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APPROXIMATE JORDAN DERIVATIONS ON HILBERT C^*- MODULES

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Abstract. In this paper we prove the generalized Hyers–Ulam–Rassias stability of Jordan derivations on Hilbert C^*- modules associated with the following generalized Jensen type functional equation

$$2\sum_{j=1}^{n} f\left(\frac{x_j}{2}+\sum_{i=1,i\neq j}^{n} x_i\right)+\sum_{i=1}^{n} f(x_j)=2n\,f\Big(\sum_{i=1}^{n} x_i\Big).$$

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [54] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [19]. It gave rise the stability theory for functional equations. The result of Hyers was extended by Aoki [2] in 1950, by considering the unbounded Cauchy differences. In 1978, Th.M. Rassias [50] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$ the mapping $f(tx)$ is continous in $t \in \mathbb{R}$. Găvruta [16] generalized the Rassias' result. Following the techniques of the proof of the corollary of Hyers [19] we observed that Hyers introduced (in 1941) the following Hyers continuity condition: about the continuity of the mapping for each fixed, and then he proved homogenouity of degree one and therefore the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers–Ulam stability problem forms (see [27]). Beginning around the year 1980. The stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4], [7], [11], [12], [13], [15], [17], [23], [25], [26], [28], [29], [31], [32]–[37], [42], [49], [51], [52]).

J.M. Rassias [45] following the spirit of the innovative approach of Hyers [19], Aoki [2] and Th.M. Rassias [50] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [44], [46] for a number of other new results).

In 2003 Cădariu and Radu applied the *fixed point method* to the investigation of the Jensen functional equation [5] (see also [6], [7], [14], [21], [39], [43], [47], [48]). They could present a short and a simple proof (different of the "*direct method*", initiated by Hyers in 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [5], for Cauchy functional equation [7] and for quadratic functional equation [6].

The following functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \tag{1.1}$$

is called a *quadratic functional equation,* and every solution of equation (1.1) is said to be a *quadratic mapping.* F. Skof [53] proved the Hyers–Ulam stability of the quadratic functional equation (1.1) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [8], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation (1.1). C. Borelli and G.L. Forti [3] generalized the stability result of the quadratic functional equation (1.1). Jun and Lee [20] proved the Hyers–Ulam stability of the Pexiderized quadratic equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y)$$

for mappings f, g, h and k. The stability problem of the quadratic equation has been extensively investigated by some mathematicians [24].

Recently P. Găvruta and L. Găvruta used a new method for investigation of Hyers–Ulam–Rassias stability of a nonlinear functional equation, Volterra integral operator and Fredholm operator. This method generalized the *fixed point method* [17].

Hilbert C^* - modules provide a natural generalization of Hilbert spaces arising when the field of scalars $\mathbb{C}$ is replaced by an arbitrary C^*-algebra. This generalization, was introduced by I. Kaplansky in [22] (see also [18]).

Definition 1.1. A *pre-Hilbert A - module* is a (right) $A-$module $\mathcal{M}$ equipped with a sesquilinear form $\langle .,. \rangle : \mathcal{M} \times \mathcal{M} \to A$ with the following properties:

(1) $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$,
(2) $\langle x, x \rangle = 0$ implies that $x = 0$,
(3) $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$,
(4) $\langle x, ya \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$ and any $a \in A$.

The mapping $\langle .,. \rangle$ is called an *$A-$valued inner product.*

Definition 1.2. A pre-Hilbert $A-$module $\mathcal{M}$ is called a Hilbert C^*-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

Definition 1.3. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a derivation on the Hilbert C^*-module $\mathcal{M}$ if it satisfies the condition $d(\langle x, y \rangle z) = \langle d(x), y \rangle z + \langle x, d(y) \rangle z + \langle x, y \rangle d(z)$ for every $x, y, z \in \mathcal{M}$.

Definition 1.4. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a Jordan derivation on the Hilbert C^*-module $\mathcal{M}$ if it satisfies the condition $d(\langle x, x \rangle y) = \langle d(x), x \rangle y + \langle x, d(x) \rangle y + \langle x, x \rangle d(y)$ for every $x, y \in \mathcal{M}$.

