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ON FIXED POINTS OF POINTWISE LIPSCHITZIAN TYPE MAPPINGS

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Abstract. The classes of pointwise Lipschitzian type mappings and asymptotic pointwise Lipschitzian type mappings are introduced and their properties are discussed. The existence of fixed points for pointwise Lipschitzian type mappings and asymptotic pointwise Lipschitzian type mappings are investigated in Banach spaces. Our results generalize and improve the corresponding results of L.P. Belluce and W.A. Kirk [Fixed point theorems for certain classes of nonexpansive mappings, *Proc. Amer. Math. Soc.*, 20 (1969) 141-146]; E. Casini and E. Maluta [Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure, *Nonlinear Anal.*, 9 (1985), 103-108]; K. Goebel and W.A. Kirk [A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, 35 (1972), 171-174]; W.A. Kirk and H.K. Xu [Asymptotic pointwise contractions, *Nonlinear Anal.*, 69 (2008), 4706-4712]; and T.C. Lim and H.K. Xu [Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Anal.*, 22(11) (1994), 1345-1355].
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1. Introduction

Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping. Then T is called

(i) *nonexpansive* if $\|Tx - Ty\| \leq ||x - y||$ for all $x, y \in C$;
(ii) *asymptotically nonexpansive* ([9]) if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n||x - y||$ for all $x, y \in C$;

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(iii) *pointwise contraction* ([3]) if there exists a function $\alpha : C \to [0,1)$ such that $\|Tx - Ty\| \le \alpha(x)\|x - y\|$ for all $x, y \in C$;
(iv) *asymptotic pointwise contraction* ([14]) if for each $n \in \mathbb{N}$, there exists a function $\alpha : C \to [0,1)$ such that $\|T^n x - T^n y\| \le \alpha_n(x)\|x - y\|$ for all $x, y \in C$, where $\alpha_n \to \alpha$ pointwise on C;
(v) *pointwise asymptotically nonexpansive* ([15]) if, for each integer $n \in \mathbb{N}$, $\|T^n x - T^n y\| \le \alpha_n(x)\|x - y\|$ for all $x, y \in C$, where $\alpha_n(x) \to \alpha(x)$ pointwise;
(vi) *asymptotically nonexpansive in the intermediate sense* ([4]) provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0; \tag{1.1}$$

(vi) *mapping of asymptotically nonexpansive type* ([13]) if

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \text{ for all } x \in C.$$

There is a class of mappings which lies strictly between the class of contraction mappings and the class of nonexpansive mappings. The class of pointwise contractions was introduced in Belluce and Kirk [3]) and later it was called “generalized contractions” in [11]. Banach’s celebrated contraction principle was extended to this larger class of mappings as follows:

Theorem 1.1. ([3, 11]) *Let C be a nonempty weakly compact convex subset of a Banach space and $T : C \to C$ a pointwise contraction. Then T has a unique fixed point, x^*, and $\{T^n x\}$ converges strongly to x^* for each $x \in C$.*

Kirk [14] combined ideas of pointwise contraction ([3]) and asymptotic contraction ([12]) and introduced the concept of an asymptotic pointwise contraction. He announced that an asymptotic pointwise contraction defined on a closed convex bounded subset of a super-reflexive Banach space has a fixed point. Recently, Kirk and Xu [15] gave a simple and elementary proof of the fact that an asymptotic pointwise contraction defined on a weakly compact convex set always has a unique fixed point (with convergence of Picard iterates). They also introduced the concept of pointwise asymptotically nonexpansive mapping and proved that every pointwise asymptotically nonexpansive mapping defined on a closed convex bounded subset of a uniformly convex Banach space has a fixed point.

The class of asymptotically nonexpansive mappings in the intermediate sense which is essentially wider than that of asymptotically nonexpansive mappings was introduced by Bruck, Kuczumow and Reich [4]. It is known ([13]) that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is a self-mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point.

