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AN EFFICIENT ALGORITHM FOR SOLVING HIGH ORDER STURM-LIOUVILLE PROBLEMS USING VARIATIONAL ITERATION METHOD

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Abstract. In this paper, a novel numerical algorithm based on generalized variational iteration method for the solution of every $2m$-order Sturm-Liouville problem for $m \geq 1$ is proposed. In this approach, a Lagrange multiplier is identified to establish suitable correction functional to construct an approximate solution which it is considered as the fixed point of the corresponding correction functional. It is proved that this algorithm converges to the corresponding exact solution. Error estimate for the algorithm is given. Numerical simulations show that this algorithm is easy to implement and produces accurate results. Numerical results are given.
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1. Introduction

The Sturm-Liouville theory has many applications in applied mathematics, physics and engineering. Many physical phenomena, both in classical mechanics and in quantum mechanics are described mathematically by second-order Sturm-Liouville problems (see [1-3] for more details). While the problems arise in the stability of hydrodynamic and magnetohydrodynamic, are almost always of high order. It is either because they involve a coupled system of ordinary differential equations, or they have been reduced to a single differential equation of order $2m$, for an integer $m > 1$, (for example see [4-6]).

The numerical techniques referred to as high-order Sturm-Liouville problems have been less widely used compared with the more familiar second-order and fourth-order types. Although there are some available software codes like "SLEIGN" [7], "SLEIGN2" [8] and "SLEIGDGE" [9] for the solution of second-order Sturm-Liouville problems and "SLEUTH" [10] for solving fourth-order Strum-Liouville problems, up to the knowledge of the authors there is no software and code for solving sixth-order Sturm-Liouville problems. In year 1998 Greenberg and Marletta [5] used the shooting method to approximate the eigenvalues of sixth-order Sturm-Liouville problems. In year 2007, Attili and Lesnic [11] used the Adomian decomposition method (ADM) to solve sixth-order Sturm-Liouville problems.

Variational iteration method (VIM) has been successfully implemented to handle linear and non-linear differential equations (for example see [12-21]). The main property of VIM is its flexibility and ability to solve non-linear differential equations accurately and conveniently. The VIM was developed by J.H. He [12-20] during the years 1998-2010. This method has been extensively applied as a powerful tool for solving various kinds of problems, such as: autonomous ordinary differential equations and approximated solutions of some nonlinear problems. Recently, Goh, Noorani and Hashim [22] used the VIM for solving the chaotic Chen system. Safari, Ganji and Sadeghi [23] implemented the VIM for the solution of Benney-Lin equation while Wazwaz [24] used VIM for solving variational problems. Although, there have been a lot of papers on the 'Variational Iteration Method' for solving problems involving ordinary differential equations, with the knowledge of the authors there is no solution for high order Sturm-Liouville problems. This paper is an application of the fixed-point iteration method to higher order ODE eigenvalue problems. In this paper, we will extend the VIM for finding the eigenvalues of $2m$-order non-singular Sturm-Liouville problem of the form

$$(-1)^m (p_m(x) y^{(m)})^{(m)} + (-1)^{m-1} (p_{m-1}(x) y^{(m-1)})^{(m-1)}$$

$$+ \cdots + (p_2(x) y'')'' - (p_1(x) y')' + p_0(x) y = E w(x) y, \quad a < x < b, \tag{1.1}$$

together with separated, self-adjoint boundary conditions imposed at $x = a$ and $x = b$. We assume that all coefficient functions are real valued. The technical conditions for the problem to be non-singular are: the interval (a, b) is finite; the coefficient functions $p_k\,(0 \leq k \leq m-1)$, $w(x)$ and $1/p_m(x)$ are in $L^1(a, b)$, and $p_m(x)$ and weight function $w(x)$ are both positive.

In the Section 2 we give some preliminary definitions for non-singular $2m$-order Sturm-Liouville problems and basic idea of VIM method. In Section 3 we will propose a novel algorithm of VIM for the $2m$-order Sturm-Liouville problems. In Section 4, while numerical results are discussed several work examples are solved to demonstrate high performance of the proposed method.

2. Preliminaries

In this section we introduce some notation and definitions necessary for this work.

2.1. $2m$-order Sturm-Liouville problems. Let us rewrite equation (1.1) in the following form

$$\begin{aligned}(-1)^m (p_m(x) y^{(m)})^{(m)} &= F(y, y', \ldots, y^{(2m-2)}, E) \\ &= (E w(x) - p_0(x)) y - \{(-1)^{m-1} (p_{m-1}(x) y)^{(m-1)} \\ &+ \cdots + (p_2(x) y'')''\}, \quad a < x < b,\end{aligned} \tag{2.1}$$

subject to some $2m$ point specified conditions at the boundary $x \in \{a, b\}$ on

$$
\begin{aligned}
&u_k = y^{(k-1)}, \ \ 1 \le k \le m, \\
&v_1 = p_1 y' - (p_2 y'')' + (p_3 y''')'' + \cdots + (-1)^{m-1}(p_m y^{(m)})^{(m-1)}, \\
&v_2 = p_2 y'' - (p_3 y''')' + (p_4 y^{(4)})'' + \cdots + (-1)^{m-2}(p_m y^{(m)})^{(m-2)}, \\
&\ \vdots \\
&v_k = p_k y^{(k)} - (p_{k+1} y^{(k+1)})' + (p_{k+2} y^{(k+2)})'' + \cdots + (-1)^{m-k}(p_m y^{(m)})^{(m-k)}, \\
&\ \vdots \\
&v_m = p_m y^{(m)}.
\end{aligned}
\tag{2.2}
$$

The eginvalues $E_k, \ \ k = 1, 2, 3, \ldots$ can be ordered as an increasing sequence, i.e.,

$$E_1 \le E_2 \le E_3 \le \cdots ,$$

where $\lim_{k \to \infty} E_k = \infty$ and each eigenvalue has multiplicity at most m. The restriction on the multiplicity arises from the fact that for each $E_k, \ \ k = 1, 2, 3, \ldots$ there are at most m linear independent solutions of the differential equation satisfying either of the endpoint conditions, [6, 25].

