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CAUCHY PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS VIA PICARD AND WEAKLY PICARD OPERATORS TECHNIQUE

JINRONG WANG*, YONG ZHOU** AND WEI WEI*

* Department of Mathematics, Guizhou University
Guiyang, Guizhou 550025, P. R. China
** Department of Mathematics, Xiangtan University
Xiangtan, Hunan 411105, P. R. China

E-mail: wjr9668@126.com; yzhou@xtu.edu.cn; wwei@gzu.edu.cn

Abstract. In this paper we study existence, uniqueness and data dependence for the solutions of Cauchy problems for fractional differential equations in Banach spaces by using Picard and weakly Picard operators technique and suitable Bielecki norms in some convenient spaces.
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1. Introduction

Fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details, one can see the monographs of Diethelm [12], Kilbas et al. [15], Lakshmikantham et al. [16], Miller and Ross [17], Podlubny [21] and Tarasov [28]. Recently, fractional differential equations (inclusions) and optimal controls in Banach spaces are studied by many researchers such as Agarwal et al. [1, 2, 3], Ahmad and Nieto [4, 5], Balachandran et al. [6, 7], Bai [8], Benchohra et al. [9], El-Borai [10], Chang and Nieto [11], Henderson and Ouahab [13], Hernández et al. [14], N'Guérékata [19], Mophou and N'Guérékata [20], Wang et al. [29, 30, 31, 32, 33, 34, 35, 36], and Zhou et al. [37, 38, 39, 40].

The aim of this paper is to obtain existence, uniqueness and data dependence results for the solutions of Cauchy problems for some fractional differential equations in Banach space. To do this we not utilize the techniques used in the papers quoted above but use the Picard and weakly Picard operators technique due to Rus [22, 23, 24, 25, 26]. To our knowledge, Picard and weakly Picard operators technique have been used to study the existence results for the solutions of some integer differential equations [18, 27].

Throughout this paper, $(X, \|\cdot\|)$ will be a Banach spaces, and $J := [0,T]$, $T > 0$. Let $C(J,X)$ be the Banach space of all continuous functions from J into X with the norm $\|x\|_C := \sup\{\|x(t)\| : t \in J\}$ for $x \in C(J,X)$. Consider the following Cauchy problem of fractional differential equation

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ t \in J, \\ x(0) = x_0 \in X, \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0,1)$, the function $f : J \times X \to X$ satisfies some assumptions that will be specified later.
Let us recall the following definitions of fractional calculus. For more details see [15].

Definition 1.1. *The fractional integral of order q with the lower limit zero for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \ t > 0, \ q > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 1.2. *The Riemann-Liouville derivative of order q with the lower limit zero for a given function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^LD^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{q+1-n}} ds, \ t > 0, \ n-1 < q < n.$$

Definition 1.3. *The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = {}^LD^q \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \ t > 0, \ n-1 < q < n.$$

Next, we recall the definition of a solution and equivalence result of the fractional Cauchy problem. For more details see [1].

Definition 1.4. *A function $x \in C^1(J,X)$ is said to be a solution of the fractional Cauchy problem (1.1) if x satisfies the equation ${}^cD^q x(t) = f(t, x(t))$ a.e. on J, and the condition $x(0) = x_0$.*

Lemma 1.5. *A function $x \in C(J,X)$ is a solution of the fractional integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \overline{f}(s) ds,$$

if and only if x is a solution of the following fractional Cauchy problem

$$\begin{cases} {}^cD^q x(t) = \overline{f}(t), \ 0 < q < 1, \ t \in J, \\ x(0) = x_0. \end{cases}$$

As a consequence of Lemma 1.5, we have the following result which is useful in what follows.

Lemma 1.6. *A function $x \in C(J, X)$ is a solution of the fractional integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \tag{1.2}$$

if and only if x is a solution of the fractional Cauchy problem (1.1).

In the present paper we consider suitable Bielecki norms in some convenient spaces and obtain existence, uniqueness and data dependence results for the solutions of the fractional Cauchy problem (1.1) via Picard and weakly Picard operators technique.

2. Preliminaries

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:
$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\}$;
$F_A = \{x \in X \mid A(x) = x\}$−the fixed point set of A;
$I(A) = \{Y \in P(X) \mid A(Y) \subseteq Y\}$;
$O_A(x) = \{x, A(x), A^2(x), \cdots, A^n(x), \cdots\}$−the A−orbit of $x \in X$;
$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$;
$H(Y, Z) = \max\left(\sup_{a\in Y} \inf_{b\in Z} d(a, b), \sup_{b\in Z} \inf_{a\in Y} d(a, b)\right)$−the Pompeiu-Hausdorff functional on $P(X)$.

