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FIXED POINT THEOREMS FOR CYCLIC φ-CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to present some fixed point results for cyclic φ-contractions in a complete metric space endowed with a partial order.
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1. Introduction

Motivated by [1], I.A. Rus in [2] introduced the following definition.

Definition 1.1. Let X be a nonempty set, m a positive integer and $T : X \longrightarrow X$ a mapping. $X = \bigcup_{i=1}^{m} X_i$ is said to be a cyclic representation of X with respect to T if

(i) X_i, $i = 1, \ldots, m$ are nonempty sets.
(ii) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Recently, in [3] the authors prove some results about fixed point theorems for operators T defined on a metric space X with a cyclic representation of X with respect to T.
Now, we recollect the main result of [3].

Definition 1.2. A function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be a comparison function if φ satisfies:

(a) φ is increasing.
(b) $(\varphi^n(t))$ converges to 0 as $n \to \infty$ for every $t \in \mathbb{R}_+$.

If we replace (b) by

(b') $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}_+$,

then φ is said to be a (c)-comparison function.

Obviously, any φ (c)-comparison function is a comparison function.

The converse is false. The function given by $\varphi(t) = \dfrac{t}{1+t}$, $t \in \mathbb{R}_+$ is a comparison function which is not a (c)-comparison function (this example appears in [4]).

Using (b) of Definition 1.2 it is easily seen that if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$ for any $t > 0$ and $\varphi(0) = 0$.

Definition 1.3. Let (X, d) be a metric space, m a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of X, $X = \bigcup_{i=1}^{m} A_i$ and $T : X \longrightarrow X$ an operator.
T is a cyclic φ-contraction if

(a) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.

(b) There exists a comparison function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$ with $i = 1, \ldots, m$, where $A_{m+1} = A_1$.

Now, we present the main result of [3].

Theorem 1.4. *Let (X, d) be a complete metric space, m a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X, $X = \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a (c)-comparison function and $T : X \longrightarrow X$ an operator. If*

(i) *$X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.*

(ii) *T is a cyclic φ-contraction.*

Then T has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$. Moreover, T is a Picard operator (this means that for any starting point $x_0 \in X$, the Picard iteration $(T^n x_0)$ converges to x^*).

The purpose of this paper is to present a version of Theorem 1.4 in the context of ordered metric spaces.
The existence of fixed point in ordered metric spaces has been considered recently in some papers (see $[5 - 14]$, among others).

In the context of ordered metric spaces, the usual contraction is weakened but at the expense that the operator is monotone.

The main tool in the fixed point theorems in ordered metric spaces combines the ideas in the contraction principle with those in the monotone iterative technique [15].

2. Fixed point results: increasing case

We start this section with the following definition.

Definition 2.1. Let $(X, \leq)$ be a partially ordered set and $T : X \longrightarrow X$ a mapping. We say that T is increasing if, for $x, y \in X$,

$$x \leq y \Rightarrow Tx \leq Ty.$$

In what follows we present the following result which is a version of Theorem 1.4 in the context of ordered metric spaces when the operator is monotone.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.*

Suppose that m is a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of X, $X = \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a (c)-comparison function and $T : X \longrightarrow X$ a increasing and continuous mapping such that

(i) *$\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.*
(ii) *$d(Tx, Ty) \leq \varphi(d(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$ with x and y comparable $(i = 1, 2, \ldots, m)$, where $A_{m+1} = A_1$.*
(iii) *There exists $x_0 \in X$ with $x_0 \leq Tx_0$.*

Then T has at least a fixed point.

Proof. If $Tx_0 = x_0$ then the proof is finished.
Suppose that $x_0 < Tx_0$.
Since T is a increasing mapping, we obtain

$$x_0 \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \ldots \leq T^n x_0 \leq T^{n+1}x_0 \leq \ldots$$

Put $x_{n+1} = Tx_n$. Then, by (i) for any $n \geq 1$ there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$ and, as x_{n-1} and x_n are comparable, by (ii) we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \varphi(d(x_n, x_{n-1})).$$