Let $L^2_w(a, b)$, be the space of functions $f(x)$ on (a, b) such that

$$\int_a^b |f(x)|^2 \, w(x) dx < \infty.$$

$L^2_w(a, b)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} w(x) dx,$$

and norm $\|f\|^2 = \langle f, f \rangle$.

2.2. **Basic idea of variational iteration method.** Consider the general nonlinear differential equation given in the form

$$Ly(x) + Ny(x) = g(x), \tag{2.3}$$

where $g(x)$ is a given function, L and N are some linear and nonlinear operators respectively. By using the variational iteration method, a correction functional can be constructed as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda \left(Ly_n(s) + N\tilde{y}_n(s) - g(s) \right) ds, \quad n \ge 0, \tag{2.4}$$

where λ is a general Lagrange multiplier [26], which can be identified optimally via the variational theory, the index n means the nth order approximation for y_n, and $\tilde{y}_n$ is a restricted variation with the property $\delta \tilde{y}_n = 0$, (see for example, [12-20]).

3. VIM generalized algorithm and convergence analysis

3.1. **VIM generalized algorithm.** In this subsection, we implement the generalized VIM for $2m$-order Sturm-Liouville problem (1.1). Let us rewrite equation (2.1) in the following form

$$\begin{aligned} y^{(2m)}(x) &= G(y(x), y'(x), \ldots, y^{(2m-1)}(x), E) \\ &= (F(y(x),\, y'(x),\, \ldots,\, y^{(2m-2)}(x),\, E) \\ &-(p'(x)y^{(2m-1)}(x) + \cdots + p_m^{(m)}(x)y^{(m)}(x))/p_m(x), \\ &\quad x \in (a,b), \end{aligned} \tag{3.1}$$

which can be written in the operator form as

$$Ly(x) + Ny(x) = 0, \tag{3.2}$$

where $Ly(x) = y^{(2m)}(x)$ and $Ny(x) = -G(y, y', \ldots, y^{(2m-1)}, E)$. According to variational iteration method [12-20], we can construct a correct functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\,(Ly_n(s) + N\tilde{y}_n(s))\,ds, \tag{3.3}$$

where λ is a general Lagrange multiplier. Now, first we determine λ and then we give details of how to calculate eigenvalues for the problem (1.1).The Lagrange multiplier λ will be identified via integration by parts from equation (3.3), i.e.,

$$\begin{aligned} y_{n+1}(x) &= y_n(x) \\ &+ \left[\lambda(s)y_n^{(2m-1)} - \lambda'(s)y_n^{(2m-2)} + \cdots + \lambda^{(2m-2)}(s)y' - \lambda^{(2m-1)}(s)y\right]_{s=0}^{s=x} \\ &+ \textstyle\int_0^x \left(\lambda^{(2m)}(s)y_n(s) + \lambda N\tilde{y}_n(s)\right)ds. \end{aligned} \tag{3.4}$$

By taking the variation on both sides of equation (3.4) with respect to y_n, and by noticing that $\delta y_n(0) = 0$, one can obtain

$$\begin{aligned} \delta y_{n+1}(x) &= (1 - \lambda^{(2m-1)}(x))\delta y_n(x) + \lambda(x)\delta y_n^{(2m-1)}(x) - \lambda'(x)\delta y_n^{(2m-2)}(x) \\ &+ \cdots + \lambda^{(2m-2)}(x)\delta y'(x) + \textstyle\int_0^x \left(\lambda^{(2m)}(s)\delta y_n(s) + \lambda N\delta\tilde{y}_n(s)\right)ds. \end{aligned} \tag{3.5}$$

Consequently, we obtain the following stationary conditions

$$\begin{aligned} &\delta y_n : \lambda^{(2m)}(s) = 0, \\ &\delta y_n : 1 - \lambda^{(2m-1)}(s)\big|_{s=x} = 0, \\ &\delta y_n' : \ \lambda^{(2m-2)}(s)\,|_{s=x} = 0, \\ &\quad\vdots \\ &\delta y_n^{(2m-2)} : \lambda'(s)\,|_{s=x} = 0\ , \\ &\delta y_n^{(2m-1)} : \lambda(s)\,|_{s=x} = 0\ . \end{aligned} \tag{3.6}$$

From equations (3.6), the Lagrange multiplier can be derived as

$$\lambda = \frac{(s-x)^{2m-1}}{(2m-1)!}. \tag{3.7}$$

Now, by substituting equation (3.7) into (3.3) we get the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^{(2m-1)}}{(2m-1)!}\left(Ly_n(s) + Ny_n(s)\right) ds, \tag{3.8}$$

where $y_0(x)$ can be chosen to be the solution of the equation $Ly(x) = y_0^{(2m)}(x) = 0$. Thus

$$y_0(x) = \sum_{i=0}^{2m-1} c_i x^i, \tag{3.9}$$

where $c_0, c_1, \ldots, c_{2m-1}$ are some constants. From (3.8), we obtain the successive approximations of problem (1.1) and the exact solution can be derived in the following form

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{3.10}$$

In fact, the solution of problem (3.1) is considered as the fixed point of the correction functional (3.8) under the suitable choice of the initial term that is given by (3.9).