Definition 2.1. *(Rus [23]). Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *(Rus [24]). Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

If A is a weakly Picard operator, then we consider the operator

$$A^\infty : X \to X,\ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

The following results are useful in what follows:

Theorem 2.3. *(Rus [22]) Let (Y, d) be a complete metric space and $A, B : Y \to Y$ two operators. We suppose the following:*
(i) A is a contraction with contraction constant α and $F_A = \{x_A^*\}$.
(ii) B has fixed points and $x_B^* \in F_B$.
(iii) There exists $\eta > 0$ such that $d(A(x), B(x)) \le \eta$, for all $x \in Y$.
Then $d(x_A^*, x_B^*) \le \frac{\eta}{1-\alpha}$.

Theorem 2.4. *(Rus [25]) Let (X, d) be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose the following:*
(i) There exists $\alpha \in [0, 1)$ such that

$$d(A^2(x), A(x)) \le \alpha d(x, A(x))$$
$$d(B^2(x), B(x)) \le \alpha d(x, B(x))$$

for all $x \in X$

(ii) There exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$ for all $x \in X$.
Then $H(F_A, F_B) \leq \frac{\eta}{1-\alpha}$ where H denotes the Pompeiu-Hausdorff functional.

Theorem 2.5. *(Rus [24]) Let (X, d) be a metric space. Then $A : X \to X$ is a weakly Picard operator if and only if there exists a partition $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ of X such that*
(i) $X_\lambda \in I(A)$,
(ii) $A\,|_{X_\lambda}: X_\lambda \to X_\lambda$ is a Picard operator, for all $\lambda \in \Lambda$.

Consider a Banach space $(X, \|\cdot\|)$, let $\|\cdot\|_B$ and $\|\cdot\|_C$ be the Bielecki and Chebyshev norms on $C(J, X)$ defined by

$$\|x\|_B = \max_{t\in J} \|x(t)\|e^{-\tau t}(\tau > 0) \text{ and } \|x\|_C = \max_{t\in J} \|x(t)\|$$

and denote by d_B and d_C their corresponding metrics.

We consider the set

$$C_L^{q-q^*}(J,X)=\Big\{x \in C(J, X) : \|x(t_1) - x(t_2)\| \leq L|t_1 - t_2|^{q-q^*} \text{ for all } t_1, t_2 \in J\Big\}$$

where $L > 0$, $q^* \in (0, q)$, and

$$C^q_{\overline{L}}(J, X) = \big\{x \in C(J, X) : \|x(t_1) - x(t_2)\| \leq \overline{L}|t_1 - t_2|^q \text{ for all } t_1, t_2 \in J\big\}$$

where $\overline{L} > 0$, and

$$C^q_{\overline{L}}(J, B_R) = \big\{x \in C(J, B_R) : \|x(t_1) - x(t_2)\| \leq \overline{L}|t_1 - t_2|^q \text{ for all } t_1, t_2 \in J\big\}$$

where $B_R = \{x \in X : \|x\| \leq R\}$ with $R > 0$.
If $d \in \{d_C, d_B\}$, then $(C(J, X), d)$, $(C_L^{q-q^*}(J, X), d)$, $(C^q_{\overline{L}}(J, X), d)$ and $(C^q_{\overline{L}}(J, B_R), d)$ are complete metric spaces.

3. Main results via Picard operators

In the sequel, we use $\|\phi\|_{L^p(J)}$ to denote the $L^p(J, \mathbb{R}_+)$ norm of ϕ whenever $\phi \in L^p(J, \mathbb{R}_+)$ for some p with $1 < p < \infty$. Let $q_i \in (0, q)$, $i = 1, 2, 3$ and the functions $m \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$, $\eta \in L^{\frac{1}{q_2}}(J, \mathbb{R}_+)$, $\mu \in L^{\frac{1}{q_3}}(J, \mathbb{R}_+)$ and $l \in C(J, \mathbb{R}_+)$.

For brevity, let

$$M = \|m\|_{L^{\frac{1}{q_1}}(J)},\ N = \|\eta\|_{L^{\frac{1}{q_2}}(J)},\ V = \|\mu\|_{L^{\frac{1}{q_2}}(J)},\ L_0 = \max_{t\in J}\{l(t)\},$$

$$\beta = \frac{q-1}{1-q_1} \in (-1, 0),\ \gamma = \frac{q-1}{1-q_2} \in (-1, 0),\ \nu = \frac{q-1}{1-q_3} \in (-1, 0).$$

Consider the fractional integral equation (1.2). We have

Theorem 3.1. *Suppose the following conditions hold:*
(C1) $f \in C(J \times X, X)$.
(C2) There exists a constant $q_1 \in (0, q)$ and function $m \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$ such that $\|f(t, x)\| \leq m(t)$ for all $x \in X$ and all $t \in J$.
(C3) There exists a constant $L > 0$ such that $L \geq \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}$.
(C4) There exists a function $l \in C(J, \mathbb{R}_+)$ such that $\|f(t, u_1)-f(t, u_2)\| \leq l(t)\|u_1-u_2\|$ for all $u_i \in X$ $(i = 1, 2)$ and all $t \in J$.
(C5) There exist constants q_1 and τ such that $\frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1} < 1$.

