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A HALPERN-LIONS-REICH-LIKE ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS

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Abstract. We prove strong convergence of a Halpern-Lions-Reich-like iterative algorithm for approximating fixed points of nonexpansive mappings in a uniformly smooth Banach space. The idea of this algorithm is then applied to solve a quadratic minimization problem in a Hilbert space.
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1. Introduction

In 1965, Halpern [4] invented an iterative algorithm for finding a fixed point of a nonexpansive mapping in the framework of Hilbert spaces. To state Halpern's algorithm, recall that a self-mapping of a closed convex subset C of a real Banach space H is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad x, y \in C. \tag{1.1}$$

The set of fixed points of T is denoted $Fix(T)$ and suppose that $Fix(T) \neq \emptyset$.

Halpern's algorithm [4] then generates a sequence $\{x_n\}$ by the recursive process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \ge 0 \tag{1.2}$$

where $u \in C$ is called an anchor, $x_0 \in C$ is an initial guess, and $\{\alpha_n\} \subset (0, 1)$ is a sequence of iteration parameters.

Halpern called a sequence $\{\alpha_n\} \subset (0, 1)$ *acceptable* if the sequence $\{x_n\}$ generated by (1.2) always converges in norm to a fixed point of T irrespective of the choice of Hilbert space H, closed convex subset C of H, nonexpansive mapping $T : C \to C$ such that $Fix(T) \neq \emptyset$, anchor $u \in C$, and starting point $x_0 \in C$. He proved that the following conditions $(H1)$ and $(H2)$ are necessary for $\{\alpha_n\}$ to be acceptable:

(H1) $\lim_{n\to\infty} \alpha_n = 0$.

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(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Halpern [4] also proved that the conditions (H1), (H2) and (H3) are sufficient for $\{\alpha_n\}$ to be acceptable, where

(H3) there is a strictly increasing sequence of positive integers, $\{n_j\}$, such that

$$\begin{cases} \frac{\alpha_{j+n_j}}{\alpha_j} \to 1, & \text{as } j \to \infty, \\ n_j \alpha_j \to \infty, & \text{as } j \to \infty. \end{cases} \tag{1.3}$$

He observed that $\alpha_n = (n+1)^{-\alpha}$ for all n, where $0 < \alpha < 1$, satisfies (H1), (H2) and (H3), hence acceptable.

In 1977, Lions [5] proved that the conditions (H1), (H2) and (L1) are sufficient for $\{\alpha_n\}$ to be acceptable, where

(L1) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_{n+1}^2 = 0$.

Note that Lions [5] is the first to extend the algorithm (1.2) to find a common fixed point of a finite family of (firmly) nonexpansive mappings.

Many researchers made contributions to the Halpern-Lions algorithm (1.2) by finding a third condition which, together with (H1) and (H2), is sufficient for $\{\alpha_n\}$ to be acceptable; each of the following conditions is such a third condition:

(W1) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ (Wittmann [15]),

(R1) $\{\alpha_n\}$ is decreasing (Reich [9]),

(X1) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_{n+1} = 0$ or equivalently, $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$ (Xu [16, 17]).

A question gives rise to whether or not the conditions (H1) and (H2) are sufficient for $\{\alpha_n\}$ to be acceptable. This question was answered negatively by Suzuki [13]. However, it is still an open question: What conditions are necessary and sufficient for $\{\alpha_n\}$ to be acceptable. If we narrow the class of nonexpansive mappings down to the class of so-called averaged nonexpansive mappings, then the conditions (H1) and (H2) are not only necessary but sufficient for $\{\alpha_n\}$ to be acceptable. Recall that a mapping $T : C \to C$ is said to be *averaged* nonexpansive if $T = (1-\lambda)I + \lambda V$, where $\lambda \in (0, 1)$ and $V : C \to C$ is nonexpansive.

On the other hand, it is interesting to extend the algorithm (1.2) to the setting of Banach spaces. In this regard, Reich [8] was the first to prove that the sequence $\{x_n\}$ generated by the algorithm (1.2) in a uniformly smooth Banach space with the choice of parameters $\alpha_n = (1+n)^{-\alpha}$ for all n, where $0 < \alpha < 1$, converges in norm to a fixed point of T. Due to this reason, we will refer the algorithm (1.2) to as the Halpern-Lions-Reich algorithm throughout the rest of this paper.