Note that exactly m conditions are specified initially at $x = a$, (these m conditions arise in different forms based on nature of the problem such as order of the highest derivative appearing in each condition must be less than $2m$). Now, if these m conditions at $x = a$ have the following form

$$y_n(a,E) = y_n'(a,E) = \cdots = y_n^{(m-1)}(a,E) = 0,$$

then the approximate solution will be

$$y_n(x,E) = \sum_{i=m}^{2m-1} c_i f_{n_i}(x,E), \quad n > 0. \tag{3.11}$$

By using other conditions at endpoint b, for example $y_n(b,E) = y_n'(b,E) =\cdots = y_n^{(m-1)}(b,E) = 0$, we get the following system

$$\begin{array}{l} \sum_{i=m}^{2m-1} c_i f_{n_i}(b,E) = 0, \\ \sum_{i=m}^{2m-1} c_i f_{n_i}'(b,E) = 0, \\ \vdots \\ \sum_{i=m}^{2m-1} c_i f_{n_i}^{(m-1)}(b,E) = 0, \end{array} \tag{3.12}$$

for $c_m, c_{m+1}, \ldots, c_{2m-1}$. By Crammer's rule, we will get a nontrivial solution for the system (3.12) if

$$M_n(E) = \det \begin{pmatrix} f_{n_m}(b,E) & f_{n_{m+1}}(b,E) & \ldots & f_{n_{2m-1}}(b,E) \\ f_{n_m}'(b,E) & f_{n_{m+1}}'(b,E) & \ldots & f_{n_{2m-1}}'(b,E) \\ \vdots & \vdots & \ldots & \vdots \\ f_{n_m}^{(m-1)}(b,E) & f_{n_{m+1}}^{(m-1)}(b,E) & \cdots & f_{n_{2m-1}}^{(m-1)}(b,E) \end{pmatrix} = 0, \tag{3.13}$$

which is a polynomial in E. Therefore the eigenvalues of the problem (1.1) are the roots of $M_n(E)$. In practice, stopping criterion for approximated eigenvalue E_k in the n-th iteration is

$$|E_{k_n} - E_{k_{n-1}}| \le \epsilon, \tag{3.14}$$

where ϵ may be made arbitrary small according to the accuracy required. Subsection 3.1 may be summarized in the following algorithm.

Algorithm 3.1.

Step 1: Use equation (3.9) and the initial conditions to construct $y_0(x)$.

Step 2: Use iteration formula (3.8) to generate the sequence $\{y_n\}_{n=1}^{K}$ for some positive integer K.

Step 3: Construct the function $M_n(E)$ as indicated in (3.13).

Step 4: Find roots of the $M_n(E)$, by using (3.14).

Note. It is obvious that, roots in Step 4 are eigenvalues of the problem (1.1).

3.2. **Convergence analysis.** In this subsection, we discuss the convergence of generalized VIM presented in the previous subsection. From (3.8) we may define the operator $\mathcal{L}$ in the following form

$$\mathcal{L}[y] = \int_0^x \frac{(s-x)^{(2m-1)}}{(2m-1)!} \left(Ly(s) + Ny(s)\right) ds. \tag{3.15}$$

By using (3.15), we can construct the following components

$$f_0 = y_0, \quad f_{n+1} = \mathcal{L}[\sum_{i=0}^{n} f_i]. \tag{3.16}$$

We conclude that, the exact solution is in the following form

$$y(x) = \lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} f_n(x). \tag{3.17}$$

Now, by using initial approximation $f_0 = y_0$ (see (3.9)), the approximation solution can be considered by taking n-terms of the series (3.17), that is $y_n(x) = \sum_{i=0}^{n} f_i(x)$.The variational iteration method makes a sequence $\{y_n\}$, here, we show that the sequence $\{y_n\}$ converges to the solution of problem (1.1). To do this, we state and prove the following theorems.

Theorem 3.2. *Let $\mathcal{L} : L^2_w(a,b) \to L^2_w(a,b)$ be an operator in a Hilbert space satisfy in (3.15). Then the series solution $y(x)$ defined by (3.17) for problem (1.1) converges if there exist $0 < \alpha < 1$ such that*

$$\|f_{n+1}\|_{L^2_w} \leq \alpha \|f_n\|_{L^2_w}. \tag{3.18}$$

Proof. Define the sequence of the partial sums s_n such that $s_0 = f_0$, $s_n = \sum_{i=0}^{n} f_i$, we see that the sequence $\{s_n\}$ is well-defined. Let us first prove that $\{s_n\}$ is a Cauchy sequence in the $L^2_w(a,b)$ space. For this purpose, we see that

$$\|s_{n+1} - s_n\|_{L^2_w} \leq \alpha \|s_n - s_{n-1}\|_{L^2_w} \leq \cdots \leq \alpha^n \|s_1 - s_0\|_{L^2_w}. \tag{3.19}$$

Then for any $m \geq n$, we have

$$\begin{aligned} \|s_m - s_n\|_{L^2_w} &\leq \|s_{n+1} - s_n\|_{L^2_w} + \|s_{n+2} - s_{n+1}\|_{L^2_w} + \cdots + \|s_m - s_{m-1}\|_{L^2_w} \\ &\leq \alpha^n[1 + \alpha + \cdots + \alpha^{m-n-1}]\|s_1 - s_0\|_{L^2_w} \\ &\leq \tfrac{\alpha^n}{1-\alpha}\|s_1 - s_0\|_{L^2_w}. \end{aligned} \tag{3.20}$$

Since $\alpha \in (0,1)$, then $\|s_m - s_n\|_{L^2_w} \to 0$ as $m, n \to \infty$. Thus $\{s_n\}$ is a Cauchy sequence in the $L^2_w(a,b)$ space, therefore the series solution converges and the proof is complete.