Since φ is increasing, we get by induction that

$$d(x_{n+1}, x_n) \leq \varphi^n(d(x_1, x_0)). \tag{2.1}$$

Thus, for every $n \in \mathbb{N}$ and $p \geq 1$, by (2.1) we have that

$$\begin{aligned} d(x_{n+p}, x_n) \leq & \quad d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \ldots + d(x_{n+1}, x_n) \\ \leq & \quad \varphi^{n+p-1}(d(x_1, x_0)) + \varphi^{n+p-2}(d(x_1, x_0)) + \ldots + \varphi^n(d(x_1, x_0) > 0. \end{aligned} \tag{2.2}$$

Since φ is a (c)-comparison function and $d(x_1, x_0) = d(Tx_0, x_0) > 0$, $\sum_{k=0}^{\infty} \varphi^k(d(x_1, x_0))$ converges and using Cauchy's criterium for convergent series, from (2.2) we obtain

$$\lim_{n \to \infty} d(x_{n+p}, x_n) = 0.$$

This shows that (x_n) is a Cauchy sequence in X.

Since X is a complete metric space, there exists $z \in X$ such that $\lim_{n \to \infty} x_n = z$.

Finally, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$$

and this finishes the proof. □

Remark 2.3. *Notice that, as* $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* T*, the fixed point* z *of the operator* T *in Theorem 2.2 satisfies that* $z \in \bigcap_{i=1}^{m} A_i$.

Remark 2.4. *Notice that in Theorem 2.2 we do not assume that the subsets* A_i *are closed.*

In what follows we prove that Theorem 2.2 is true for T not necessarily continuous if we assume the following condition in X (which appears in Theorem 1.4 of [6]):

$$\text{if } (x_n) \text{ is a increasing sequence in } X \text{ such that } x_n \to x \text{ then } x_n \leq x, \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Theorem 2.5. *If in Theorem 2.2 we substitute the continuity of* T *by condition (2.3) we obtain the same conclusion.*

Proof. Following the proof of Theorem 2.2 we only have to check that $Tz = z$.

As $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, the sequence (x_n) appearing in Theorem 2.2 has a infinite number of terms in each A_i, for $i = 1, \ldots, m$.

Suppose that $z \in A_{i_n}$ for certain $i_n \in \{1, \ldots, m\}$.

We take a subsequence (x_{n_k}) of (x_n) in $A_{i_n - 1}$ (where $A_0 = A_m$) converging to z.

As (x_{n_k}) is a increasing sequence in X with $x_{n_k} \to z$, condition (2.3) gives us that $x_{n_k} \leq z$ for all $k \in \mathbb{N}$.

Using the contractive condition ((ii) of Theorem 2.2) and the fact that $\varphi(t) < t$, for $t > 0$, we get

$$d(x_{n_k+1}, T(z)) = d(Tx_{n_k}, Tz) \leq \varphi(d(x_{n_k}, z)) < d(x_{n_k}, z).$$

Letting $k \to \infty$ in the last inequality we obtain

$$d(z, Tz) \leq d(z, z) = 0$$

and this proves that z is a fixed point of T.

This finishes the proof. □

Now, we present an example where it can be appreciated that assumptions in Theorem 2.2 do not guarantee uniqueness of the fixed point.

Example 2.6. Consider $(\mathbb{R}^2, d_2)$, where d_2 is the Euclidean distance and with the order given by $R = \{(x, x) : x \in \mathbb{R}^2\}$.

Let A_i $(i = 1, 2)$ be the closed subsets of $\mathbb{R}^2$ given by

$$A_1 = \{(x, y) : x \geq 0\} \qquad \text{and} \qquad A_2 = \{(x, y) : x \leq 0\}.$$

Obviously, $\mathbb{R}^2 = A_1 \cup A_2$.

We consider the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (-x, y)$.

Obviously, $A_1 \cup A_2$ is a cyclic representation of $\mathbb{R}^2$ with respect to T and with $m = 2$.

Moreover, T is a increasing and continuous operator since elements in X are only comparable to themselves.

This fact says us that T satisfies (ii) of Theorem 2.2 for any (c)-comparison function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.
As $(0, 0) \leq T(0, 0) = (0, 0)$, assumptions of Theorem 2.2 are satisfied and the operator T has as fixed points the set $\{(0, y) : y \in \mathbb{R}\}$.

In what follows we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.2 and 3.