While searching new iterative algorithms, Yao, et al [20] introduced an iterative algorithm that generates a sequence $\{x_n\}$ through the recursion:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = 1$ for all n. We shall call it a Halpern-Lions-Reich-like algorithm. Yao, et al [20] proved that if, in addition, there hold the conditions:

(i) $\alpha_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\gamma_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ generated by (1.4) converges in norm to a fixed point of T.

Nevertheless, it is recently pointed out in [10] that Yao, et al's result above is false, that is, the conditions (i) and (ii) are insufficient to guarantee the strong convergence of the sequence $\{x_n\}$. It is proved in [10] that if, in addition to the condition (i), there hold the conditions:

(iii) $\beta_n \to 0$ as $n \to \infty$,
(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(v) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then the sequence $\{x_n\}$ generated by (1.4) does converge in norm to a fixed point of T.

It is of interest to investigate the strong convergence of the Halpern-Lions-Reich-like algorithm (1.4) under appropriate conditions to be imposed on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$. The purpose of this paper is twofold. First, we will prove a strong convergence theorem for the Halpern-Lions-Reich-like algorithm (1.4) under different conditions from those of Sangago [10]. Secondly, we will apply our convergence result to solve a quadratic minimization problem.

2. Preliminaries

Let X be a real uniformly smooth Banach space and C a closed convex subset of X. Let $J : X \to X^*$ be the (normalized) duality map defined by

$$J(x) \in X^*, \quad \|J(x)\| = \|x\|, \quad \langle x, J(x)\rangle = \|x\|^2.$$

Note that the uniform smoothness of X implies that J is uniformly continuous on bounded sets in the norm-to-norm topology.

Let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. For each fixed anchor $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique fixed point of the contraction

$$T_t x := tu + (1 - t)Tx, \quad x \in C. \tag{2.1}$$

The following theorem is known, the Hilbert space counterpart of which is proved by Browder [1].

Theorem 2.1 [8] If X is a uniformly smooth Banach space, then $\{x_t\}$ converges in norm, as $t \to 0$, to a fixed point of T; moreover, the operator $Q : C \to Fix(T)$ defined by

$$Q(u) := \|\cdot\| - \lim_{t\to 0} x_t, \quad u \in C \tag{2.2}$$

defines the unique sunny nonexpansive retraction from C onto $Fix(T)$; that is, Q satisfies the properties:

(i) $\langle Qu - u, J(p - u)\rangle \geq 0, \quad u \in C, \quad p \in Fix(T)$.
(ii) $\|Qu - Qv\|^2 \leq \langle u - v, J(Qu - Qv)\rangle, \quad u, v \in C$.

To prove our main result in the next section, we need the following two lemmas.

Lemma 2.2 [16] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ or $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 [12] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n) y_n, \quad n \ge 0 \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Assume

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.4}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The following lemma is straightforward, but convenient in use.

Lemma 2.4 In a real smooth Banach space, there holds the inequality for all x, y:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

3. Convergence of a Halpern-Lions-Reich-like Algorithm

Recall that our Halpern-Lions-Reich-like algorithm generate a sequence $\{x_n\}$ through the recursion:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = 1$ for all n.

Yao, et al [20] claimed that the conditions

(a) $\alpha_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\gamma_n \to 0$ as $n \to \infty$,

were sufficient to guarantee the strong convergence of the sequence $\{x_n\}$ generated by (1.4). But the fact is that their conclusion is incorrect, as the counterexamples of Sangago [10] showed. Sangago [10] did not figure out the cause of the incorrectness in the proof given in Yao, et al [20]. So let us briefly review the main points of the proof of Yao, et al [20]. Let $t \in (0,1)$ and $n \ge 1$ be given and let $z_{t,n}$ be the unique fixed point of the contraction

$$T_{t,n} z := \frac{(1-\alpha_n)t}{\gamma_n + t\beta_n} u + \frac{(1-t)\gamma_n}{\gamma_n + t\beta_n} T z, \quad z \in C. \tag{3.2}$$

Then one has that

$$\lim_{t\to 0} z_{t,n} = p \in Fix(T), \quad n \ge 1. \tag{3.3}$$

Indeed, $p = Qu$, where $Q : C \to Fix(T)$ is the unique sunny nonexpansive retraction from C onto $Fix(T)$ as defined in Theorem 2.1.

