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A BEST PROXIMITY POINT THEOREM FOR SUZUKI TYPE CONTRACTION NON-SELF-MAPPINGS

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Abstract. Let A and B be nonempty subsets of a metric space X. The purpose of this paper is to prove the existence and uniqueness of a best proximity point for a non-self-mapping $T : A \to B$ such that T is a contraction mapping in the sense of Suzuki. Examples are given to support the usability of our results.

Key Words and Phrases: best proximity point; fixed point; contraction non-self-mapping; P-property.
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1. Introduction

Let A and B be two nonempty subsets of a metric space X. A non-self-mapping $T : A \to B$ is said to be a contraction if there exists a constant $k \in [0, 1)$, such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in A$. The well-known Banach contraction principle states that if A is a complete subset of X and T is a contraction self-mapping, then the fixed point equation $Tx = x$ has exactly one solution.

The Banach contraction principle is a very important tool in nonlinear analysis and there are many extensions of this principle; see, e.g., [9] and the references therein. One of the interesting generalizations of the Banach contraction principle which characterizes the metric completeness is due to Suzuki. He proved the following fixed point theorem.

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Theorem 1.1 ([14]) *Define a nondecreasing function* $\theta : [0,1) \to (\frac{1}{2}, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1) \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases} \tag{1.1}$$

Then for a metric space (X, d)*, the following are equivalent:*
(i) X *is complete.*
(ii) Every mapping T *on* X *satisfying the following proposition has a fixed point:*
- *There exists* $r \in [0,1)$ *such that* $\theta(r)d(x, Tx) \leq d(x, y)$ *implies* $d(Tx, Ty) \leq rd(x, y)$ *for all* $x, y \in X$*.*

(iii) There exists $r \in (0,1)$ *such that every mapping* T *on* X *satisfying the following proposition has a fixed point:*
- $\frac{1}{10000}d(x, Tx) \leq d(x, y)$ *implies* $d(Tx, Ty) \leq rd(x, y)$ *for all* $x, y \in X$*.*

Remark 1.2 *We note that for every* $r \in [0,1), \theta(r)$ *is the best constant.*

Now consider the non-self-mapping $T : A \to X$, in which A is a nonempty subset of a metric space (X, d). Clearly, the fixed point equation $Tx = x$ may not have solution. Hence, it is contemplated to find an approximate $x \in A$ such that the error $d(x, Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea. Here we state the following well-known best approximation theorem due to Kay Fan.

Theorem 1.3 ([8]) *Let* A *be a nonempty compact convex subset of a normed linear space* X *and* $T : A \to X$ *be a continuous function. Then there exists* $x \in A$ *such that*

$$\|x - Tx\| = dist(Tx, A) := \inf\{\|Tx - a\| : a \in A\}.$$

A point $x \in A$ in the above theorem is called a best approximant point of T in A. The notion of best proximity point for non-self-mappings is introduced in a similar fashion:

Definition 1.4 *Let* A *and* B *be nonempty subsets of a metric space* (X, d) *and* $T : A \to B$ *be a non-self mapping. A point* $p \in A$ *is called a best proximity point of* T *if* $d(p, Tp) = dist(A, B)$*.*

In fact best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A} d(x, Tx), \tag{1.2}$$

has at least one solution.

Existence and convergence of best proximity points for various classes of mappings is an interesting subject in optimization theory which attracted the attention of many authors; see [1, 2, 4, 5, 6, 7, 11, 15] and references therein.

The aim of this paper is to study the existence and uniqueness of best proximity point for non-self-mappings which are contraction in the sense of Suzuki.

2. Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). In this work, we adopt the following notations and definitions.

$$A_0 = \{x \in A : d(x, y) = dist(A, B), \quad \text{for some} \quad y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = dist(A, B), \quad \text{for some} \quad x \in A\}.$$

We note that if A intersects B, then A_0 and B_0 are nonempty. Also, if A and B are nonempty weakly compact convex subsets of a Banach space X, then (A_0, B_0) must be a nonempty pair in X; meaning that both components are nonempty. Moreover, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B respectively, provided that A and B are closed subsets of a Banach space X such that $dist(A, B) > 0$ (see [12]).