Then the fractional Cauchy problem (1.1) has a unique solution x^* in $C_L^{q-q_1}(J,X)$, and this solution can be obtained by the successive approximation method, starting from any element of $C_L^{q-q_1}(J,X)$.

Proof. Consider the operator

$$A:(C_L^{q-q_1}(J,X),\|\cdot\|_B)\to(C_L^{q-q_1}(J,X),\|\cdot\|_B)$$

defined by

$$A(x)(t)=x_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f\left(s,x(s)\right)ds.$$

It is easy to see the operator A is well defined due to (C1).

Firstly, we check that $Ax\in C(J,X)$ for every $x\in C_L^{q-q_1}(J,X)$.

For any $\delta>0$, every $x\in C_L^{q-q_1}(J,X)$, by (C2), Hölder inequality,

$$\begin{aligned}
&\|A(x)(t+\delta)-A(x)(t)\|\\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t((t-s)^{q-1}-(t+\delta-s)^{q-1})m(s)ds\\
&+\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}m(s)ds\\
\leq\ & \frac{1}{\Gamma(q)}\left(\int_0^t[(t-s)^{q-1}-(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^t(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}\\
&+\frac{1}{\Gamma(q)}\left(\int_t^{t+\delta}[(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_t^{t+\delta}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}\\
\leq\ & \frac{M}{\Gamma(q)}\left(\int_0^t(t-s)^{\beta}-(t+\delta-s)^{\beta}ds\right)^{1-q_1}+\frac{M}{\Gamma(q)}\left(\int_t^{t+\delta}(t+\delta-s)^{\beta}ds\right)^{1-q_1}\\
\leq\ & \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}+\frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}\\
\leq\ & \frac{3M}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}.
\end{aligned}$$

It is easy to see that the right-hand side of the above inequality tends to zero as $\delta\to0$. Therefore $Ax\in C(J,X)$.

Secondly, we show that $Ax\in C_L^{q-q_1}(J,X)$.

Without lose of generality, for any $t_1<t_2$, $t_1,t_2\in J$, applying (C2) and Hölder inequality, we have

$$\begin{aligned}
&\|A(x)(t_2)-A(x)(t_1)\|\\
&\leq\frac{1}{\Gamma(q)}\left\|\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\right\|\\
&\leq\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]\|f(s,x(s))\|ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|f(s,x(s))\|ds\\
&\leq\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]m(s)ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}m(s)ds
\end{aligned}$$

$$\leq \frac{1}{\Gamma(q)}\left(\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^{t_1}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}$$
$$+\frac{1}{\Gamma(q)}\left(\int_{t_1}^{t_2}[(t_2-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_{t_1}^{t_2}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}$$
$$\leq \frac{M}{\Gamma(q)}\left(\int_0^{t_1}(t_1-s)^{\beta}-(t_2-s)^{\beta}ds\right)^{1-q_1}+\frac{M}{\Gamma(q)}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta}ds\right)^{1-q_1}$$
$$\leq \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}\left(t_1^{1+\beta}-t_2^{1+\beta}+(t_2-t_1)^{1+\beta}\right)^{1-q_1}$$
$$+\frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}(t_2-t_1)^{(1+\beta)(1-q_1)}$$
$$\leq \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|t_1-t_2|^{(1+\beta)(1-q_1)}\leq \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|t_1-t_2|^{q-q_1}.$$

Similarly, for any $t_1 > t_2$, $t_1, t_2 \in J$, we also have the above inequality. This implies that Ax is belong to $C_L^{q-q_1}(J, X)$ due to (C3).

Thirdly, A is continuous.

For that, let $\{x_n\}$ be a sequence of B_R such that $x_n \to x$ in B_R. Then, $f(s, x_n(s)) \to f(s, x(s))$ as $n \to \infty$ due to (C1). On the one other hand using (C2), we get for each $s \in [0, t]$, $\|f(s, x_n(s)) - f(s, x(s))\| \leq 2m(s)$. On the other hand, using the fact that the functions $s \to 2(t-s)^{q-1}m(s)$ is integrable on $[0, t]$, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1}\|f(s, x_n(s)) - f(s, x(s))\|ds \to 0.$$

For all $t \in J$, we have

$$\|A(x_n)(t) - A(x)(t)\| \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s, x_n(s)) - f(s, x(s))\|ds.$$

Thus, $Ax_n \to Ax$ as $n \to \infty$ which implies that A is continuous.