Theorem 3.3. *If the series solution (3.17) generated by using iteration formula (3.8) converges, then it converges to an exact solution of the problem (1.1).*

Proof. If the series (3.17) converges, we can write $y(x) = \lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} f_n(x)$, then

$$\lim_{n\to\infty} f_n = 0. \tag{3.21}$$

We can write,

$$\sum_{n=0}^{m} [f_{n+1} - f_n] = (f_1 - f_0) + \cdots + (f_{m+1} - f_m) = f_{m+1} - f_0. \tag{3.22}$$

Hence,

$$\sum_{n=0}^{\infty} [f_{n+1} - f_n] = \lim_{n\to\infty} f_n - f_0 = -f_0. \tag{3.23}$$

Now, by applying linear operator $L = \frac{d^{2m}}{dx^{2m}}$, to both sides of (3.23) and since $f_0 = y_0 = \sum_{i=0}^{2m-1} c_i x^i$, see (3.9), we get

$$\sum_{n=0}^{\infty} L[f_{n+1} - f_n] = -Lf_0 = 0. \tag{3.24}$$

Now, from (3.16), we have

$$L[f_{n+1} - f_n] = L[\mathcal{L}[\sum_{i=0}^{n} f_i] - \mathcal{L}[\sum_{i=0}^{n-1} f_i]], \tag{3.25}$$

and by using (3.15), we obtain

$$\begin{aligned} L[f_{n+1} - f_n] = L\{\textstyle\int_0^x \frac{(s-x)^{(2m-1)}}{(2m-1)!}(L[\sum_{i=0}^{n} f_i] + N[\sum_{i=0}^{n} f_i] \\ -L[\sum_{i=0}^{n-1} f_i] - N[\sum_{i=0}^{n-1} f_i])ds\}, \quad n \geq 1. \end{aligned} \tag{3.26}$$

Now, since the linear differential operator $L = \frac{d^{2m}}{dx^{2m}}$ is the left inverse to $2m$-fold integral operator, then (3.26) becomes

$$L[f_{n+1} - f_n] = L[f_n] + N[f_n]. \tag{3.27}$$

Thus

$$\sum_{n=0}^{m} L[f_{n+1} - f_n] = L\sum_{i=0}^{m} [f_n] + N\sum_{i=0}^{m} [f_n], \tag{3.28}$$

and so,

$$\sum_{n=0}^{\infty} L[f_{n+1} - f_n] = L\sum_{n=0}^{\infty} [f_n] + N\sum_{n=0}^{\infty} [f_n]. \tag{3.29}$$

Therefore, from (3.24) and (3.29), we see that $y(x) = \sum_{n=0}^{\infty} f_n(x)$ must be an exact solution.

Theorem 3.4. *For the approximation solution $y_n = \sum_{i=0}^{n} f_i$, if the series solution (3.17) is convergent to an exact solution $y(x)$, then the error estimate is*

$$e_n = \frac{\alpha^n}{1-\alpha}\|f_1\|_{L_w^2}. \tag{3.30}$$

Proof. From Theorem 3.1, letting $m \to \infty$ in (3.20), we get

$$\|y(x) - \sum_{i=0}^{n} f_i\|_{L_w^2} \leq \frac{\alpha^n}{1-\alpha}\|f_1\|_{L_w^2}, \tag{3.31}$$

and this completes the proof.
Note that, Theorem 3.2 is a special case of Banach fixed point theorem [27].

4. Work examples

In this section the efficiency of the generalized variational iteration method (proposed in Section 3) is illustrated. Three classes of work examples of second-order, fourth-order and sixth-order Sturm-Liouville problems are discussed. In order to compare our results with others, each problem represent a specific example of eigenvalue problems that are frequently studied in the context of Sturm-Liouville operators. In Examples 4.1-4.3, ϵ is chosen to be 10^{10}.

Example 4.1. Consider the following second-order eigenvalue problem

$$y^{(2)}(x) + Ey(x) = 0, \quad x \in (0,1), \tag{4.1}$$

subject to boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \tag{4.2}$$

Thus, from (3.7) the Lagrange multiplier becomes

$$\lambda = s - x.$$

and corresponding to (3.8), the correction functional for equation (4.1) is give by

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x)\left(\frac{d^2}{ds^2}y_n(s) + Ey_n(s)\right) ds. \tag{4.3}$$

Now, choose y_0 so that $L(y_0) = 0$ and $y'(0) = 0$. Simple calculations implies that $y_0(x) = c_0$, where c_0 is a constant. By applying iteration formula (4.3), we get the following approximations,

$$\begin{aligned} y_1(x) &= c_0(1 - E\tfrac{x^2}{2}), \\ y_2(x) &= c_0(1 - E\tfrac{x^2}{2} + E^2\tfrac{x^4}{24}), \\ &\vdots \end{aligned}$$

It is easy to see that

$$y_n(x,E) = c_0 f_{n_0}(x,E), \quad n > 0. \tag{4.4}$$

Using the boundary condition at 1, we get an equation of the form

$$y_n(1,E) = c_0 f_{n_0}(1,E) = 0. \tag{4.5}$$

TABLE 1. Convergence behavior of the first three eigenvalues for Example 4.1

n	E_1	E_2	E_3
7	2.4674011001	22.2316528426	46.2659547508
8	2.4674011003	22.2047858586	–
9	2.4674011003	22.2067176578	58.8534769990
10	2.4674011003	22.2066046809	62.3855801420
11	2.4674011003	22.2066101110	61.6158764684
12	2.4674011003	22.2066098926	61.6917738329
13	2.4674011003	22.2066099000	61.6844723409
14	2.4674011003	22.2066098998	61.6850672413
15	2.4674011003	22.2066098998	61.6850250561
16	2.4674011003	22.2066098998	61.6850276793
17	2.4674011003	22.2066098998	61.6850275351
18	2.4674011003	22.2066098998	61.6850275421
19	2.4674011003	22.2066098998	61.6850275418
20	2.4674011003	22.2066098998	61.6850275418