In [10], Sankar Raj introduced a notion of P-property on a nonempty pair of subsets of a metric space as follows.

Definition 2.1 [10] *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if and only if*

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 2.2 ([10]) Let A, B be two nonempty closed convex subsets of a Hilbert space X. Then (A, B) satisfies the P-property.

Example 2.3 Let A, B be two nonempty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $dist(A, B) = 0$. Then (A, B) has the P-property.

Example 2.4 ([3]) Let A, B be two nonempty bounded, closed and convex subsets of a uniformly convex Banach space X. Then (A, B) has the P-property.

The following theorem was established by Sankar Raj.

Theorem 2.5 ([10]) *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping, that is, $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$, for all $x, y \in A$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty), \psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. Assume that $T(A_0) \subseteq B_0$ and the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

If in the above theorem $\psi(t) = (1 - k)t$, for some $k \in (0, 1)$ and $t \geq 0$, then we deduce the following result.

Corollary 2.6 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a contraction non-self-mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

3. The main result

Now, we state the main result of this paper.

Theorem 3.1 *Define a strictly decreasing function η from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by*

$$\eta(r) = \frac{1}{1 + r}.$$

Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and there exists $r \in [0, 1)$ such that

$$\eta(r)d^*(x, Tx) \le d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \le rd(x, y), \tag{3.1}$$

for all $x, y \in A$, where $d^*(a, b) := d(a, b) - dist(A, B)$, for all $(a, b) \in A \times B$. If the pair (A, B) has the P-property, then T has a unique best proximity point.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. Again, since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. Thus we have a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Since (A, B) has the P-property, we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.3}$$

By (3.2) we have

$$d(x_0, Tx_0) \le d(x_0, x_1) + d(x_1, Tx_0) = d(x_0, x_1) + dist(A, B),$$

and so $d^*(x_0, Tx_0) \le d(x_0, x_1)$. Since $\eta(r) \le 1$, we conclude that

$$\eta(r)d^*(x_0, Tx_0) \le d^*(x_0, Tx_0) \le d(x_0, x_1).$$

Now by (3.1) we obtain $d(Tx_0, Tx_1) \le rd(x_0, x_1)$. Similarly,

$$d(x_1, Tx_1) \le d(x_1, x_2) + d(x_2, Tx_1) = d(x_1, x_2) + dist(A, B).$$

Thus $d^*(x_1, Tx_1) \le d(x_1, x_2)$. This implies that

$$\eta(r)d^*(x_1, Tx_1) \le d^*(x_1, Tx_1) \le d(x_1, x_2).$$

Hence, by (3.1), (3.3) we have

$$d(Tx_1, Tx_2) \le rd(x_1, x_2) = rd(Tx_0, Tx_1) \le r^2 d(x_0, x_1).$$

Continuing this process, we deduce that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \le r^n d(x_0, x_1).$$

Therefore, $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete and and A is closed, we can find $p \in A$ such that $x_n \to p$. We now claim that

$$d^*(p, Tx) \leq rd(p, x) \quad \text{for all} \quad x \in A \quad \text{with} \quad x \neq p. \tag{3.4}$$

Since $x_n \to p$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, p) \leq \frac{1}{3} d(x, p) \quad \text{for all} \quad n \geq N_1.$$

We now have

$$\begin{aligned}
\eta(r) d^*(x_n, Tx_n) &\leq d^*(x_n, Tx_n) = d(x_n, Tx_n) - dist(A, B) \\
&\leq d(x_n, p) + d(p, x_{n+1}) + d(x_{n+1}, Tx_n) - dist(A, B) \\
&= d(x_n, p) + d(p, x_{n+1}) \\
&\leq \frac{2}{3} d(x, p) = d(x, p) - \frac{1}{3} d(x, p) \\
&\leq d(x, p) - d(x_n, p) \leq d(x_n, x).
\end{aligned}$$