Moreover, for all $x, z \in C_L^{q-q_1}(J, X)$, using (C4) and Hölder inequality we have

$$\begin{aligned}
\|A(x)(t) - A(z)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s, x(s) - f(s, z(s))\|ds \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}l(s)\|x(s) - z(s)\|ds \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\max_{s\in[0,t]}\{l(s)\}\left[\|x(s) - z(s)\|e^{-\tau s}\right]e^{\tau s}ds \\
&\leq \frac{L_0}{\Gamma(q)}\|x - z\|_B\int_0^t (t-s)^{q-1}e^{\tau s}ds \\
&\leq \frac{L_0}{\Gamma(q)}\|x - z\|_B\left(\int_0^t (t-s)^{\beta}ds\right)^{1-q_1}\left(\int_0^t e^{\frac{\tau s}{q_1}}ds\right)^{q_1} \\
&\leq \frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1}e^{\tau t}\|x - z\|_B.
\end{aligned}$$

It follows that

$$\|A(x)(t) - A(z)(t)\|e^{-\tau t} \leq \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \|x - z\|_B$$

for all $t \in J$. So we have

$$\|A(x) - A(z)\|_B \leq \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \|x - z\|_B$$

for all $x,\ z \in C_L^{q-q_1}(J,X)$. The operator A is of Lipschitz type with constant

$$L_A = \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \tag{3.1}$$

and $0 < L_A < 1$ due to (C5). By applying the Contraction Principle to this operator we obtain that A is a Picard operator. This completes the proof. $\square$

Example: Consider the fractional Cauchy problem

$$\begin{cases} {}^cD^q x(t) = x(t),\ q = \frac{1}{2}, \\ x(0) = 0 \in X, \end{cases} \tag{3.2}$$

on $[0,1]$. Set $L_0 = 1$, $T = 1$, $q_1 = \frac{1}{3}$, then $\beta = -\frac{3}{4}$. Indeed

$$\frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1 \iff \frac{qL_0}{\Gamma(q+1)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1,$$

which implies that we must choose a suitable $\tau_0 > 0$ such that $\frac{\frac{1}{2}}{\Gamma(\frac{3}{2})} \frac{1}{(\frac{1}{4})^{\frac{2}{3}}} \left(\frac{\frac{1}{3}}{\tau_0}\right)^{\frac{1}{3}} < 1$. Noting that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, for $\tau_0 = \frac{16}{9} > \frac{16}{3\sqrt{\pi^3}}$ we have the condition (C5) in Theorem 3.1.

Theorem 3.2. *Suppose the following conditions hold:*

(C1) $f \in C(J \times X, X)$.

(C2′) There exists a constant $\overline{M} > 0$ such that $\|f(t,x)\| \leq \overline{M}$ for all $x \in X$ and all $t \in J$.

(C3′) There exists a constant $\overline{L} > 0$ such that $\overline{L} \geq \frac{2\overline{M}}{\Gamma(q+1)}$.

(C4′) There exists a constant $\overline{L}_0 > 0$ such that $\|f(t,u_1) - f(t,u_2)\| \leq \overline{L}_0\|u_1 - u_2\|$ for all $u_i \in X$ $(i = 1,2)$ and all $t \in J$.

(C5′) There exist constants q_1 and τ such that $\overline{L}_{\overline{A}} = \frac{\overline{L}_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1$.

Then the fractional Cauchy problem (1.1) has a unique solution x^* in $C_{\overline{L}}^q(J,X)$, and this solution can be obtained by the successive approximation method, starting from any element of $C_{\overline{L}}^q(J,X)$.

Proof. Consider the following continuous operator

$$\overline{A} : (C_{\overline{L}}^q(J,X), \|\cdot\|_B) \to (C_{\overline{L}}^q(J,X), \|\cdot\|_B)$$

defined by

$$\overline{A}(x)(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\left(s, x(s)\right) ds.$$

As the proof in Theorem 3.1, applying the given conditions one can verify that

$$\|\overline{A}(x)-\overline{A}(z)\|_B \le \frac{\overline{L}_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1}\|x-z\|_B$$

for all $x,\ z \in C_L^q(J,X)$. So, the operator $\overline{A}$ is a Picard operator. The proof is completed. □

Similarly as above, we can prove

Theorem 3.3. *Suppose the following conditions hold:*

(C1′) $f \in C(J \times B_R, X)$.

(C2″) There exists a constant $\overline{M}(R) > 0$ *such that* $\|f(t,x)\| \le \overline{M}(R)$ *for all* $x \in B_R$ *and all* $t \in J$ *with* $R \ge \|x_0\| + \frac{\overline{M}(R)T^q}{\Gamma(q+1)}$.

(C3″) There exists a constant $\overline{L} > 0$ *such that* $\overline{L} \ge \frac{2\overline{M}(R)}{\Gamma(q+1)}$.

(C4″) There exists a constant $\overline{L}_0 > 0$ *such that* $\|f(t,u_1)-f(t,u_2)\| \le \overline{L}_0\|u_1-u_2\|$ *for all* $u_i \in B_R$ $(i=1,2)$ *and all* $t \in J$.