The key step of the proof of Yao, et al [20] is the following inequality

$$\limsup_{n\to\infty} \langle u - p, J(x_n - p)\rangle \le 0. \tag{3.4}$$

To achieve this, they interchanged the order in the following iterated limits

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle u - z_{t,n}, J(x_n - z_{t,n})\rangle \leq 0 \tag{3.5}$$

by using the uniform smoothness of the space X (equivalently, the norm-to-norm uniform continuity over bounded sets of the normalized duality map J). This however requires that the limit in (3.3) be uniformly over $n \geq 1$, which fails to be true, in general, under the conditions (a) and (b) of Yao, et al [20]. To illustrate this, we use the counterexample in [10].

Example 3.1 [10] Take $X = \mathbb{R}$ to be the real line equipped with the absolute value as norm, $C = [-1, 1]$, and $T : C \to C$ to be the reflection: $Tx = -x$ for $x \in C$. Then T is nonexpansive and $x = 0$ is the unique fixed point of T. Furthermore, take $u = 1$ and $x_0 = \frac{1}{3}$, and take $\alpha_n = \gamma_n \in (0, \frac{1}{3})$ for all n so that $\beta_n = 1 - 2\alpha_n \in (\frac{2}{3}, 1)$. It is then easily seen that the sequence $\{x_n\}$ generated by the algorithm (3.1) is a constant:

$$x_n \equiv \frac{1}{3}, \quad n \geq 1.$$

Hence, the sequence $\{x_n\}$ fails to converge to a fixed point of T.

In this case, it is not hard to find that the unique fixed point $z_{t,n}$ of the contraction $T_{t,n}$ defined in (3.2) is given by

$$z_{t,n} = \frac{(1-\alpha_n)t}{2\alpha_n + t(\beta_n - \alpha_n)}.$$

It is immediately clear that

$$\lim_{t\to 0} z_{t,n} = 0 \in Fix(T), \quad \lim_{n\to\infty} z_{t,n} = 1 \notin Fix(T).$$

This shows that the limit in (3.3) fails to be uniform over $n \geq 1$, and consequently, the order of the iterated limits in (3.5) cannot be interchanged. As a matter of fact, we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle u - z_{t,n}, J(x_n - z_{t,n})\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}(1 - z_{t,n})(\frac{1}{3} - z_{t,n}) = 0,$$

$$\limsup_{n\to\infty}\limsup_{t\to 0}\langle u - z_{t,n}, J(x_n - z_{t,n})\rangle = \limsup_{n\to\infty}\limsup_{t\to 0}(1 - z_{t,n})(\frac{1}{3} - z_{t,n}) = \frac{1}{3}.$$

Moreover, the relation (3.4) fails to hold; indeed, we have

$$\limsup_{n\to\infty}\langle u - p, J(x_n - p)\rangle = \limsup_{n\to\infty} x_n = \frac{1}{3}.$$

We will provide with a new selection of the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ by avoiding usage of the sequence $\{z_{t,n}\}$. Below is our convergence result on the algorithm (3.1).

Theorem 3.2 Let X be a real uniformly smooth Banach space, C a closed convex subset of X, and $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be generated by the Halpern-Lions-Reich-like algorithm (3.1). Assume that the sequences (α_n), (β_n), and (γ_n) satisfy the following conditions:

(i) $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ are such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$.

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \beta_n = \beta \in (0,1)$.

Then (x_n) converges in norm to Qu, where Q is the sunny nonexpansive retraction from C onto $Fix(T)$ defined by (2.2).

Proof. 1. $\{x_n\}$ is bounded. Indeed, take a fixed point p of T to get

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(u-p)+\beta_n(x_n-p)+\gamma_n(Tx_n-p)\| \\ &\le \alpha_n\|u-p\|+(\beta_n+\gamma_n)\|x_n-p\|\| \\ &\le \max\{\|u-p\|,\|x_n-p\|\}.\end{aligned}$$

So an induction gives

$$\|x_n-p\| \le \max\{\|u-p\|,\|x_0-p\|\}, \quad n\ge 0.$$

2. $\|x_n-Tx_n\|\to 0$. To see this we put

$$y_n=\frac{\alpha_n u+\gamma_n Tx_n}{1-\beta_n} \tag{3.6}$$

so that we have

$$x_{n+1}=\beta_n x_n+(1-\beta_n)y_n. \tag{3.7}$$

It is not hard to find that

$$\begin{aligned}y_{n+1}-y_n &= \frac{\alpha_{n+1}u+\gamma_{n+1}Tx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n Tx_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1}}{1-\beta_{n+1}}(Tx_{n+1}-Tx_n) \\ &\quad+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)Tx_n.\end{aligned}$$