In this paper, for a fixed positive integer $n \geq 2$, we introduce a new functional equation, which is called an *generalized Jensen type functional equation* and whose solution is said to be an *generalized Jensen type mapping,*

$$2\sum_{j=1}^{n} f(\frac{x_j}{2} + \sum_{i=1, i\neq j}^{n} x_i) + \sum_{i=1}^{n} f(x_i) = 2nf(\sum_{i=1}^{n} x_i). \tag{1.2}$$

We investigate the Hyers–Ulam–Rassias stability of Jordan derivations on Hilbert C^*-modules.

Throughout this paper assume that $n_0 \in \mathbb{N}$ is a positive integer. Suppose that $\mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$ and that $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta};\ 0 \leq \theta \leq \frac{2\pi}{n_o}\}$. We have $\mathbb{T}^1 = \mathbb{T}^1_{\frac{1}{1}}$. Moreover, in this paper, $\mathcal{M}$ denotes a Hilbert C^*- module with norm $\|.\|$.

Let X and Y be vector spaces. For a given mapping $f : X \to Y$ and for a fixed positive integer $n \geq 2$, we define

$$D_\mu f(x_1, \cdots, x_n) := 2\sum_{j=1}^{n} f(\frac{\mu x_j}{2} + \sum_{i=1, i\neq j}^{n} \mu x_i) + \sum_{i=1}^{n} \mu f(x_i) - 2nf(\sum_{i=1}^{n} \mu x_i).$$

for all $\mu \in T^1_{\frac{1}{n_o}}$ and all $x_1, \cdots, x_n \in X$.

2. Stability of Jordan derivations on Hilbert C^*- modules: direct method

In this section, using an idea of Gavruta [16], we used direct method to prove the stability of Jordan derivations on Hilbert C^*- modules in the spirit of Hyers, Ulam and Rassias.

Lemma 2.1. *Let X and Y be real vector spaces. A mapping $f : X \to Y$ satisfies (1.2) for all $x_1, \cdots, x_n$ if and only if the mapping f is additive.*

Proof. Putting $x_1 = \cdots = x_n = 0$ in (1.2), we get that $f(0) = 0$. Let j and k be fixed integers with $1 \leq j < k \leq n$. Setting $x_i = 0$ for all $1 \leq i \leq n, i \neq j, k$ in (1.2), we have

$$2f(\frac{x_j}{2} + x_k) + 2f(x_j + \frac{x_k}{2}) + f(x_j) + f(x_k) = 4f(x_j + x_k) \tag{2.1}$$

for all $x_j, x_k \in X$. Replacing x_j by $2x_j$ and x_k by $2x_j$ in (2.1), respectively, we get

$$2f(x_j + 2x_k) + 2f(2x_j + x_k) + f(2x_j) + f(2x_k) = 4f(2x_j + 2x_k) \tag{2.2}$$

for all $x_j, x_k \in X$. Putting $x_k = 0$ in (2.2), we conclude that $f(2x_j) = 2f(x_j)$ for all $x_j \in X$, there for we obtain that

$$f(x_j + 2x_k) + f(2x_j + x_k) + f(x_j) + f(x_k) = 4f(x_j + x_k) \tag{2.3}$$

for all $x_j, x_k \in X$. Replacing x_j by $x_j - x_k$in (2.3), we have

$$f(x_j + x_k) + f(2x_j - x_k) + f(x_j - x_k) + f(x_k) = 4f(x_j) \tag{2.4}$$

for all $x_j, x_k \in X$. Letting $x_j = 0$ in (2.4), we conclude that $f(-x_k) = -f(x_k)$ for all $x_j \in X$. This means that f is an odd function. Replacing x_k by $-x_k$ in (2.4), and using the oddness of f, we obtain that

$$f(x_j - x_k) + f(2x_j + x_k) + f(x_j + x_k) - f(x_k) = 4f(x_j) \tag{2.5}$$

for all $x_j, x_k \in X$. Replacing x_j and x_k by x_k and x_j in (2.5), respectively, and using the oddness of f, we get

$$-f(x_j - x_k) + f(x_j + 2x_k) + f(x_j + x_k) - f(x_j) = 4f(x_k) \tag{2.6}$$

for all $x_j, x_k \in X$. Adding (2.5) to (2.6) and using (2.3),we get that

$$f(x_j + x_k) = f(x_j) + f(x_k)$$

for all $x_j, x_k \in X$. Therefore, $f : X \to Y$ is an additive mapping.