(C5′) There exist constants q_1 *and* τ *such that* $\overline{L}_{\overline{A}} = \frac{\overline{L}_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1} < 1$.

Then the fractional Cauchy problem (1.1) has a unique solution x^* *in* $C_L^q(J,B_R)$, *and this solution can be obtained by the successive approximation method, starting from any element of* $C_L^q(J,B_R)$.

Consider the following new fractional Cauchy problem

$$\begin{cases} {}^cD^q x(t) = g(t,x(t)),\ t \in J, \\ x(0) = y_0 \in X, \end{cases} \tag{3.3}$$

where $g \in C(J \times X, X)$. By Lemma 1.6, a function $x \in C(J,X)$ is a solution of the fractional integral equation

$$x(t) = y_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g\left(s,x(s)\right) ds, \tag{3.4}$$

if and only if x is a solution of the fractional Cauchy problem (3.3).

Now, we consider both fractional integral equation (1.2) and (3.4). We have

Theorem 3.4. *Suppose the following:*

(D1) All conditions in Theorem 3.1 are satisfied and $x^* \in C_L^{q-q_1}(J,X)$ *is the unique solution of the fractional integral equation (1.2).*

(D2) With the same function m *as in Theorem 3.1,* $\|g(t,x)\| \le m(t)$ *for all* $x \in X$ *and all* $t \in J$.

(D3) With the same function l *as in Theorem 3.1,* $\|g(t,u_1)-g(t,u_2)\| \le l(t)\|u_1 - u_2\|$ *for all* $u_i \in X$ $(i=1,2)$ *and all* $t \in J$.

(D4) $L \ge \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}$.

(D5) There exists a constant $q_2 \in (0,q)$ *and function* $\eta \in L^{\frac{1}{q_2}}(J,\mathbb{R}_+)$ *such that* $\|f(t,u)-g(t,u)\| \le \eta(t)$ *for all* $u \in X$ *and all* $t \in J$.

Then, if y^* *is the solution of the fractional integral equation (3.4),*

$$\|x^*-y^*\|_B \le \frac{\|x_0-y_0\| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}}{1-L_A} \tag{3.5}$$

where L_A is given by (3.1) with $\tau = \tau_0 > 0$ such that $0 < L_A < 1$.

Proof. Consider the following two operators

$$A,\ B : (C_L^{q-q_1}(J,X), \|\cdot\|_B) \to (C_L^{q-q_1}(J,X), \|\cdot\|_B)$$

defined by

$$A(x)(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\left(s, x(s)\right) ds,$$
$$B(x)(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds,$$

on J. We have

$$\begin{aligned}
\|A(x)(t) - B(x)(t)\| &\le \|x_0 - y_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f\left(s, x(s)\right) - g\left(s, x(s)\right)\| ds \\
&\le \|x_0 - y_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \eta(s) ds \\
&\le \|x_0 - y_0\| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}},
\end{aligned}$$

for $t \in J$. It follows that

$$\|A(x) - B(x)\|_B \le \|x_0 - y_0\| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}.$$

So we can apply Theorem 2.3 to obtain (3.5) which completes the proof. □

Remarks. (a) All the results obtained in Theorem 3.1 hold even if the condition (C2) is replaced by the following:

(C2-E) There exists a constant $q_1 \in [0, q)$ and function $m \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$ such that $\|f(t,x)\| \le m(t)$ for all $x \in X$ and all $t \in J$.

In fact, we only need extend the space $L^p(J, \mathbb{R}_+)(1 < p < \infty)$ to $L^p(J, \mathbb{R}_+)(1 \le p \le \infty)$ where $L^p(J, \mathbb{R}_+)(1 \le p \le \infty)$ be the Banach space of all Lebesgue measurable functions $\phi : J \to \mathbb{R}_+$ with $\|\phi\|_{L^p(J)} < \infty$ and the norm $\|\cdot\|_{L^p(J)}$ is defined by

$$\|\phi\|_{L^p(J)} := \begin{cases} \left(\int_J (\phi(t))^p dt \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \\ \inf\limits_{\mathrm{mes}(\bar{J})=0} \left\{ \sup\limits_{t \in J - \bar{J}} (\phi(t)) \right\}, \ p = \infty, \end{cases}$$

where $mes(\bar{J})$ is the Lebesgue measure of $\bar{J}$.

(b) The results obtained in this section can be generalized to study existence, uniqueness and data dependence for the solutions of the following problems.

(i) Problem with linear modification of the argument

$$\begin{cases} {}^cD^q x(t) = f\left(t, x(\lambda t)\right), \ 0 < \lambda < 1, \ t \in J, \\ x(0) = x_0 \in X. \end{cases}$$

(ii) Nonlocal problems

$$\begin{cases} {}^cD^q x(t) = f\left(t, x(t)\right), \ t \in J, \\ x(0) = x_0 + G(x), \end{cases}$$

where $G : C(J, X) \to X$ is the nonlocal term.

