)y(s)ds$$

where

$$k(t,s)=\begin{cases} \dfrac{s^2}{2}+\dfrac{\sum_{k=0}^{i-1}\beta_k(s-\xi_k)(\frac{1}{2}-t)}{1-\sum_{i=0}^{m-1}\beta_i\xi_i}, t\le s,\ \xi_{i-1}\le s\le \xi_i \\ \\ \dfrac{s^2}{2}+t-s+\dfrac{\sum_{k=0}^{i-1}\beta_k(s-\xi_k)(\frac{1}{2}-t)}{1-\sum_{i=0}^{m-1}\beta_i\xi_i},\ t\ge s,\ \xi_{i-1}\le s\le \xi_i. \end{cases}$$

Considering that f can be extended continuously on $[0,1]\times(-\infty,+\infty)$, condition (C1) of Lemma 2.1 is fulfilled.
Define the cone of nonnegative functions

$$C=\{x\in X: x(t)\ge 0,\ t\in\ [0,1]\},$$

and

$$\Omega_1=\{x\in X: r>|x|>M\|x\|, t\in[0,1]\},$$
$$\Omega_2=\{x\in X: \|x\|<R\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets, furthermore

$$\overline{\Omega}_1=\{x\in X: r\ge|x|\ge M\|x\|, t\in[0,1]\}\subset\Omega_2, C\cap\overline{\Omega}_2\backslash\Omega_1\ne 0.$$

Let $J=I$ and $(\gamma x)(t)=|x(t)|$ for $x\in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that (C3) of Lemma 2.1 holds.

Next we confirm that (C2) of Lemma 2.1 holds. For this purpose, suppose that there exists $x_0\in C\cap\partial\Omega_2\cap domL$ and $\lambda_0\in(0,1)$ such that $Lx_0=\lambda_0 Nx_0$. Then

$$x_0''(t)+\lambda_0 f(t,x_0)=0$$

for all $t\in(0,1)$. Let $t_1\in[0,1]$ be such that $x_0(t_1)=R$. This gives

$$0\ge x''(t_1)=-\lambda_0 f(t_1,x_0(t_1)),$$

which contradicts to $(H2)$. Thus (C2) holds.

For $x\in KerL\cap\Omega_2$, define

$$H(x,\lambda)=x-\lambda|x|-\frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x|)ds,$$

where $x\in KerL\cap\Omega_2$ and $\lambda\in[0,1]$. Suppose $H(x,\lambda)=0$. In view of $(H1)$ we obtain

$$c=\lambda|c|+\frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|c|)ds$$

$$\geq \lambda|c| - \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)\kappa|c|ds = \lambda|c|(1-\kappa) \geq 0.$$

Hence $H(x,\lambda) = 0$ implies $c \geq 0$. Furthermore, if $H(R,\lambda) = 0$, we get

$$0 \leq R(1-\lambda) = \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,R)ds,$$

contradicting $(H2)$. Thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0,1]$. Therefore

$$d_B(H(x,0), KerL \cap \Omega_2, 0) = d_B(H(x,1), KerL \cap \Omega_2, 0) = d_B(I, KerL \cap \Omega_2, 0) = 1.$$

This ensures

$$d_B([I - (P + JQN)\gamma]|_{KerL}, KerL \cap \Omega_2, 0) = d_B(H(x,1), KerL \cap \Omega_2, 0) \neq 0.$$

Let $x \in \overline{\Omega}_2 \backslash \Omega_1$ and $t \in [0,1]$. From condition $H1$, we see

$$(\Psi_\gamma x)(t) = \int_0^1 |x(t)|dt + \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|)ds$$

$$+ \int_0^1 k(t,s)[f(s,|x(s)|) - \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(\tau)f(\tau,|x(\tau)|)d\tau]ds$$

$$= \int_0^1 |x(t)|dt + \int_0^1 U(t,s)f(s,|x(s)|)ds \geq \int_0^1 |x(s)|ds - \kappa\int_0^1 U(t,s)|x(s)|ds$$

$$= \int_0^1 (1 - \kappa U(t,s))|x(s)|ds \geq 0.$$

Hence $\Psi_\gamma(\overline{\Omega}_2)\backslash\Omega_1 \subset C$. Moreover, since for $x \in \partial\Omega_2$, we have

$$(P + JQN)\gamma x = \int_0^1 |x(s)|ds + \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|)ds$$

$$\geq \int_0^1 (1 - \frac{\kappa}{\int_0^1 G(s)ds}G(s))|x(s)|ds$$

$$\geq 0,$$

which means $(P + JQN)\gamma(\partial\Omega_2) \subset C$. This ensures that conditions (C6), (C7) of Lemma 2.1 hold.