Since $c_0 \neq 0$, this show that the last equation have nonzero solution if

$$f_{n_0}(1, E) = 0, \tag{4.6}$$

which is a polynomial in E. The eigenvalues of problem (4.1)-(4.2) are the roots of (4.6). The first three eigenvalues of problem (4.1)-(4.2) are calculated and given in the Table 1. This Table shows that after at most 20 iterations these 3 eigenvalues converge to correct solution up to 10 decimal points.
In Fig. 1, the first three normalized eigenfunctions correspond to eigenvalues E_1, E_2 and E_3 are plotted, where the normalized eigenfunction $\overline{y}_k$ is given by

$$\overline{y}_k(x, E_k) = \frac{y_k(x, E_k)}{\int_0^1 |y_k(x, E_k)| dx}, \quad k = 1, 2, 3. \tag{4.7}$$

Example 4.2. Consider the following fourth- order Sturm-Liouville problem

$$y^{(4)}(x) = Ey(x), \ \ x \in (0,\, 1), \tag{4.8}$$

subject to

$$y(0) = y'(0) = 0, \ \ y(1) = y''(1) = 0. \tag{4.9}$$

In elasticity, the fourth-order Sturm-Liouville equations are associated to the steady state Euler-Bernoulli equation for the deflection y of a vibrating beam, with the other quantities involved having physical meaning, for example $p > 0$ is the flexural rigidity of the beam, py'' is the bending moment and $Ew - q$ is the frequency of vibration, (for example see [6, 28, 29]). Let us choose $y_0(x)$ so that $Ly_0(x) = 0$, $y(0) = y'(0) = 0$. A simple calculation implies that $y_0(x) = c_2x^2 + c_3x^3$, where c_2, c_3 are some constants. By using (3.7) and (3.8), the iteration formula for (4.8), can be constructed as,

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^3}{6}(y_n^{(4)}(s) - Ey_n(s))\, ds. \tag{4.10}$$

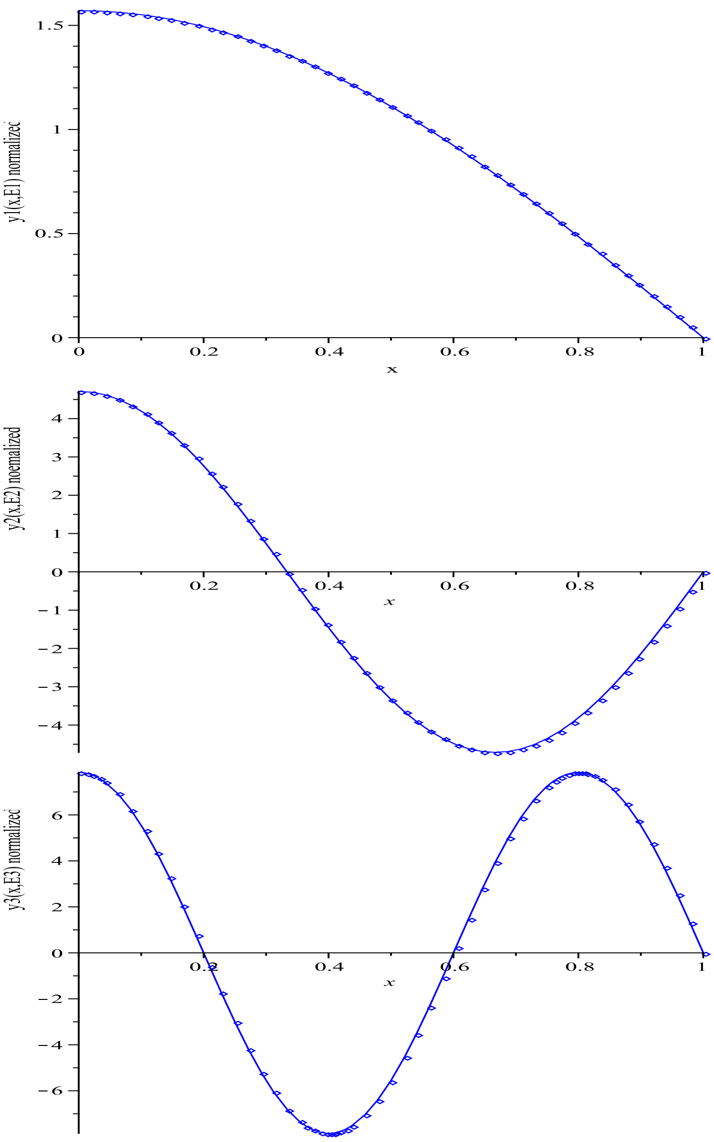

FIGURE 1. The first three normalized eigenfunctions for Example 4.1, where $\diamond$ and - stand for approximated and exact solutions

By applying iteration formula (4.10) we get the following approximations,

$$y_1(x) = c_2\left(x^2 + E\frac{x^6}{360}\right) + c_3\left(x^3 + E\frac{x^7}{840}\right),$$
$$y_2(x) = c_2\left(x^2 + E\frac{x^6}{360} + E^2\frac{x^{10}}{1814400}\right) + c_3\left(x^3 + E\frac{x^7}{840} + E^2\frac{x^{11}}{6652800}\right),$$
$$\vdots$$