The converse is obviously true. □

Now we prove the generalized Hyers–Ulam –Rassias stability of Jordan derivations on Hilbert C^*- modules for the functional equation $D_\mu f(x_1, \cdots, x_n) = 0$.

Theorem 2.2. *Let X be a real vector space and Y be a Banach space. Suppose that $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^k} \varphi(2^k x_1, \cdots, 2^k x_n) = 0, \tag{2.7}$$

$$\widetilde{\varphi_j}(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(0, \cdots, 0, \underbrace{2^k x}_{j\ th}, 0, \cdots, 0) < \infty, \tag{2.8}$$

and

$$\|D_1 f(x_1, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n) \tag{2.9}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized Jensen type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{1}{2} \widetilde{\varphi_j}(2x) \tag{2.10}$$

for all $x \in X$.

Proof. For convenience, set

$$\varphi_j(x) := \varphi(0, \cdots, 0, \underbrace{x}_{j\ \text{th}}, 0, \cdots, 0)$$

for all $x \in X$ and all $1 \le j \le n$. For each $1 \le k \le n$ with $k \ne j$, let $x_k = 0$ and $x_j = 2x$ in (2.9), then we get the following inequality

$$\|2f(x) - f(2x)\|_Y \le \varphi_j(2x) \tag{2.11}$$

for all $x \in X$. Replacing x by $2^k x$ in (2.11) and dividing both sides of (2.11) by 2^{k+1}, we get

$$\left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x) \right\|_Y \le \frac{1}{2^{k+1}} \varphi_j(2^{k+1}x)$$

for all $x \in X$ and all $k \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^m}f(2^m x)\Big\|_Y &\le \sum_{l=m}^{k}\Big\|\frac{1}{2^{l+1}}f(2^{l+1}x)-\frac{1}{2^l}f(2^l x)\Big\|_Y \\ &\le \frac{1}{2}\sum_{l=m}^{k}\frac{1}{2^l}\varphi_j(2^{l+1}x)\end{aligned} \tag{2.12}$$

for all $x \in X$ and all integers $k \ge m \ge 0$. It follows from (2.8) and (2.12) that the sequence $\{\frac{f(2^k x)}{2^k}\}$ is a Cauchy sequence in Y for all $x \in X$, and thus converges by the completeness of Y. So we can define the mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty}\frac{f(2^k x)}{2^k}$$

for all $x \in X$. Letting $m = 0$ and taking the limit as $k \to \infty$ in (2.12), we obtain the desired inequality (2.10).

It follows from (2.7) and (2.9) that

$$\begin{aligned}\|D_1 L(x_1,\cdots,x_n)\|_Y &= \lim_{k\to\infty}\frac{1}{2^k}\|D_1 f(2^k x_1,\cdots,2^k x_n)\|_Y \\ &\le \lim_{k\to\infty}\frac{1}{2^k}\varphi(2^k x_1,\cdots,2^k x_n) = 0\end{aligned}$$

for all $x_1,\cdots,x_n \in X$. Therefore the mapping $L : X \to Y$ satisfies the equation (1.2). Hence, by Lemma 2.1, L is a generalized Jensen type additive mapping.

To prove the uniqueness of L, let $L' : X \to Y$ be another generalized Jensen type additive mapping satisfying (2.10). By Lemma 2.1, the mapping L' is additive. Therefore it follows from (2.10) that

$$\begin{aligned}\|L(x)-L'(x)\|_Y &= \lim_{k\to\infty}\frac{1}{2^k}\big\|f(2^k x)-L'(2^k x)\big\|_Y \\ &\le \frac{1}{2}\lim_{k\to\infty}\frac{1}{2^k}\sum_{l=0}^{\infty}\frac{1}{2^l}\varphi_j(2^{l+k+1}x) \\ &= \frac{1}{2}\lim_{k\to\infty}\sum_{l=k}^{\infty}\frac{1}{2^l}\varphi_j(2^{l+1}x) = 0.\end{aligned}$$