(iii) Integro-Differential equations of mixed type

$$\begin{cases} {}^cD^q x(t) = F\left(t, x(t), \int_0^t K_1(t,s)x(s)ds, \int_0^T K_2(t,s)x(s)ds\right), \ t \in J, \\ x(0) = x_0, \end{cases}$$

where $F \in C(J \times X^3, X)$, $K_i \in C(D_i, \mathbb{R})$ $(i = 1, 2)$, $D_1 = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$, $D_2 = J \times J$.

4. Main results via weakly Picard operators

Now, we consider another fractional integral equation

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{4.1}$$

on J, where $f \in C(J \times X, X)$ is as in the fractional Cauchy problem (1.1). We have

Theorem 4.1. *Suppose that for the fractional integral equation (4.1) the same conditions as in Theorem 3.1 are satisfied. Then this equation has solutions in $C_L^{q-q_1}(J, X)$. If $\mathcal{S} \subset C_L^{q-q_1}(J, X)$ is its solutions set, then card $\mathcal{S}$= card X.*

Proof. Consider the operator

$$A_* : (C_L^{q-q_1}(J, X), \|\cdot\|_B) \to (C_L^{q-q_1}(J, X), \|\cdot\|_B)$$

defined by

$$A_*(x)(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

This is a continuous operator, but not a Lipschitz one. We can write

$$C_L^{q-q_1}(J, X) = \bigcup_{\alpha \in X} X_\alpha, \ X_\alpha = \left\{x \in C_L^{q-q_1}(J, X) : x(0) = \alpha\right\}.$$

We have that X_α is an invariant set of A_* and we apply Theorem 3.1 to $A_*|_{X_\alpha}$. By using Theorem 2.5 we obtain that A_* is a weakly Picard operator.

Consider the operator

$$A_*^\infty : C_L^{q-q_1}(J, X) \to C_L^{q-q_1}(J, X), \ A_*^\infty(x) = \lim_{n\to\infty} A_*^n(x).$$

From $A_*^{n+1}(x) = A_*(A_*^n(x))$ and the continuity of A_*, $A_*^\infty(x) \in F_{A_*}$. Then

$$A_*^\infty(C_L^{q-q_1}(J, X)) = F_{A_*} = \mathcal{S} \text{ and } \mathcal{S} \ne \emptyset.$$

So, *card $\mathcal{S}$= card X.* □

Theorem 4.2. *Suppose that for the fractional integral equation (4.1) the same conditions as in Theorem 3.2 are satisfied. Then this equation has solutions in $C_{\overline{L}}^q(J, X)$. If $\mathcal{S} \subset C_{\overline{L}}^q(J, X)$ is its solutions set, then card $\mathcal{S}$= card X.*

Proof. As the proof in Theorem 4.1, we need to consider the continuous operator (but not a Lipschitz one)

$$\overline{A}_*:(C^{q}_{\overline{L}}(J,X),\|\cdot\|_B)\to(C^{q}_{\overline{L}}(J,X),\|\cdot\|_B)$$

defined by

$$\overline{A}_*(x)(t)=x(0)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f\left(s,x(s)\right)ds.$$

We can write $C^{q}_{\overline{L}}(J,X)=\bigcup_{\alpha\in X}\overline{X}_\alpha,\ \overline{X}_\alpha=\left\{x\in C^{q}_{\overline{L}}(J,X):x(0)=\alpha\right\}$. We have that $\overline{X}_\alpha$ is an invariant set of $\overline{A}_*$ and we apply Theorem 3.2 to $\overline{A}_*|_{X_\alpha}$. By using Theorem 2.5 we obtain that $\overline{A}_*$ is a weakly Picard operator. Consider the operator $\overline{A}_*^{\infty}:C^{q}_{\overline{L}}(J,X)\to C^{q}_{\overline{L}}(J,X)$, $\overline{A}_*^{\infty}(x)=\lim_{n\to\infty}\overline{A}_*^{n}(x)$. From $\overline{A}_*^{n+1}(x)=\overline{A}_*(\overline{A}_*^{n}(x))$ and the continuity of $\overline{A}_*$, $\overline{A}_*^{\infty}(x)\in F_{\overline{A}_*}$. Then $\overline{A}_*^{\infty}(C^{q}_{\overline{L}}(J,X))=F_{\overline{A}_*}=\mathcal{S}$ and $\mathcal{S}\neq\emptyset$. So, *card* $\mathcal{S}$= *card* X. □

Similarly as above, we can prove

Theorem 4.3. *Suppose that for the fractional integral equation (4.1) the same conditions as in Theorem 3.3 are satisfied. Then this equation has solutions in* $C^{q}_{\overline{L}}(J,B_R)$*. If* $\mathcal{S}\subset C^{q}_{\overline{L}}(J,B_R)$ *is its solutions set, then card* $\mathcal{S}$= *card* B_R.