It turns out that

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \le \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-1\right|M+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|M \\ +\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|M,\end{aligned}$$

where $M>0$ is selected so that $M\ge\max\{\|u\|,2\|x_j\|,\|Tx_j\|\}$ for all j. Since $\alpha_n\to 0$ and $\beta_n\to\beta\in(0,1)$, $\gamma_n\to 1-\beta\in(0,1)$ and we get

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.8}$$

Due to (3.8) together with the assumption $\beta_n\to\beta\in(0,1)$, we can apply Lemma 2.3 to the relation (3.7) to get

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

Noticing also from (3.6)

$$\|y_n-Tx_n\|=\frac{\alpha_n}{1-\beta_n}\|u-Tx_n\|\le\frac{2\alpha_n M}{1-\beta_n}\to 0,$$

we obtain

$$\|x_n-Tx_n\|\le\|x_n-y_n\|+\|y_n-Tx_n\|\to 0.$$

3. $\limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle \le 0$, with $q = Qu$.

Let $t \in (0,1)$ and let $z_t \in C$ solves the fixed point equation

$$z_t = tu + (1-t)Tz_t.$$

Then $q = \lim_{t\to 0} z_t$ in the norm topology.

We have

$$\begin{aligned}\|z_t - x_n\|^2 &= \|(1-t)(Tz_t - x_n) + t(u-x_n)\|^2\\ &\le (1-t)^2\|Tz_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle\\ &\le (1-t)^2(\|Tz_t - Tx_n\| + \|Tx_n - x_n\|)^2\\ &\quad + 2t(\langle u - z_t, J(z_t - x_n)\rangle + \|z_t - x\|^2)\\ &\le (1+t^2)\|z_t - x_n\|^2 + M\|Tx_n - x_n\|\\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle.\end{aligned}$$

Here M is such that

$$M \ge 2\|z_t - x_n\| + \|Tx_n - x_n\| \quad \text{for all } n \text{ and } t \in (0,1).$$

It turns out that

$$\langle u - z_t, J(x_n - z_t)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{M}{2t}\|Tx_n - x_n\|. \tag{3.9}$$

Let K be a bounded set such that

$$\{x_n - z_t, x_n - q, z_t - u\} \subset K \quad \text{for all } n \text{ and } t \in (0,1)$$

and let $d := \sup\{\|u\| : u \in K\} < \infty$. Since the duality map J is uniformly continuous in the norm topology, there exists, given $\varepsilon > 0$, a $\delta > 0$ (we assume also that $\delta < \varepsilon$) such that

$$u, v \in K, \quad \|u - v\| < \delta \quad \Rightarrow \quad \|J(u) - J(v)\| < \varepsilon.$$

In particular, since $z_t \to q$ in norm, there exists $t_0 > 0$ small enough so that

$$\|z_t - q\| < \delta < \varepsilon \quad \text{for all } 0 < t < t_0. \tag{3.10}$$

It turns out that

$$\|J(x_n - z_t) - J(x_n - q)\| < \varepsilon \quad \text{for all } n \text{ and } 0 < t < t_0. \tag{3.11}$$

It follows from (3.9)-(3.11) that, for all n and $0 < t < t_0$,

$$\begin{aligned}\langle u - q, J(x_n - q)\rangle &= \langle u - z_t, J(x_n - z_t)\rangle + \langle z_t - q, J(x_n - q)\rangle\\ &\quad + \langle u - z_t, J(x_n - q) - J(x_n - z_t)\rangle\\ &\le \langle u - z_t, J(x_n - z_t)\rangle + 2d\varepsilon\\ &\le \frac{td^2}{2} + \frac{M}{2t}\|Tx_n - x_n\| + 2d\varepsilon.\end{aligned}$$

Consequently, for $0 < t < t_0$,

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \frac{td^2}{2} + 2d\varepsilon. \tag{3.12}$$