At last, we confirm that (C5) is satisfied. Taking $u_0(t) \equiv 1$ on [0,1], we see

$$u_0 \in C\backslash\{0\}, C(u_0) = \{x \in C | x(t) > 0 \ on [0,1]\}$$

and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$, we have

$$x(t) > 0, \ t \in [0,1], \ 0 < \|x\| \leq r \ and \ x(t) \geq M\|x\| \ on \ [0,1].$$

Therefore, in view of (H3), we obtain for all $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^1 x(s)ds + \int_0^1 U(t_0,s)f(s,x(s))ds \\ &\geq M\|x\| + \int_0^1 U(t_0,s)g(s)h(x(s))ds \\ &= M\|x\| + \int_0^1 U(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)ds \\ &\geq M\|x\| + \frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)M^a\|x\|^a ds \\ &\geq M\|x\| + (1-M)\|x\| = \|x\|.\end{aligned}$$

So $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, which means that condition (C5) of Lemma 2.1 holds.

Thus by Lemma 2.1, we confirm that the equation $Lx = Nx$ has a solution x, which implies that problem (1.1) with resonance condition $\sum_{i=1}^{m-2} \beta_i = 1$ has at least one positive solution.

4. Example

In this section we give an example to illustrate the main results of the paper. Consider the multi-point boundary value problem

$$\begin{cases} x''(t) + \left(-\frac{1}{3}t^2 + \frac{1}{3}t + \frac{1}{3}\right)(x^2 - 4x + 3)\sqrt{x^2 - 6x + 10} = 0,\ t \in (0,1) \\ (5.1) \\ x'(0) = \frac{1}{4}x'\left(\frac{1}{4}\right) + \frac{1}{4}x'\left(\frac{1}{2}\right),\ x(1) = \frac{1}{3}x\left(\frac{1}{4}\right) + \frac{2}{3}x\left(\frac{1}{2}\right) \end{cases}$$

where $\alpha_1 = \alpha_2 = \frac{1}{4},\ \beta_1 = \frac{1}{3},\ \beta_2 = \frac{2}{3},\ \xi_1 = \frac{1}{4},\ \xi_2 = \frac{1}{2}$ and

$$G(s) = \begin{cases} \frac{7}{6},\ 0 \leq s < \frac{1}{4} \\ \frac{23}{24} - \frac{1}{3}s,\ \frac{1}{4} \leq s < \frac{1}{2} \\ 1 - s,\ \frac{1}{2} \leq s \leq 1 \end{cases}$$

By a simple computation, we have

$$\int_0^1 G(s)ds = \frac{5}{8},\ \kappa = \frac{15}{28},\ \int_0^1 U(0,s)ds = 1.$$

We take $R = \frac{6}{5},\ r = \frac{1}{2},\ t_0 = 0,\ a = 1,\ M = \frac{1}{2}$ and

$$g(t) = -\frac{1}{3}t^2 + \frac{1}{3}t + \frac{1}{3},\ h(x) = \sqrt{x^2 - 6x + 10}\ .$$

It's easy to check that

$$\frac{1}{3} \le g(t) \le \frac{5}{12} < \frac{15}{28},\ t \in [0,1],\ x^2 - 4x + 3 \ge -x,\ x \in [0, \frac{6}{5}].$$

We see that

(1) $f(t,x) > -\frac{15}{28}x$, for all $(t,x) \in [0,1] \times [0,R]$,

(2) $f(t,R) < 0$, for all $t \in [0,1]$,

(3) $f(t,x) \ge g(t)h(x)$ for all $[t,x] \in [0,1] \times \left(0, \frac{1}{2}\right]$ and $\frac{h(x)}{x} = \frac{\sqrt{x^2 - 6x + 10}}{x}$ is non-increasing on $\left(0, \frac{1}{2}\right]$ with

$$\frac{h(r)}{r^a} \int_0^1 U(0,s)g(s)ds \ge \frac{\sqrt{29}}{3} \int_0^1 U(0,s)ds = \frac{\sqrt{29}}{3} \ge 1 = \frac{1-M}{M^a}.$$

Thus all the conditions of Theorem 3.1 are satisfied. This ensures that resonance problem (5.1) has at least one solution, positive on [0,1].

Remark 4.1. To our best knowledge, the early methods for positive solutions about second-order m-point boundary value problems such as in [14, 15] and [17, 18, 19, 20, 21] for $p = 2$ are not applicable to this problem. We generalize the results of positive solutions for problem (1.1).