In order to study data dependence for the solutions set of the fractional integral equation (4.1) we consider both (4.1) and the following fractional integral equation

$$x(t)=x(0)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g\left(s,x(s)\right)ds$$

on J where $g\in C(J\times X,X)$. Let $\mathcal{S}_1$ be the solutions set of this equation.

Theorem 4.4. *Suppose the following conditions:*

(E1) There exists a function $l\in C(J,\mathbb{R}_+)$ *such that*

$$\|f(t,u_1)-f(t,u_2)\|\le l(t)\|u_1-u_2\| \text{ and } \|g(t,u_1)-g(t,u_2)\|\le l(t)\|u_1-u_2\|$$

for all $u_i\in X$ $(i=1,2)$ *and all* $t\in J$.

(E2) There exists a constant $q_1,q_3\in(0,q)$ *and functions* $m\in L^{\frac{1}{q_1}}(J,\mathbb{R}_+)$, $\mu\in L^{\frac{1}{q_3}}(J,\mathbb{R}_+)$ *such that*

$$\|f(t,x)\|\le m(t) \text{ and } \|g(t,x)\|\le\mu(t)$$

for all $x\in X$ *and all* $t\in J$.

(E3) There exists a constant $L>0$ *such that*

$$L\ge\frac{2\max\{M,V\}}{\Gamma(q)\min\left\{(1+\beta)^{1-q_1},(1+\nu)^{1-q_3}\right\}}.$$

(E4) There exists a constant $q_2\in(0,q)$ *and functions* $\eta\in L^{\frac{1}{q_2}}(J,\mathbb{R}_+)$

$$\|f(t,u)-g(t,u)\|\le\eta(t)$$

for all $u\in X$ *and all* $t\in J$.

(E5) $\frac{L_0T^q}{\Gamma(q+1)}<1$.

Then

$$H_{\|\cdot\|_C}(\mathcal{S},\mathcal{S}_1) \;\leq\; \frac{\frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}}{1-\frac{L_0T^q}{\Gamma(q+1)}}$$

where by $H_{\|\cdot\|_C}$ *we denote the Pompeiu-Hausdorff functional with respect to* $\|\cdot\|_C$ *on* $C_L^{q-q_1}(J,X)$.

Proof. Consider the operator

$$B_* : (C_L^{q-q_1}(J,X),\|\cdot\|_B) \to (C_L^{q-q_1}(J,X),\|\cdot\|_B)$$

defined by

$$B_*(x)(t) = x(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g\,(s,x(s))\,ds, \text{ for } t\in J.$$

Because of (E1)-(E3), $A_*, B_* : (C_L^{q-q_1}(J,X),\|\cdot\|_B) \to (C_L^{q-q_1}(J,X),\|\cdot\|_B)$ two orbitally continuous operators. Moreover, we have

$$\begin{aligned}\|A_*^2(x)(t)-A_*(x)(t)\| &\leq \frac{L_0}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A_*(x)(s)-x(s)\|ds\\ &\leq \frac{L_0T^q}{\Gamma(q+1)}\|A_*(x)-x\|_C.\end{aligned}$$

for all $x\in C_L^{q-q_1}(J,X)$. Similarly,

$$\|B_*^2(x)(t)-B_*(x)(t)\| \;\leq\; \frac{L_0T^q}{\Gamma(q+1)}\|B_*(x)-x\|_C$$

for all $x\in C_L^{q-q_1}(J,X)$. It follows that

$$\|A_*^2(x)-A_*(x)\|_C \leq \frac{L_0T^q}{\Gamma(q+1)}\|A_*(x)-x\|_C,$$

$$\|B_*^2(x)-B_*(x)\|_C \leq \frac{L_0T^q}{\Gamma(q+1)}\|B_*(x)-x\|_C.$$

Because of (E4),

$$\begin{aligned}\|A_*(x)-B_*(x)\|_C &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\eta(s)ds\\ &\leq \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}},\end{aligned}$$

for all $x\in C_L^{q-q_1}(J,X)$.

By (E5) and applying Theorem 2.4 we obtain

$$H_{\|\cdot\|_C}(F_{A_*},F_{B_*}) \;\leq\; \frac{\frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}}{1-\frac{L_0T^q}{\Gamma(q+1)}}$$

and the theorem is proved. $\square$

Theorem 4.5. *Suppose the following conditions:*

(E1′) There exists a constant $L_* > 0$ such that

$$\|f(t,u_1) - f(t,u_2)\| \le L_* \|u_1 - u_2\| \text{ and } \|g(t,u_1) - g(t,u_2)\| \le L_* \|u_1 - u_2\|$$

for all $u_i \in X$ $(i = 1,2)$ and all $t \in J$.