So $L(x) = L'(x)$ for all $x \in X$. It completes the proof. □

Theorem 2.3. *Let $f : \mathcal{M} \to \mathcal{M}$ be a mapping satisfying $f(0) = 0$ for which there exists a control function $\varphi : \mathcal{M}^n \to [0,\infty)$ satisfying (2.7), (2.8) and*

$$\|D_\mu f(x_1,\cdots,x_n)\| \le \varphi(x_1,\cdots,x_n) \tag{2.13}$$

for all $\mu \in T^1_{\frac{1}{n_o}}$ and all $x_1,\cdots,x_n \in \mathcal{M}$. Let $\psi : \mathcal{M}^2 \to [0,\infty)$ be a function such that

$$\lim_{k\to\infty}\frac{1}{2^k}\psi(2^k x, 2^k y) = 0 \tag{2.14}$$

and

$$\|f(<x,x>y)-<f(x),x>y-<x,f(x)>y \\ -<x,x>f(y)\| \le \psi(x,y) \qquad (2.15)$$

for all $x,y \in \mathcal{M}$, then there exists a unique Jordan derivation $H:\mathcal{M}\to\mathcal{M}$ such that

$$\|f(x)-H(x)\| \le \frac{1}{2}\widetilde{\varphi_j}(2x) \qquad (2.16)$$

for all $x \in \mathcal{M}$.

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique generalized Jensen type additive mapping $H:\mathcal{M}\to\mathcal{M}$. The mapping $H:\mathcal{M}\to\mathcal{M}$ is defined by $H(x)=\lim_{k\to\infty}\frac{1}{2^k}f(2^kx)$ for all $x\in\mathcal{M}$.

By the assumption, we have

$$\begin{aligned}
&\big\|D_\mu H(0,\cdots,0,\underbrace{x}_{j\text{ th}},0,\cdots,0)\big\| \\
&=\lim_{k\to\infty}\frac{1}{2^k}\big\|D_\mu f(0,\cdots,0,\underbrace{2^kx}_{j\text{ th}},0,\cdots,0)\big\| \\
&\le \lim_{k\to\infty}\frac{1}{2^k}\varphi(0,\cdots,0,\underbrace{2^kx}_{i\text{ th}},0,\cdots,0)=0
\end{aligned}$$

for all $x\in\mathcal{M}$ and all $\mu\in\mathbb{T}^1_{\frac{1}{n_o}}$. Then

$$H(\mu x)=\mu H(x)$$

for all $x\in\mathcal{M}$ and all $\mu\in\mathbb{T}^1_{\frac{1}{n_o}}$. By a similar method to the proof of [10], one can show that the mapping $H:\mathcal{M}\to\mathcal{M}$ is $\mathbb{C}-$linear. It follows from (2.14) and (2.15) that

$$\begin{aligned}
&\|H(<x,x>y)-<H(x),x>y-<x,H(x)>y-<x,x>H(y)\| \\
&=\lim_{k\to\infty}\frac{1}{2^{3k}}\|f(<2^kx,2^kx>2^ky)-<f(2^kx),2^kx>2^ky \\
&\qquad -<2^kx,f(2^kx)>2^ky-<2^kx,2^kx>f(2^ky)\| \\
&\le \lim_{k\to\infty}\frac{1}{2^{3k}}\psi(2^kx,2^ky)
\end{aligned}$$

for all $x,y\in\mathcal{M}$. So

$$H(<x,x>y)=<H(x),x>y+<x,H(x)>y+<x,x>H(y)$$

for all $x,y\in\mathcal{M}$. Therefore the mapping $H:\mathcal{M}\to\mathcal{M}$ is a Jordan derivation. □

Theorem 2.4. *Let $\epsilon\ge 0$ and $\{p_k\}_{k\in J}$ be real numbers such that $p_k>0$ for all $k\in J$, where $J\subseteq\{1,2,\cdots,n\}$ and $|J|\ge 3$. Let $f:\mathcal{M}\to\mathcal{M}$ be a mapping for which there is a function $\psi:\mathcal{M}^2\to[0,\infty)$ satisfying (2.14), (2.15) and*

$$\|D_\mu f(x_1,\cdots,x_n)\| \le \epsilon\prod_{k\in J}\|x_k\|^{p_k}, \qquad (2.17)$$

for all $x_1, \cdots, x_n \in \mathcal{M}$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then the mapping* $f : \mathcal{M} \to \mathcal{M}$ *is a Jordan derivation.*