TABLE 2. Convergence behavior of the first six eigenvalues for Example 4.2

n	E_1	E_2	E_3	E_4	E_5	E_6
2	241.0407900573	1118.7986354445				
3	237.7407649918	2067.3174169404	–			
4	237.7211107807	2473.9208558105	–	–		
5	237.7210675562	2496.2291044578	–	–	–	
6	237.7210675140	2496.4860247337	10976.9214218253	18841.8594651662	–	–
7	237.7210675140	2496.4874325568	10868.9861525959	27469.5565766394	–	–
8	237.7210675140	2496.4874368486	10867.5937509176	31471.1337284258	–	–
9	237.7210675140	2496.4874368565	10867.5822966476	31774.8223630040	–	–
10	237.7210675140	2496.4874368565	10867.5822368856	31780.0417309785	75154.7207471462	96008.93627618767
11	237.7210675140	2496.4874368565	10867.5822366781	31780.0959997769	74017.3083240608	128854.1479108191
12	237.7210675140	2496.4874368565	10867.5822366776	31780.0963876107	74001.0462005716	146530.9300723852
13	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895840	74000.8508171968	148589.3856542964
14	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8491003179	148633.8757297095
15	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490890426	148634.4728921409
16	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490889861	148634.4789663363
17	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490889859	148634.4790142247
18	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490889859	148634.4790145220
19	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490889859	148634.4790145235
20	237.7210675140	2496.4874368565	10867.5822366776	31780.0963895913	74000.8490889859	148634.4790145235

We see that

$$y_n(x,E) = c_2 f_{n_2}(x,E) + c_3 f_{n_3}(x,E).$$

Now by using the boundary conditions at 1, we get the following system

$$y_n(1,E) = c_2 f_{n_2}(1,E) + c_3 f_{n_3}(1,E) = 0,$$

$$y_n(1,E) = c_2 f''_{n_2}(1,E) + c_3 f''_{n_3}(1,E) = 0.$$

Hence, we have nonzero solution for c_2 and c_3, if

$$M(E) = \det \begin{pmatrix} f_{n_2}(1,E) & f_{n_3}(1,E) \\ f''_{n_2}(1,E) & f''_{n_3}(1,E) \end{pmatrix} = 0. \tag{4.11}$$

By computing roots of (4.11), we can obtain the eigenvalues of problem (4.8)-(4.9). In this example, the first eigenvalue E_1 is obtained after sixth iteration and the second eigenvalue E_2 is obtained in the ninth iteration. Following the same approach, the remaining eigenvalues E_k, $k = 3, 4, 5, 6$ are obtained, and these are listed in Table 2. The first three normalized eigenfunctions are plotted in Fig. 2. Our numerical results are close to those values obtained by ADM, (see Table 1 in Ref. [29]). In Table 2 it is shown that at most 20 iterations are needed that all of 6 eigenvalues converge to the correct solution up to 10 decimal points.

Example 4.3. Consider the following sixth-order boundary value problem

$$-y^{(6)}(x) = Ey(x), \;\; x \in (0, \pi), \tag{4.12}$$

subject to homogeneous boundary value conditions

$$\begin{aligned} y(0) = y''(0) = y^{(4)}(0) = 0, \\ y(\pi) = y''(\pi) = y^{(4)}(\pi) = 0. \end{aligned} \tag{4.13}$$

Let $L(y) = -y^6$ and $N(y) = -Ey(x)$. As an initial approximating solution, let us choose y_0 so that $L(y_0) = 0$ and $y(0) = y''(0) = y^{(4)}(0) = 0$, this implies that

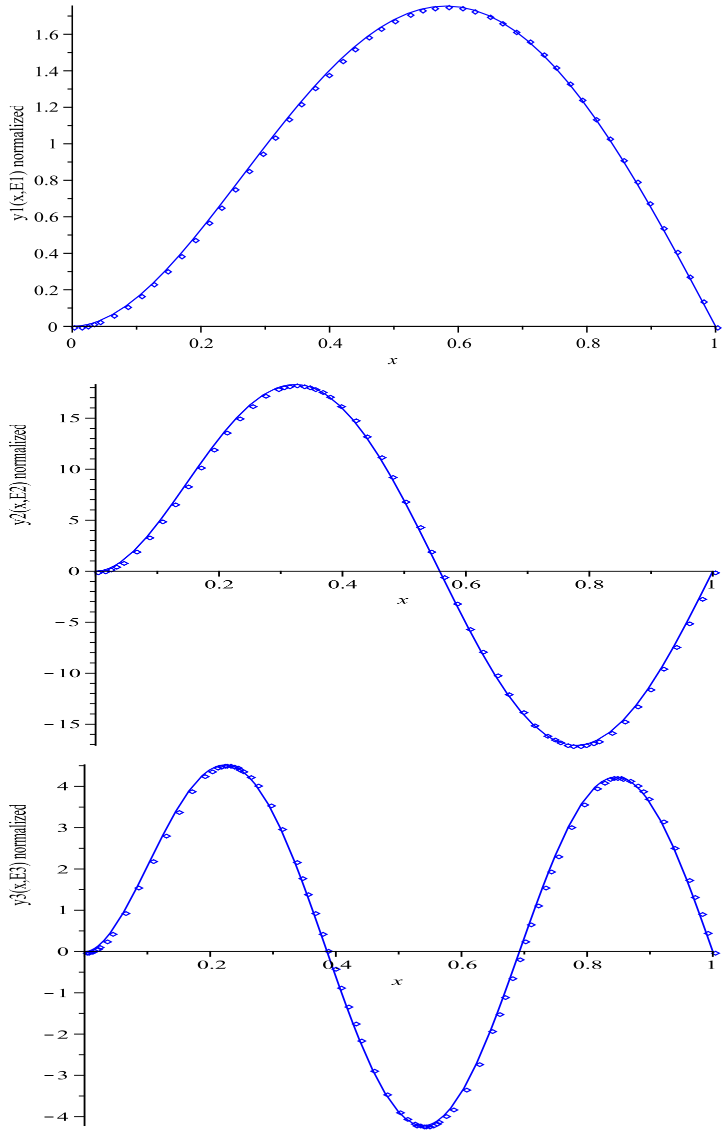

FIGURE 2. The first three normalized eigenfunctions for Example 4.2, where $\diamond$ and - stand for approximated and exact solutions