(E2′) There exists a constant $M_* > 0$ such that

$$\|f(t,x)\| \le M_* \text{ and } \|g(t,x)\| \le M_*$$

for all $x \in X$ and all $t \in J$.

(E3′) There exists a constant $\overline{L} > 0$ such that

$$\overline{L} \ge \frac{2M_*}{\Gamma(q+1)}.$$

(E4′) There exists a constant $\eta_* > 0$ such that

$$\|f(t,u) - g(t,u)\| \le \eta_*$$

for all $u \in X$ and all $t \in J$.

(E5′) $\frac{L_* T^q}{\Gamma(q+1)} < 1$.

Then we have

$$\overline{H}_{\|\cdot\|_C}(\mathcal{S}, \mathcal{S}_1) \ \le \ \frac{\frac{\eta_* T^q}{\Gamma(q+1)}}{1 - \frac{L_* T^q}{\Gamma(q+1)}}$$

where by $\overline{H}_{\|\cdot\|_C}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C^q_{\overline{L}}(J,X)$.

Proof. Consider the operator

$$\overline{B}_* : (C^q_{\overline{L}}(J,X), \|\cdot\|_B) \to (C^q_{\overline{L}}(J,X), \|\cdot\|_B)$$

defined by

$$\overline{B}_*(x)(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds, \text{ for } t \in J.$$

Applying (E1′)-(E3′), $\overline{A}_*, \overline{B}_* : (C^{q-q_1}_{\overline{L}}(J,X), \|\cdot\|_B) \to (C^{q-q_1}_{\overline{L}}(J,X), \|\cdot\|_B)$ two orbitally continuous operators. Moreover, we have

$$\begin{aligned}
\|\overline{A}_*^2(x)(t) - \overline{A}_*(x)(t)\| \ &\le \ \frac{L_* T^q}{\Gamma(q+1)} \|\overline{A}_*(x) - x\|_C, \\
\|\overline{B}_*^2(x)(t) - \overline{B}_*(x)(t)\| \ &\le \ \frac{L_* T^q}{\Gamma(q+1)} \|\overline{B}_*(x) - x\|_C,
\end{aligned}$$

for all $x \in C^q_{\overline{L}}(J,X)$. It follows that

$$\|\overline{A}_*^2(x) - \overline{A}_*(x)\|_C \le \frac{L_* T^q}{\Gamma(q+1)} \|\overline{A}_*(x) - x\|_C$$

$$\|\overline{B}_*^2(x) - \overline{B}_*(x)\|_C \le \frac{L_* T^q}{\Gamma(q+1)} \|\overline{B}_*(x) - x\|_C.$$

Because of (E4′), we obtain

$$\|\overline{A}_*(x) - \overline{B}_*(x)\|_C \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\eta_* ds \leq \frac{\eta_* T^q}{\Gamma(q+1)},$$

for all $x \in C^q_{\overline{L}}(J, X)$.
By (E5′) and applying Theorem 2.4 we obtain the result and the theorem is proved. □
Similarly as above, we can prove

Theorem 4.6. *Suppose the following:*
(E1′′) There exists a constant $L_* > 0$ *such that*

$$\|f(t,u_1) - f(t,u_2)\| \leq L_*\|u_1 - u_2\| \text{ and } \|g(t,u_1) - g(t,u_2)\| \leq L_*\|u_1 - u_2\|$$

for all $u_i \in B_R$ $(i = 1, 2)$ *and all* $t \in J$.
(E2′′) There exists a constant $M_*(R) > 0$ *such that*

$$\|f(t,x)\| \leq M_*(R) \text{ and } \|g(t,x)\| \leq M_*(R)$$

for all $x \in B_R$ *and all* $t \in J$ *with*

$$R \geq \|x(0)\| + \frac{M_*(R)T^q}{\Gamma(q+1)}.$$

(E3′′) There exists a constant $\overline{L} > 0$ *such that*

$$\overline{L} \geq \frac{2M_*(R)}{\Gamma(q+1)}.$$

(E4′) There exists a constant $\eta_* > 0$ *such that*

$$\|f(t,u) - g(t,u)\| \leq \eta_*$$

for all $u \in B_R$ *and all* $t \in J$.
(E5′) $\frac{L_* T^q}{\Gamma(q+1)} < 1$.
Then

$$\overline{H}_{\|\cdot\|_C}(\mathcal{S}, \mathcal{S}_1) \leq \frac{\frac{\eta_* T^q}{\Gamma(q+1)}}{1 - \frac{L_* T^q}{\Gamma(q+1)}}$$

where by $\overline{H}_{\|\cdot\|_C}$ *we denote the Pompeiu-Hausdorff functional with respect to* $\|\cdot\|_C$ *on* $C^q_{\overline{L}}(J, B_R)$.

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