Proof. It follows from (2.17) that $f(0) = 0$. Since $|J| \geq 3$, letting $\mu = 1$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i, j$, in (2.17), we get

$$2f(\frac{x_i}{2} + x_j) + 2f(x_i + \frac{x_j}{2}) + f(x_i) + f(x_j) = 4f(x_i + x_j)$$

for all $x_i, x_j \in \mathcal{M}$. By the same reasoning as in the proof of Lemma 2.1, the mapping f is additive. So by letting $x_i = x$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i$, in (2.17), we get that $f(\mu x) = \mu f(x)$ for all $x \in \mathcal{M}$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. By a similar method to the proof of [10], the mapping f is $\mathbb{C}-$linear. Hence, it follows from (2.14) and (2.15) that

$$\begin{aligned}
&\|f(< x, x > y)- < f(x), x > y- < x, f(x) > y- < x, x > f(y)\| \\
&= \lim_{k\to\infty} \frac{1}{2^{3k}} \|f(< 2^k x, 2^k x > 2^k y)- \\
&\qquad < f(2^k x), 2^k x > 2^k y- < 2^k x, f(2^k x) > 2^k y- < 2^k x, 2^k x > f(2^k y)\| \\
&\leq \lim_{k\to\infty} \frac{1}{2^{3k}} \psi(2^k x, 2^k y)
\end{aligned}$$

for all $x, y \in \mathcal{M}$. So

$$f(< x, x > y) =< f(x), x > y- < x, f(x) > y- < x, x > f(y)$$

for all $x, y \in \mathcal{M}$. Therefore, the mapping $f : \mathcal{M} \to \mathcal{M}$ is a Jordan derivation. □

Corollary 2.5. *Let* $\{\epsilon_i\}_{i\in J}$ *and* $\{p_i\}_{i\in J}$ *be real numbers such that* $\epsilon_i \geq 0$ *and* $0 < p_i < 1$ *for all* $i \in J = \{1, 2, \cdots, n\}$. *Let* $f : \mathcal{M} \to \mathcal{M}$ *be a mapping for which there is a function*

$$\|D_\mu f(x_1, \cdots, x_n)\| \leq \sum_{i=1}^{n} \epsilon_i \|x_i\|^{p_i}, \tag{2.18}$$

$$\begin{aligned}
&\|f(< x, x > y)- < f(x), x > y- < x, f(x) > y \\
&\qquad - < x, x > f(y)\| \leq (\epsilon_r \|x\|^{p_r} + \epsilon_s \|y\|^{p_s}),
\end{aligned} \tag{2.19}$$

for all $\mu \in T^1_{\frac{1}{n_o}}$ *and all* $x, y, x_1, \cdots, x_n \in \mathcal{M}$. *Then there exists a unique Jordan derivation* $H : \mathcal{M} \to \mathcal{M}$ *such that*

$$\|f(x) - H(x)\| \leq \frac{2^{p_i}}{2 - 2^{p_i}} \epsilon_i \|x\|^{p_i}$$

for all $x \in \mathcal{M}$ *and for all* $i \in J$.

Proof. The result follows from Theorem 2.3 by taking

$$\varphi(x_1, \cdots, x_n) = \sum_{i=1}^{n} \epsilon_i \|x_i\|^{p_i}$$

and

$$\psi(x, y) = (\epsilon_r \|x\|^{p_r} + \epsilon_s \|y\|^{p_s})$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{M}$. It follows from (2.18) that $f(0) = 0$. □

3. Stability of Jordan derivations on Hilbert C^*-modules: fixed point method

In this section, by using the idea of P. Găvruta and L. Găvruta [17], we prove the generalized Hyers–Ulam–Rassias stability of Jordan derivations on Hilbert C^*-modules for the functional equation $D_\mu f(x_1, \cdots, x_n) = 0$.

We apply the following theorem:

Theorem 3.1. *(Banach) Let (X, d) be a complete metric space and $T : X \to X$ a contraction, i.e. there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X.$$