$y_0(x) = c_1x + c_3x^3 + c_5x^5$, where c_1, c_3 and c_5 are some constants. From equation (3.7), the Lagrange multiplier can be derived as

$$\lambda = \frac{(s-x)^5}{5!},$$

and corresponding to equation (3.8), the iteration formula for equation (4.12) is given by,

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^5}{5!}\left(-y_n^{(6)}(s) - Ey_n(s)\right) ds. \tag{4.14}$$

By applying iteration formula (4.14) we get the following approximations,

$$\begin{aligned}
y_1(x) &= c_1(x + E\tfrac{x^7}{5040}) + c_3(x^3 + E\tfrac{x^9}{60480}) + c_5(x^5 + E\tfrac{x^{11}}{332640}),\\
y_2(x) &= c_1\{x + \tfrac{Ex^7}{5040} + \tfrac{E}{5!}(\tfrac{Ex^{13}}{51891840} + \tfrac{x^7}{21})\} + c_3\{x^3 + \tfrac{Ex^9}{60480}\\
&\quad + \tfrac{E}{5!}(\tfrac{Ex^{15}}{1816214400} + \tfrac{x^9}{252})\} + c_5\{x^5 + \tfrac{Ex^{11}}{332640}\\
&\quad + \tfrac{E}{5!}(\tfrac{Ex^{17}}{24700515840} + \tfrac{x^{11}}{1386})\},\\
&\;\;\vdots
\end{aligned}$$

It is easy to see that

$$y_n(x, E) = c_1 f_{n_1}(x, E) + c_3 f_{n_3}(x, E) + c_5 f_{n_5}(x, E), \quad n > 0. \tag{4.15}$$

Using the boundary conditions at $x = \pi$, namely, $y(\pi) = y''(\pi) = y^{(4)}(\pi) = 0$, we get three equations of the form

$$\begin{aligned}
c_1 f_{n_1}(\pi, E) + c_3 f_{n_3}(\pi, E) + c_5 f_{n_5}(\pi, E) = 0,\\
c_1 f''_{n_1}(\pi, E) + c_3 f''_{n_3}(\pi, E) + c_5 f''_{n_5}(\pi, E) = 0,\\
c_1 f^{(4)}_{n_1}(\pi, E) + c_3 f^{(4)}_{n_3}(\pi, E) + c_5 f^{(4)}_{n_5}(\pi, E) = 0.
\end{aligned}$$

We see that the last system have nonzero values for c_1, c_3 and c_5, if

$$M_n(E) = \det\begin{pmatrix} f_{n_1}(\pi, E) & f_{n_3}(\pi, E) & f_{n_5}(\pi, E)\\ f''_{n_1}(\pi, E) & f''_{n_3}(\pi, E) & f''_{n_5}(\pi, E)\\ f^{(4)}_{n_1}(\pi, E) & f^{(4)}_{n_3}(\pi, E) & f^{(4)}_{n_5}(\pi, E)\end{pmatrix} = 0. \tag{4.16}$$

TABLE 3. Convergence behavior of the first ten eigenvalues for Example 4.3

n	E_1	E_2	E_3	E_4	E_5
1	0.9597480757				
2	1.0000129882	56.2788904610			
3	1.0000000053	64.0198038788	532.1493875851		
4	1.0000000055	63.9999944246	731.3614449310	2356.0755682967	
5	1.0000000055	64.0000004693	728.9978232917	4228.0495427143	6840.02472508924
6	1.0000000055	64.0000004688	729.0000049614	4095.8036686190	–
7	1.0000000055	64.0000004688	729.0000042846	4096.0002036061	15616.0898538767
8	1.0000000055	64.0000004688	729.0000042847	4096.0000657743	15625.0108035041
9	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999494920
10	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559372
11	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
12	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
13	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
14	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
15	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
16	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
17	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
18	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
19	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352
20	1.0000000055	64.0000004688	729.0000042847	4096.0000658174	15624.9999559352

n	E_6	E_7	E_8	E_9	E_{10}
1					
2					
3					
4					
5					
6	–				
7	–	–			
8	46394.7837800722	–	–		
9	46656.4933211556	113176.2893575077	11363438.29672751	18065366.34480531	
10	46655.9997476830	117664.8009831108	229578.4234499600	21843555.68644232	34962659.843431151
11	46656.0002155877	117648.9820664145	262548.0507885136	406294.3648678014	43744463.825209270
12	46656.0002153382	117649.0034775570	262143.2888042216	542345.4768564463	642819.68618286682
13	46656.0002153383	117649.0034604007	262144.0032822110	531421.9696541301	–
14	46656.0002153383	117649.0034604091	262144.0024903922	531440.9607325989	999577.37782224018
15	46656.0002153383	117649.0034604091	262144.0024909532	531440.9332967993	1000001.1422732220
16	46656.0002153383	117649.0034604091	262144.0024909529	531440.9333229243	1000000.3748335370
17	46656.0002153383	117649.0034604091	262144.0024909529	531440.9333229070	1000000.3757690812
18	46656.0002153383	117649.0034604091	262144.0024909529	531440.9333229070	1000000.3757682898
19	46656.0002153383	117649.0034604091	262144.0024909529	531440.9333229070	1000000.3757682903
20	46656.0002153383	117649.0034604091	262144.0024909529	531440.9333229070	1000000.3757682903