By using (3.1), we obtain

$$d(Tx_n, Tx) \leq rd(x_n, x) \quad \text{for all} \quad n \geq N_1.$$

Thus

$$\begin{aligned}
d(p, Tx) &= \lim_{n \to \infty} d(x_n, Tx) \leq \lim_{n \to \infty} [d(x_n, Tx_n) + d(Tx_n, Tx)] \\
&\leq \lim_{n \to \infty} [d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) + rd(x_n, x)] \\
&\leq \lim_{n \to \infty} [dist(A, B) + r^n d(x_0, x_1) + rd(x_n, x)] \\
&= dist(A, B) + rd(p, x).
\end{aligned}$$

Now, we conclude that $d^*(p, Tx) \leq rd(p, x)$, that is, (3.4) holds. On the other hand,

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, p) + d(p, Tx_n) \\
&\leq d(x_n, p) + rd(p, x_n) + dist(A, B).
\end{aligned}$$

This implies that $d^*(x_n, Tx_n) \leq (1 + r) d(x_n, p)$ and so

$$\eta(r) d^*(x_n, Tx_n) \leq d(x_n, p).$$

Therefore, by (3.1) we must have

$$d(Tx_n, Tp) \leq rd(x_n, p) \to 0.$$

Hence, $Tx_n \to Tp$. Then by using (3.2) we have

$$d(p, Tp) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = dist(A, B),$$

and so p is a best proximity point of T. We conclude the proof by showing that the best proximity point of T is unique. Suppose that $q \in A$ is another best proximity point of T, that is, $d(q,Tq) = dist(A,B)$. Again, since (A,B) has the P-property, $d(p,q) = d(Tp,Tq)$. Besides,

$$\eta(r)d^*(p,Tp) = 0 \leq d(p,q).$$

By (3.1) we obtain

$$d(p,q) = d(Tp,Tq) \leq rd(p,q),$$

which ensures that $p = q$.

Remark 3.2. We note that, Theorem 3.1 is a real extension of Corollary 2.6. Indeed, the assumption of continuity of the non-self-mapping T in Theorem 2.5 and Corollary 2.6 is essential, whereas, if the non-self-mapping T satisfies the condition (3.1), it might not be continuous.

Let us illustrate Remark 3.2 with the next example in which the sets A and B are non-convex.

Example 3.3. Suppose that $X = \mathbb{R}^2$ with the usual metric. Let

$$A := \{(x,0) : 0 \leq x \leq \frac{1}{10}\} \cup \{(1,2)\},$$

$$B := \{(x,1) : 0 \leq x \leq \frac{1}{10}\} \cup \{(1,1)\}.$$

Thus, (A,B) is a nonempty closed non-convex pair of subsets of X. Also, we note that $dist(A,B) = 1$ and $A_0 = A, B_0 = B$. Moreover, it is easy to see that (A,B) has the P-property. Define a non-self-mapping $T : A \to B$ as follows:

$$T(1,2) = (\frac{1}{100},1) \ \ \& \ \ T(x,0) = \begin{cases} (1,1) & \text{if} \ \ x = 0, \\ (\frac{1}{100},1) & \text{if} \ \ x \neq 0. \end{cases}$$

We claim that T satisfies the condition (3.1).

Case 1. if $\mathbf{x} = (0,0)$ and $\mathbf{y} = (y,0)$ where, $0 < y \leq \frac{1}{10}$, then for each $r \in [0,1)$

$$\begin{aligned} \frac{1}{1+r}d^*(\mathbf{x},T\mathbf{x}) &= \frac{1}{1+r}d^*((0,0),(1,1)) = \frac{1}{1+r}[\sqrt{2}-1] \\ &> \frac{1}{2}[\sqrt{2}-1] > \frac{1}{10} \geq y = d((0,0),(y,0)) = d(\mathbf{x},\mathbf{y}), \end{aligned}$$

which implies that (3.1) holds.