Then there exists a unique $a \in X$ such that $Ta = a$. Moreover, $a = lim_{n\to\infty} T^n x$, and

$$d(a, x) \leq \frac{1}{1-\alpha} d(x, Tx), \quad for\ any\ x \in X.$$

Theorem 3.2. *Let $f : \mathcal{M} \to \mathcal{M}$ be a mapping satisfying $f(0) = 0$ for which there exists a control function $\varphi : \mathcal{M}^n \to (0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^k} \varphi\big(2^k x_1, \cdots, 2^k x_n\big) = 0, \tag{3.1}$$

$$\|D_\mu f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n), \tag{3.2}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x_1, \cdots, x_n \in \mathcal{M}$. Let $\psi : \mathcal{M}^2 \to [0, \infty)$ be a function such that

$$\lim_{k\to\infty} \frac{1}{2^k} \psi(2^k x, 2^k y) = 0 \tag{3.3}$$

and

$$\|f(< x, x > y) - < f(x), x > y - < x, f(x) > y \\ - < x, x > f(y)\| \leq \psi(x, y) \tag{3.4}$$

for all $x, y \in \mathcal{M}$. If for some $1 \leq j \leq n$, there exists a Lipschitz constant $0 \leq \mathcal{L} < 1$ such that

$$\varphi(0, \cdots, 0, \underbrace{x}_{j\ th}, 0, \cdots, 0) \leq 2\,\mathcal{L}\varphi(0, \cdots, 0, \underbrace{\frac{1}{2}x}_{j\ th}, 0, \cdots, 0)$$

for all $x \in \mathcal{M}$, then there exists a unique Jordan derivation $H : \mathcal{M} \to \mathcal{M}$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{2 - 2\mathcal{L}} \varphi(0, \cdots, 0, \underbrace{2x}_{j\ th}, 0, \cdots, 0) \tag{3.5}$$

for all $x \in \mathcal{M}$.

Proof. For convenience, set

$$\varphi_j(x) := \varphi(0, \cdots, 0, \underbrace{x}_{j\ \text{th}}, 0, \cdots, 0)$$

for all $x \in \mathcal{M}$ and all $1 \leq j \leq n$. Consider the set

$$\mathcal{X} := \left\{ g : \mathcal{M} \to \mathcal{M}, \quad \sup_{x \in \mathcal{M}} \frac{\|g(x) - f(x)\|}{\varphi_j(2x)} < \infty \right\}$$

and introduce the *metric* on $\mathcal{X}$:

$$d(g, h) = \sup_{x \in \mathcal{M}} \frac{\|g(x) - h(x)\|}{\varphi_j(2x)}.$$

Then $(\mathcal{X}, d)$ is complete. Now we consider the linear mapping $J : \mathcal{X} \to \mathcal{X}$ such that $Jg(x) := \frac{1}{2} g(2x)$ for all $x \in \mathcal{M}$. For any $g, h \in \mathcal{X}$, we have

$$\begin{aligned} d(g, h) < \mathcal{C} &\Longrightarrow \frac{\|g(x) - h(x)\|}{\varphi_j(2x)} \leq \mathcal{C}, \qquad \forall x \in \mathcal{M} \\ &\Longrightarrow \frac{\left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\|}{\varphi_j(4x)} \leq \frac{1}{2} \mathcal{C} \\ &\Longrightarrow \frac{\left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\|}{\varphi_j(2x)} \leq \mathcal{L} \mathcal{C} \\ &\Longrightarrow d(Jg, Jh) \leq \mathcal{L} \mathcal{C}. \end{aligned}$$

Therefore, we see that

$$d(Jg, Jh) \leq \mathcal{L} d(g, h), \qquad \forall g, h \in \mathcal{X}.$$

This means J is a strictly contractive self-mapping of $\mathcal{X}$, with the Lipschitz constant $\mathcal{L}$.

Letting $\mu = 1$, $x_j = 2x$ and for each $1 \leq k \leq n$ with $k \neq j$, $x_k = 0$ in (3.2), we get

$$\frac{\left\| 2f(x) - f(2\,x) \right\|}{\varphi_j(2x)} \leq 1 \tag{3.6}$$

for all $x \in \mathcal{M}$. So

$$\frac{\left\| f(x) - \frac{1}{2} f(2x) \right\|}{\varphi_j(2x)} \leq \frac{1}{2}$$

for all $x \in \mathcal{M}$. Hence $d(f, Jf) \leq \dfrac{1}{2}$.

By Theorem 3.1, there exists a unique mapping $H : \mathcal{M} \to \mathcal{M}$ such that

$$H(2x) = 2H(x) \tag{3.7}$$

for all $x \in \mathcal{M}$, i.e., H is a unique fixed point of J. Moreover,

$$H(x) = \lim_{m \to \infty} \frac{1}{2^m} f\left(2^m x\right) \tag{3.8}$$

for all $x \in \mathcal{M}$. So, We can conclude that $d(H, f) \leq \dfrac{1}{1 - L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (3.5) holds.