The first ten eigenvalues of problem (4.12)-(4.13) are given in Table 3. The first normalized eigenfunction $\overline{y}_1$ is given by

$$\overline{y}_1(x, E_1) = \frac{y_1(x, E_1)}{\int_0^\pi |y_1(x, E_1)| dx}, \tag{4.17}$$

and it is plotted in Fig. 3. Here the first estimated eigenvalue E_1 is obtained in the

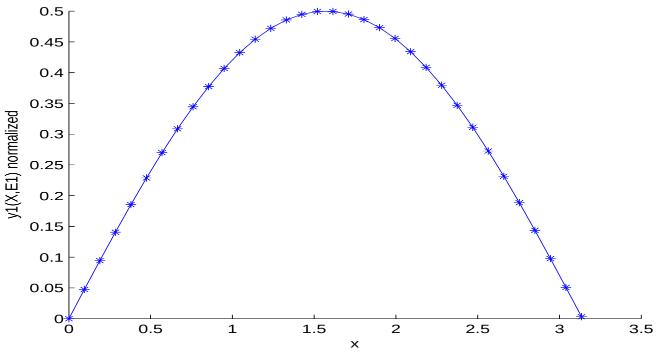

FIGURE 3. The first normalized eigenfunction $\overline{y}_1(x, E_1)$ for Example 4.3, where $*$ and - stand for approximated and exact solutions

fourth iteration and the second eigenvalue E_2 is obtained in the sixth iteration. The other eigenvalues are found in the same way (see Table 3). These results are more accurate than the results obtained by using the ADM (see Table 2 in Ref. [11]) and results obtained by using shooting method (see Table 6.1 in Ref. [5]). It is well known that the exact eigenvalues are given by $E_k = k^6$.

Example 4.4. We wanted to test our algorithm on a problem whose differential equation exhibits stiffness in at least part of its rang: we chose the following problems

(i) Consider the following second-order Sturm-Liouville problem

$$-y''(x) + \alpha x^2 y(x) = Ey(x), \quad x \in (0,5), \tag{4.18}$$

subject to

$$y(0) = y(5) = 0. \tag{4.19}$$

Let $L(y) = y''$ and $N(y) = (\alpha x^2 y - Ey)$. Choose y_0 so that $L(y_0) = 0$ and $y(0) = 0$. Simple calculation implies $y_0(x) = c_0$. The iteration formula (3.8) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x)\left(\frac{d^2}{ds^2}y_n(s) + \alpha x^2 y_n(s) - Ey_n(s)\right) ds. \tag{4.20}$$

For $\alpha = 0.01$, the first two eigenvalues are: $E_1 = 0.4637357700$, $E_2 = 1.6597620115$.

(ii) Consider the following fourth-order Sturm-Liouville problem

$$y^{(4)}(x) - 0.02x^2 y'' - 0.04xy' + (0.0001x^4 - 0.02)y = Ey(x), \quad x \in (0,5), \tag{4.21}$$

subject to

$$y(0) = y''(0) = 0, \quad y(5) = y''(5) = 0. \tag{4.22}$$

By using Algorithm 3.1, the first fifth eigenvalues are: $E_1 = 0.2150508644$, $E_2 = 2.7548099347$, $E_3 = 13.2153515406$, $E_4 = 40.9508197591$ and $E_5 = 99.0534781381$. This result show that, the eigenvalues of this problem are the squares of eigenvalues of problem (4.18)-(4.19).

(iii) Consider the following sixth-order Sturm-Liouville problem

$$\begin{aligned} -y^{(6)}(x) + (3\alpha^2 x^2 y'')'' + ((8\alpha - 3\alpha^2 x^4)y')' + (\alpha^3 x^6 - 14\alpha^2 x^2)y \\ = Ey(x), \quad x \in (0,\, 5), \end{aligned} \tag{4.23}$$

subject to homogeneous boundary value conditions

$$\begin{aligned} y(0) = y''(0) = y^{(4)}(0) = 0, \\ y(5) = y''(5) = y^{(4)}(5) = 0. \end{aligned} \tag{4.24}$$

By using Algorithm 3.1, the first fifth eigenvalues for $\alpha = 0.01$ are: $E_1 = 0.0997267728$, $E_2 = 4.5723214092$, $E_3 = 48.0416683775$, $E_4 = 262.0590748452$ and $E_5 = 985.8390701194$. We see that, the eigenvalues of this problem are the cubes of the eigenvalues of the second-order problem (4.18)-(4.19).

5. Conclusion

This paper suggests an effective numerical algorithm for the high order Sturm-Liouville problem and the results are of high accuracy. We have proposed a numerical technique based on fixed point variational iteration method for computing eigenvalues of the general $2m$-order Sturm-Liouville problems for $m \geq 1$. First, a generalization VIM algorithm for $m = 1$ is explained for fourth and sixth order Sturm-Liouville problems numerically. Second, theoretical, convergence and numerical aspects of the generalization of VIM for $2m$-order Sturm-Liouville problems is discussed for general case $m \geq 1$. In this process a general formula for the Lagrange multiplier λ is given. Three different cases for (i) $m = 1$, (ii) $m = 2$ and (iii) $m = 3$ for the solution of second, fourth and sixth order Sturm-Liouville problems are discussed. Numerical results (obtained from proposed method in this paper) are compared with results that obtained by exact solution, Adomian decomposition method and shooting method.

Numerical results show that the variational iteration method is an efficient tool to compute eigenvalues of high-order Sturm-Liouville problems.

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