This condition says:

$$\text{for } x, y \in \bigcap_{i=1}^{m} A_i \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.4}$$

Theorem 2.7. *Adding condition (2.4) to the assumptions of Theorem 2.2 (or Theorem 2.5) we obtain uniqueness of the fixed point.*

Proof. Suppose that $z, y \in X$ are two fixed points of T. By Remark 2.3, $z, y \in \bigcap_{i=1}^{m} A_i$.

We distinguish two cases.

Case 1. Suppose that z is comparable to y.

By (ii) of Theorem 2.2 we have

$$d(z, y) = d(Tz, Ty) \leq \varphi(d(z, y)) < d(z, y)$$

which is impossible. Therefore, $z = y$.

Case 2. Suppose that z is not comparable to y.

By condition (4), there exists $x \in X$ comparable to z and y.

The increasing character of T implies that $T^n x$ is comparable to $T^n z = z$ for $n = 0, 1, 2, \ldots$.

As $z \in \bigcap_{i=1}^{m} A_i$, by using (ii) of Theorem 2.2, we have

$$d(z, T^n x) = d(T^n z, T^n x) \leq \varphi(d(T^{n-1} z, T^{n-1} x)) \leq \varphi(d(z, T^{n-1} x)).$$

Taking into account the monotonicity of φ and using mathematical induction, we get

$$d(z, T^n x) \leq \varphi^n(d(z, x)).$$

Since φ is a (c)-comparison function

$$\lim_{n\to\infty} d(z, T^n x) \leq \lim_{n\to\infty} \varphi^n(d(z, x)) = 0.$$

Consequently, $\lim_{n\to\infty} T^n x = z$.

Using a similar argument we can prove that $\lim_{n\to\infty} T^n x = y$.

Finally, the uniqueness of the limit gives us $y = z$.

This finishes the proof. $\square$

3. Fixed point results: nonincreasing case

In this section we present a parallel study for nonincreasing functions.

We start with the following definition.

Definition 3.1. Let $(X,\leq)$ be a partially ordered set and $T : X \longrightarrow X$ an operator. We say that T is nonincreasing if for $x, y \in X$,

$$x \leq y \Rightarrow Tx \geq Ty.$$

Theorem 3.2. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space.*

Suppose that m is a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of X, $X = \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a (c)-comparison function and $T : X \longrightarrow X$ a nonincreasing operator such that

- (i) *$\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.*
- (ii) *$d(Tx,Ty) \leq \varphi(d(x,y))$ for any $x \in A_i$, $y \in A_{i+1}$ with x and y comparable $(i = 1,2,\ldots,m)$, where $A_{m+1} = A_1$.*
- (iii) *There exists x_0 with x_0 comparable to Tx_0.*

Suppose also that either T is continuous or

X is such that if $(x_n) \subset X$ with $x_n \to x$ then there exists a subsequence (x_{n_k}) of (x_n) such that every term of the subsequence (x_{n_k}) is comparable to the limit x. (3.1)

Then T has at least a fixed point.

Proof. If $Tx_0 = x_0$ then the proof is finished.

Suppose that $x_0 \neq Tx_0$.

Put $x_{n+1} = Tx_n$.

As T is nonincreasing, for any $n \geq 1$, x_{n-1} and x_n are comparable and, using a similar argument that in the proof of Theorem 2.2, we can prove that (x_n) is a Cauchy sequence and, therefore, (x_n) is convergent to certain $z \in X$.

In the case that T is continuous it is easily seen that z is a fixed point.

Suppose that condition (3.1) holds.

As $z \in X = \bigcup_{i=1}^{m} A_i$, there exists $i_n \in \{1,2,\ldots,m\}$ such that $z \in A_{i_n}$.

On the other hand, as $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, the sequence (x_n) has infinite terms in each A_i for $i = 1,2,\ldots,m$.

We take a subsequence (x_{n_k}) of (x_n) in A_{i_n-1} (where $A_0 = A_m$).

As $(x_{n_k}) \to x$ by condition (3.1), we can find a subsequence of (x_{n_k}) which we will follow denoting by (x_{n_k}) whose terms are comparable to z.