Letting $t \to 0$ in (3.12) yields immediately that

$$\limsup_{n\to\infty}\langle u-q, J(x_n-q)\rangle \le 0. \tag{3.13}$$

4. $x_n \to q$ in norm. Applying Lemma 2.4, we get

$$\begin{aligned} \|x_{n+1}-q\|^2 &\le \|\beta_n(x_n-q)+\gamma_n(Tx_n-q)+\alpha_n(u-q)\|^2 \\ &\le \|\beta_n(x_n-q)+\gamma_n(Tx_n-q)\|^2+2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\ &\le (\beta_n\|x_n-q\|+\gamma_n\|x_n-q\|)^2+2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\ &\le (1-\alpha_n)\|x_n-q\|^2+2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle. \end{aligned} \tag{3.14}$$

By applying Lemma 2.2 to (3.14) we conclude that $\|x_n-q\|^2 \to 0$, as required. □

4. A Quadratic Minimization Problem

Consider the quadratic minimization problem in a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle \tag{4.1}$$

where C is the fixed point set $Fix(T)$ of a nonexpansive mapping T on H and u is a given point in H. Assume $Fix(T)$ is nonempty. Assume also A is strongly positive; that is, there is a constant $\gamma > 0$ with the property

$$\langle Ax, x\rangle \ge \gamma\|x\|^2 \text{ for all } x \in H. \tag{4.2}$$

Then the minimization (4.1) has a unique solution $x^* \in C$ which satisfies the optimality condition

$$\langle Ax^*-u, x-x^*\rangle \ge 0, \quad x \in C. \tag{4.3}$$

In [19, 7] it is proved that the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n u, \quad n \ge 0 \tag{4.4}$$

converges in norm to the solution x^* of (4.1) provided the sequence $\{\alpha_n\}$ in (0,1) satisfies the conditions (H1) and (H2), and additionally, either condition (W1) or (X1) stated in Section 1. Below following the idea presented in Section 3, we will demonstrate a new algorithm that generates a sequence strongly converging to the solution x^* of (4.1) under the conditions (H1) and (H2) only. Given an anchor $u \in H$ and a starting point $x_0 \in H$. Let $\{\alpha_n\} \subset (0,1)$ be given. Let a sequence $\{\beta_n\}$ be also given in $(0,1)$ such that $\underline{\beta} \le \beta_n \le \overline{\beta}$ for all n and some $0 < \underline{\beta} \le \overline{\beta} < 1$. Define a sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = (I-\alpha_n A)(\beta_n x_n + (1-\beta_n)Tx_n) + \alpha_n u, \quad n \ge 0. \tag{4.5}$$

Lemma 4.1 [7] Assume A is a strongly positive linear bounded operator on a real Hilbert space H with coefficient $\gamma > 0$ (i.e., $\langle Ax, x\rangle \ge \gamma\|x\|^2$ for all $x \in H$) and $0 < \alpha \le \|A\|^{-1}$. Then $\|I-\alpha A\| \le 1-\alpha\gamma$.

Lemma 4.2 [3] Let H be a Hilbert space, K a closed convex subset of H, and $T: K \to K$ a nonexpansive mapping with $Fix(T) \ne \emptyset$. If $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)x = 0$.

Theorem 4.3 Suppose A is a strongly positive linear bounded operator with coefficient $\gamma > 0$ as given in (4.2). Suppose the sequence $\{\alpha_n\}$ of parameters satisfies the conditions $(H1)$ and $(H2)$. Then the sequence $\{x_n\}$ generated by the algorithm (4.5) converges in norm to the unique solution x^* of the minimization problem (4.1).

Proof. First we claim that $\{x_n\}$ is bounded. As a matter of fact, take a $p \in Fix(T)$ and use Lemma 4.1 to deduce (as $\alpha_n \to 0$ we assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$ for all n)

$$\begin{aligned} \|x_{n+1} - p\| &= \|(I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n - p) + \alpha_n(u - Ap)\| \\ &\leq (1 - \gamma\alpha_n)\|x_n - p\| + \alpha_n\|u - Ap\| \\ &\leq \max\{\|x_n - p\|, (1/\gamma)\|u - Ap\|\}. \end{aligned}$$

By induction we can get

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{1}{\gamma}\|u - Ap\|\right\}, \quad n \geq 0.$$

Hence, $\{x_n\}$ is bounded. Next rewrite x_{n+1} in the form:

$$x_{n+1} = (1 - \gamma_n)x_n + \gamma_n y_n, \tag{4.6}$$

where

$$\gamma_n = 1 - (1 - \alpha_n)\beta_n \tag{4.7}$$

and

$$y_n = \frac{\alpha_n\beta_n}{\gamma_n}(I - A)x_n + \frac{1 - \beta_n}{\gamma_n}(I - \alpha_n A)Tx_n + \frac{\alpha_n}{\gamma_n}u. \tag{4.8}$$