Case 2. if $\mathbf{x} = (0,0)$ and $\mathbf{y} = (1,2)$ then we have

$$d(T\mathbf{x},T\mathbf{y}) = d((1,1),(\frac{1}{100},1)) = \frac{99}{100}, \ \ d(\mathbf{x},\mathbf{y}) = d((0,0),(1,2)) = \sqrt{5}.$$

Thus the relation $d(T\mathbf{x},T\mathbf{y}) \leq rd(\mathbf{x},\mathbf{y})$ holds for each $r \in [\frac{99}{100\sqrt{5}},1)$.

Case 3. If $\mathbf{x} = (x,0)$ with $0 < x \leq \frac{1}{10}$ and $\mathbf{y} = (1,2)$, then it is easy to see that $d(T\mathbf{x},T\mathbf{y}) = 0$.

The above argument shows that all conditions of Theorem 3.1 hold and hence T has a unique best proximity point. This point is $p = (\frac{1}{100},0)$. We note that, the non-self-mapping T is not continuous and hence, the existence of best proximity point for T can not be established by Theorem 2.5.

The following result demonstrates the existence and uniqueness of fixed point for non-self mappings.

Corollary 3.4 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty and $dist(A, B) = 0$. Assume that $T : A \to B$ is a non-self-mapping such that $T(A_0) \subseteq B_0$ and T satisfies the condition (3.1) of Theorem 3.1. Then T has a unique fixed point in $A \cap B$.*

Example 3.5 Let $X = \mathbb{R}^2$ and define the metric d on X by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|, \forall\ (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Obviously, (X, d) is a complete metric space. Let $A = \{(3, 3), (4, 6), (5, 6)\}$ and $B = \{(3, 4), (3, 5)\}$. Define $T : A \to B$ with $T(3, 3) = T(4, 6) = (3, 4)$ and $T(5, 6) = (3, 5)$. Then T satisfies the assumptions of Theorem 3.1.

Proof. It is easy to see that $d(T\mathbf{x}, T\mathbf{y}) \leq \frac{1}{5}d(\mathbf{x}, \mathbf{y})$, if $(\mathbf{x}, \mathbf{y}) \neq ((4, 6), (5, 6))$. On the other hand, in case $(\mathbf{x}, \mathbf{y}) = ((4, 6), (5, 6))$ we have $d(T\mathbf{x}, T\mathbf{y}) = 1$ and $d(\mathbf{x}, \mathbf{y}) = 1$. That is, T is not a contraction, since

$$\frac{1}{1+r}d^*((4, 6), T(4, 6)) = \frac{2}{1+r} > 1 = d((4, 6), (5, 6)),$$

for every $r \in [0, 1)$. Hence T satisfies the condition (3.1) in this case. Note that, the other conditions of Theorem 3.1 hold and hence, T has a unique best proximity point and this point is $p = (3, 3)$.

Example 3.6 Consider $X = \mathbb{R}^2$ with the metric d defined in Example 3.5. Let $A = \{(0, 0), (4, 5), (5, 4)\}$ and $B = \{(0, 0), (0, 4), (4, 0)\}$. Define $T : A \to B$ as follows:

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } \ x_1 \leq x_2, \\ (0, x_2) & \text{if } \ x_2 < x_1. \end{cases}$$

We can see that $d(T\mathbf{x}, T\mathbf{y}) \leq \frac{4}{9}d(\mathbf{x}, \mathbf{y})$, if $(\mathbf{x}, \mathbf{y}) \neq ((4, 5), (5, 4))$. Also, in case $(\mathbf{x}, \mathbf{y}) = ((4, 5), (5, 4))$, we have

$$\frac{1}{1+r}d((4, 5), T(4, 5)) = \frac{5}{1+r} > 2 = d((4, 5), (5, 4)),$$

for every $r \in [0, 1)$. This implies that T satisfies the condition (3.1). It is easy to see that the other conditions of Corollary 3.4 hold. Hence T has a unique fixed point and this point is $p = (0, 0)$.

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