Now, using the same reasoning that in the proof of Theorem 2.5, we prove that z is a fixed point of T. $\square$

Remark 3.3. *Example 2.6 proves that assumptions on Theorem 3.2 do not guarantee uniqueness of the fixed point.*

Theorem 3.4. *Adding condition (2.4) to the assumptions of Theorem 3.2 we obtain uniqueness of the fixed point.*

Proof. The proof is similar to Theorem 2.7 and we omit it. □

4. Fixed point results: compact case

In this section, we prove that if X is compact, the conclusions of Theorems 2.2 and 2.5 are true under the weaker assumption that φ is a comparison function.

Theorem 4.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.*

Moreover, suppose that m is a positive integer, $A_1, A_2, \dots, A_m$ nonempty subsets of X, $X = \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a comparison function and $T : X \longrightarrow X$ maps comparable elements into comparable elements such that

(*i*) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of X with respect to T.*

(*ii*) $d(Tx, Ty) \leq \varphi(d(x, y))$ *for any $x \in A_i$, $y \in A_{i+1}$ $(i = 1, 2, \dots, m)$, with x and y comparable, where $A_{m+1} = A_1$.*

If there exists $x_0 \in X$ with x_0 comparable to Tx_0 then $\inf\{d(x, Tx) : x \in X\} = 0$.

If, in addition $(X, \leq)$ satisfies (2.4), X is compact and T is continuous then T has a unique fixed point.

Proof. If $Tx_0 = x_0$ then it is obvious that $\inf\{d(x, Tx) : x \in X\} = 0$.

Suppose that $x_0 < Tx_0$ (the same argument serves for $Tx_0 < x_0$).

Since that T applies comparable elements into comparable elements, the consecutive terms of the sequence $(T^n x_0)$ are comparable and by assumptions (*i*) and (*ii*) we can obtain

$$d(T^{n+1}x_0, T^n x_0) \leq \varphi(d(T^n x_0, T^{n-1}x_0)).$$

Using the monotonicity of φ, the fact that $\varphi < t$ for $t > 0$ and mathematical induction, we have

$$d(T^{n+1}x_0, T^n x_0) \leq \varphi^n(d(Tx_0, x_0)).$$

Now, using (*b*) of the definition of comparison function

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = 0.$$

Therefore, $\inf\{d(x, Tx) : x \in X\} = 0$.

This finishes the first part of the proof.

Now, suppose that X is compact and T is continuous.

As the mapping

$$X \longrightarrow \mathbb{R}^+$$
$$x \mapsto d(x, Tx)$$

is obviously continuous and since X is compact we can find $z \in X$ such that

$$d(z, Tz) = \inf\{d(x, Tx) : x \in X\}.$$

Finally, the first part of the theorem gives us

$$d(z, Tz) = 0$$

and, consequently, z is a fixed point of T.

The uniqueness of the fixed point is proved as in Theorem 2.7. $\square$

5. Examples

We begin this section with some examples which prove that if at least one of the assumptions of Theorem 2.2 is not satisfied then the conclusion of this theorem is false.

Example 5.1. We consider $\mathbb{N}^*$, the set of the natural number without zero, with the usual distance given by $d(x, y) = |x - y|$ and the usual order. Obviously, $(\mathbb{N}^*, d)$ is a complete metric space.

We consider the closed subsets of $\mathbb{N}^*$ defined by

$$A_1 = \{n \in \mathbb{N}^* : n \text{ even } \},$$

$$A_2 = \{n \in \mathbb{N}^* : n \text{ odd } \}.$$

Obviously, $\mathbb{N}^* = A_1 \cup A_2$.

Let $T : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ be the operator defined by $T(n) = n + 1$. It is easily seen that T is continuous and increasing and that $\mathbb{N}^* = A_1 \cup A_2$ is a cyclic representation of $\mathbb{N}^*$ with respect to T.

Notice that for any $n \in \mathbb{N}^*$, $n \leq T(n) = n + 1$.

On the other hand, for $p \in A_1$ and $q \in A_2$ with $p < q$ we have

$$d(Tp, Tq) = |p - q|,$$

and for any $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, φ a (c)-comparison function

$$\varphi(d(p, q)) = \varphi(|p - q|) < |p - q|.$$

Thus, assumption (ii) of Theorem 2.2 is not satisfied.

In this case, it is obvious that T has not fixed point.

Example 5.2. Consider the same set $(\mathbb{N}^*, d)$ that in Example 5.1 and the same operator $T : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ given by $T(n) = n + 1$ and the same subsets A_i $(i = 1, 2)$.