It follows from (3.1), (3.2) and (3.8) that

$$\Big\|2\sum_{j=1}^{n}H(\frac{x_j}{2}+\sum_{i=1,i\neq j}^{n}x_i)+\sum_{i=1}^{n}H(x_i)-2nH(\sum_{i=1}^{n}x_i)\Big\|$$
$$=\lim_{m\to\infty}\frac{1}{2^m}\Big\|2\sum_{j=1}^{n}f(2^{m-1}x_j+\sum_{i=1,i\neq j}^{n}2^m x_i)+\sum_{i=1}^{n}f(2^m x_i)-2nf(\sum_{i=1}^{n}2^m x_i)\|$$
$$\leq\lim_{m\to\infty}\frac{1}{2^m}\varphi(2^m x_1,\cdots,2^m x_n)$$

for all $x_1,\cdots,x_n\in\mathcal{M}$. So

$$2\sum_{j=1}^{n}H(\frac{x_j}{2}+\sum_{i=1,i\neq j}^{n}x_i)+\sum_{i=1}^{n}H(x_i)=2nH(\sum_{i=1}^{n}x_i)$$

for all $x_1,\cdots,x_n\in\mathcal{M}$. By Lemma 2.1, the mapping $H:\mathcal{M}\to\mathcal{M}$ is Cauchy additive, i.e., $H(x+y)=H(x)+H(y)$ for all $x,y\in\mathcal{M}$.

By a similar method to the proof of [10], one can show that the mapping $H:\mathcal{M}\to\mathcal{M}$ is $\mathbb{C}-$linear.

It follows from (3.3) and (3.4) that

$$\|H(<x,x>y)-<H(x),x>y-<x,H(x)>y-<x,x>H(y)\|$$
$$=\lim_{k\to\infty}\frac{1}{2^{3k}}\|f(<2^kx,2^kx>2^ky)-<f(2^kx),2^kx>2^ky$$
$$-<2^kx,f(2^kx)>2^ky-<2^kx,2^kx>f(2^ky)\|$$
$$\leq\lim_{k\to\infty}\frac{1}{2^{3k}}\psi(2^kx,2^ky)$$

for all $x,y\in\mathcal{M}$. So

$$H(<x,x>y)=<H(x),x>y-<x,H(x)>y-<x,x>H(y)$$

for all $x,y\in\mathcal{M}$.

Thus $H:\mathcal{M}\to\mathcal{M}$ is a Jordan derivations on Hilbert C^*-modules satisfying (3.5), as desired. □

Corollary 3.3. *Let $\theta>0$, $\{\epsilon_i\}_{i\in J}$ and $\{p_i\}_{i\in J}$ be real numbers such that $\epsilon_i\geq 0$ and $0<p_i<1$ for all $i\in J=\{1,2,\cdots,n\}$. Let $f:\mathcal{M}\to\mathcal{M}$ be a mapping satisfying $f(0)=0$ for which*

$$\|D_\mu f(x_1,\cdots,x_n)\|\leq\theta+\sum_{i=1}^{n}\epsilon_i\|x_i\|^{p_i},\tag{3.9}$$

$$\|f(<x,x>y)-<f(x),x>y-<x,f(x)>y$$
$$-<x,x>f(y)\|\leq(\epsilon_r\|x\|^{p_r}+\epsilon_s\|y\|^{p_s}),\tag{3.10}$$

for all $\mu\in\mathbb{T}^1_{\frac{1}{n_o}}$ and all $x,y,x_1,\cdots,x_n\in\mathcal{M}$. Then there exists a unique Jordan derivation $H:\mathcal{M}\to\mathcal{M}$ such that

$$\|f(x)-H(x)\|\leq\frac{1}{2-2^{p_i}}\theta+\frac{2^{p_i}}{2-2^{p_i}}\epsilon_i\|x\|^{p_i}$$

for all $x \in \mathcal{M}$ and for all $i \in J$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, \cdots, x_n) := \theta + \sum_{i=1}^{n} \epsilon_i \|x_i\|^{p_i}$$

and

$$\psi(x, y) = (\epsilon_r \|x\|^{p_r} + \epsilon_s \|y\|^{p_s})$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{M}$. We can choose $L = \frac{1}{2^{1-p_i}}$ to get the desired result. □

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