Since $\alpha_n \to 0$, it is easily seen that $\liminf_{n\to\infty} \gamma_n \geq 1 - \overline{\beta} > 0$ and $\limsup_{n\to\infty} \gamma_n \leq 1 - \underline{\beta} < 1$. Since also $\{x_n\}$ is bounded, equation (4.8) shows that $\{y_n\}$ is bounded. Set

$$z_n := \frac{1}{\gamma_n}(\beta_n(I - A)x_n - (1 - \beta_n)ATx_n + u).$$

Then $\{z_n\}$ is bounded and from (4.8), y_n can be rewritten as

$$y_n = \alpha_n z_n + \frac{1 - \beta_n}{\gamma_n}Tx_n = \alpha_n z_n + \left(1 - \frac{\alpha_n\beta_n}{\gamma_n}\right)Tx_n \tag{4.9}$$

since $\frac{1-\beta_n}{\gamma_n} = 1 - \frac{\alpha_n\beta_n}{\gamma_n}$, due to (4.7). We can now compute

$$\begin{aligned} y_{n+1} - y_n &= \alpha_{n+1}z_{n+1} - \alpha_n z_n + \left(1 - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right)Tx_{n+1} - \left(1 - \frac{\alpha_n\beta_n}{\gamma_n}\right)Tx_n \\ &= \alpha_{n+1}z_{n+1} - \alpha_n z_n + \left(1 - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right)(Tx_{n+1} - Tx_n) \\ &\quad + \left(\left(1 - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right) - \left(1 - \frac{\alpha_n\beta_n}{\gamma_n}\right)\right)Tx_n \\ &= \left(1 - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right)(Tx_{n+1} - Tx_n) \\ &\quad + \left(\frac{\alpha_n\beta_n}{\gamma_n} - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right)Tx_n + \alpha_{n+1}z_{n+1} - \alpha_n z_n. \end{aligned}$$

It turns out that

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\|x_{n+1} - x_n\| + \alpha_{n+1}\|z_{n+1}\| + \left|\frac{\alpha_n\beta_n}{\gamma_n} - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right| \|Tx_n\| + \alpha_n\|z_n\|. \qquad (4.10)$$

Since $\alpha_n \to 0$ as $n \to \infty$, it is immediately clear from (4.10) that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Consequently, we can apply Lemma 2.3 to assert that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \qquad (4.11)$$

By (4.9) we have

$$\|y_n - Tx_n\| = \alpha_n \left\| z_n - \frac{\beta_n}{\gamma_n} Tx_n \right\| \to 0. \qquad (4.12)$$

This together with (4.11) yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (4.13)$$

Lemma 4.3 then implies that $\omega_w(x_n) \subset Fix(T) = C$. Here

$$\omega_w(x_n) = \{z : \exists x_{n_j} \to z \text{ weakly}\}$$

is the set of weak ω-limit points of the sequence $\{x_n\}$.

Let x^* be the unique solution to the minimization (4.1). Then by the definition of the algorithm (4.5), we can write

$$x_{n+1} - x^* = (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n - x^*) + \alpha_n(u - Ax^*).$$

Apply Lemma 2.4 (as J is the identity in a Hilbert space) and use Lemma 4.1 to get (noting $\|\beta_n x_n + (1 - \beta_n)Tx_n - x^*\| \le \|x_n - x^*\|$)

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|(I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n - x^*)\|^2 \\ &\quad + 2\alpha_n\langle u - Ax^*, x_{n+1} - x^*\rangle \\ &\le (1 - \gamma\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - Ax^*, x_{n+1} - x^*\rangle. \end{aligned} \qquad (4.14)$$

However, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - Ax^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle u - Ax^*, x_{n_j} - x^*\rangle$$

and also $x_{n_j} \to p$ weakly. Then, since $p \in Fix(T) = C$, we get from the optimality condition (4.3),

$$\limsup_{n\to\infty}\langle u - Ax^*, x_n - x^*\rangle = \langle u - Ax^*, p - x^*\rangle \le 0. \qquad (4.15)$$

Therefore, applying Lemma 2.2 to (4.14) and noticing (4.15), we conclude that $\|x_n - x^*\|^2 \to 0$. □

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