But now we consider in $\mathbb{N}^*$ the order given by $R = \{(n, n) : n \in \mathbb{N}^*\}$.

In this case, assumption (i) of Theorem 2.2 is obviously satisfied. Moreover, since elements in $\mathbb{N}^*$ are only comparable to themselves, assumption (ii) of Theorem 2.2 is also satisfied.

On the other hand, assumption (iii) of Theorem 2.2 fails since for any $n \in \mathbb{N}^*$, n and $T(n)$ are not comparable.

Obviously, T has not fixed point.

Now, we present an example which can be treated by Theorem 2.2 and it does not satisfy assumptions of Theorem 1.4.

Example 5.3. We consider $\mathbb{N}^*$, the set of the natural numbers without zero, with the usual distance given by $d(x,y) = |x-y|$ and the usual order. Obviously, $(\mathbb{N}^*, d)$ is a complete metric space.

We consider the closed subsets of $\mathbb{N}^*$ defined by

$$A_1 = \{n \in \mathbb{N}^* : n \text{ even } \}$$
$$A_2 = \{n \in \mathbb{N}^* : n \text{ odd } \}$$

Obviously, $\mathbb{N}^* = A_1 \cup A_2$.

Let $T : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ be the operator defined by $T(n) = n+1$. It is easily seen that T is continuous and increasing and that $\mathbb{N}^* = A_1 \cup A_2$ is a cyclic representation of $\mathbb{N}^*$ with respect to T.

Notice that for any $n \in \mathbb{N}^*$, $n \leq T(n) = n+1$.

On the other hand, for $p \in A_1$ and $q \in A_2$ with $p < q$ we have

$$d(Tp, Tq) = |p-q|,$$

and for any $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, φ a (c)-comparison function

$$\varphi(d(p,q)) = \varphi(|p-q|) < |p-q|.$$

Thus, assumption (ii) of Theorem 2.2 is not satisfied.

In this case, it is obvious that T has not fixed point.

Example 5.4. Consider the same set $(\mathbb{N}^*, d)$ that in Example 5.1 and the same operator

$T : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ given by $T(n) = n+1$, and the same subsets $A_i (i = 1,2)$.

But now we consider in $\mathbb{N}^*$ the order given by $R = \{(n,n) : n \in \mathbb{N}^*\}$.

In this case, assumption (i) of Theorem 2.2 is obviously satisfied. Moreover, since elements in $\mathbb{N}^*$ are only comparable to themselves, assumption (ii) of Theorem 2.2 is also satisfied.

On the other hand, assumption (iii) of Theorem 2.2 fails since for any $n \in \mathbb{N}^*$, n and $T(n)$ are not comparable.

Obviously, T has not fixed point.

Now, we present an example which can be treated by Theorem 2.2 and it does not satisfy assumptions of Theorem 1.4.

Example 5.5. Let $X = \{(0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$ with the Euclidean distance d_2. (X, d_2) is, obviously, a complete metric space.

We consider in X the order $\leq$ given by $R = \{(x,x) : x \in X\}$.

Notice that the elements in X are only comparable to themselves. Let $T : X \longrightarrow X$ be the operator given by $T(1,0) = (0,1)$; $T(0,1) = (1,0)$, $T(1,1) = (1,1)$.

If we take $A_1 = \{(0,1), (1,1)\}$ and $A_2 = \{(1,0), (1,1)\}$ then it is easily proved that $A_1 \cup A_2$ is a cyclic representation of X with respect to T (in this case $m = 2$).

The condition (ii) of Theorem 2.2 is obviously satisfied since the elements in X are only comparable to themselves.

Moreover, as $(1,1) \leq T(1,1)$, Theorem 2.2 gives us the existence of a fixed point for T (which it is obviously the point $(1,1)$).

On the other hand,

$$\sqrt{2} = d_2(T(1,0), T(0,1)) = d_2((0,1),(1,0))$$

and if $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a (c)-comparison function

$$\varphi(d((1,0),(0,1))) = \varphi(\sqrt{2}) < \sqrt{2}.$$

Consequently, T is not a cyclic φ-contraction for any φ (c)-comparison function and, consequently, this example cannot be treated by Theorem 1.4.

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