Acknowledgement. The work is sponsored by the Anhui Provincial Natural Science Foundation (10040606Q50).

References

[1] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differ. Eq., **23**(1987), 803-810.

[2] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differential Eq., **23**(1987), 979-987.

[3] C.P. Gupta, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differ-ential equation*, J. Math. Anal. Appl., **168**(1992), 540-551.

[4] C.P. Gupta, *A sharper condition for the Solvability of a three-point second order boundary value problem*, J. Math. Anal. Appl., **205**(1997), 586-579.

[5] C.P. Gupta, *A generalized multi-point boundary value problem for second order ordinary differential equation*, Appl. Math. Comput., **89**(1998), 133-146.

[6] R. Ma, *Positive solutions of a nonlinear three-point boundary valve problem*, Electron. J. Diff. Eqs., **34**(1999), 1-8.

[7] L. Yang et.al., *Multiplicity results for second-order m-point boundary value problem*, J. Math. Anal. Appl., **324**(2006), 532-542.

[8] R.I. Avery et. al., *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations*, Comput. Math. Appl., **42**(2001), 695-704.

[9] X. He and W. Ge, *Triple solutions for second-order three-point boundary value problems*, J. Math. Anal. Appl., **268**(2002), 256-265.

[10] X. Xian, D. O'Regan, *Multiplicity of sign-changing solutions for some four-point boundary value problem*, Nonlinear Anal., **69**(2008), 434-447.

[11] R.P. Agarwal, D. O'Regan, *Some new existence results for differential and integral equations*, Nonlinear Anal., **29**(1997), 679-692.

[12] J. Henderson, *Double solutions of three-point boundary-value problems for second-order equations*, Electron. J. Diff. Eqs., **115**(2004), 1-7.

[13] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Boston, 1999.
[14] R. Ma, *Existence of Solutions of nonlinear m-point boundary-value problems*, J. Math. Anal. Appl., **256**(2001), 556-567.
[15] L. Yang, X. Liu, M. Jia, *Multiplicity results for second order m-point boundary value problem*, J. Math. Anal. Appl., **324**(2006), 532-542.
[16] R.I. Avery, A.C. Peterson, *Three positive fixed points of nonlinear operators on ordered Banach spaces*, Comput. Math. Appl., **42**(2001), 313-322.
[17] B. Su et. al., *Three positive solutions for multi-point one-dimensional p-Laplacian boundary value problems with dependence on the first order derivative*, Math. Comput. Model., **45**(2007), 1170-1178.
[18] Y. Wang et. al., *Multiple positive solutions for multi-point boundary value problems with one-dimensional p-Laplacian*, J. Math. Anal. Appl., **327**(2007), 1381-1395.
[19] Y. Wang et. al., *Existence of multiple positive solutions for one-dimensional p-Laplacian*, J. Math. Anal. Appl., **315**(2006), 144-153.
[20] D. Ma et. al., *Existence and iteration of monotone positive solutions for multi-point boundary value problems with p-Laplacian operators*, Comput. Math. Appl., **50**(2005), 729-739.
[21] D. Ji et. al., *Positive solutions for one-dimensional p-Laplacian boundary value problems with sign changing nonlinearity*, Nonlinear Anal., (Preprint).
[22] W. Feng and J.R. Webb, *Solvability of three point boundary value problems at resonance*, Nonlinear Anal., **30**(1997), 3227-3238.
[23] R. Ma, *Existence results of a m-point boundary value problem at resonance*, J. Math. Anal. Appl., **294**(2004), 147-157.
[24] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, in: NSFCBMS Regional Conference Series in Mathematics, Amer. Math. Soc., Providence, RI, 1979.
[25] B. Liu, *Solvability of multi-point boundary value problem at resonance (IV)*, Appl. Math. Comput., **143**(2003), 275-299.
[26] C. Bai, J. Fang, *Existence of positive solutions for three-point boundary value problems at resonance*, J. Math. Anal. Appl., **291**(2004), 538-549.
[27] C.T. Cremins, *A fixed-point index and existence theorems for semi-linear equations in cones*, Nonlinear Anal., **46**(2001), 789-806.
[28] G. Infante, M. Zima, *Positive solutions of multi-point boundary value problems at resonance*, Nonlinear Anal., **69**(2008), 2458-2465.
[29] D. O'Regan, M. Zima, *Leggett-Williams norm-type theorems for coincidences*, Arch. Math., **87**(2006), 233-244.

Received: June 9, 2011; Accepted: December 12, 2011.