On the other hand, if $c_n := \max\{\sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|), 0\}$, then (1.1) reduces to relation

$$\|T^n x - T^n y\| \le \|x - y\| + c_n \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The classes of mappings more general than the class of mappings satisfying (1.2) were studied in Alber, Chidume and Zegeye [2] as the class of total asymptotically nonexpansive mappings and in Sahu [19] as the class of nearly Lipschitzian mappings.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be *total asymptotically nonexpansive* ([2]) if there exist nonnegative real sequences $\{\mu_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} c_n = 0$ and strictly increasing continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + c_n \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ ([19]) if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \tag{1.4}$$

for all $x, y \in C$. The infimum of constants k_n in (1.4) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$. A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(ii) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$,
(iii) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$,
(iv) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,
(v) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \eta(T^n) = 1$.

Remark 1.2. (a) If $\mu_n = 0$, then (1.3) reduces to (1.2).
(b) If S is asymptotically nonexpansive self-mapping and T is total asymptotically nonexpansive self-mapping defined on a bounded set C, then both S and T are nearly nonexpansive mappings. To see this, let T be an asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ defined on a bounded set C with diameter $diam(C)$. Fix $a_n := (k_n - 1)diam(C)$. Then,

$$\|T^n x - T^n y\| \leq \|x - y\| + (k_n - 1)\|x - y\| \leq \|x - y\| + a_n$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

The corresponding Lipschitzian type mappings (for instance, contraction type mappings) concerning asymptotically nonexpansive mapping in the intermediate sense and total asymptotically nonexpansive mapping are not defined in Bruck, Kuczumow and Reich [4] and Alber, Chidume and Zegeye [2]. The notion of nearly Lipschitzian mappings allows to define different classes of Lipschitzian type mappings, for example, nearly contraction, nearly nonexpansive, nearly asymptotically nonexpansive, nearly uniformly L-Lipschitzian, etc. Therefore, the fixed point theory of nearly Lipschitzian mappings is of fundamental importance. Some properties and existence and convergence results for nearly Lipschitzian mappings are studied in [19, 20].

Motivated and inspired by Kirk and Xu [15], the purpose of this paper is to characterize the existence of fixed points of pointwise nearly Lipschitzian mappings in

Banach spaces. It is shown that the asymptotic center of every bounded orbit of a pointwise asymptotically nonexpansive mapping is a fixed point of the mapping in a uniformly convex Banach space. As a consequence of our results, we derive a result which is an improvement of Casini and Maluta [6] and Lim and Xu [16, Theorem 1] results in a Banach space with weak uniformly normal structure. The results presented in this paper are significant improvements upon those results which are known for Lipschitzian and non-Lipschitzian mappings.

2. Preliminaries

Let C be a nonempty subset of a Banach space X, $\{x_n\}$ a bounded sequence in X and $T : C \to C$ a mapping. We denote by $F(T)$ the set of fixed points of T. T is said to be *asymptotically regular at point* $x \in C$ if $T^n x - T^{n+1}x \to 0$ as $n \to \infty$ and T is said to be *asymptotically regular on* C if T is asymptotically regular at each point of C, i.e. if for all $x \in C$, $T^n x - T^{n+1}x \to 0$ as $n \to \infty$. The mapping T is said to be *demicontinuous* if, whenever a sequence $\{y_n\}$ in C converges strongly to $y \in C$, then $\{Ty_n\}$ converges weakly to Ty. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}_+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \ x \in X.$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $\mathcal{Z}_a(C, \{x_n\})$. This set may be empty, a singleton, or certain infinitely many points.

A Banach space is said to satisfy property (I) if the asymptotic center of every bounded sequence in X with respect to closed convex set of X consists of exactly one point. Uniformly convex Banach spaces are examples of this type of Banach spaces (see [1]).

The number $diam_a(\{x_n\}) = \limsup_{k\to\infty}(\sup\{\|x_n - x_m\| : n, m \geq k\})$ is called an asymptotic diameter of $\{x_n\}$. The normal structure coefficient $N(X)$ of a Banach space X is defined ([5]) by

$$N(X) = \inf\left\{\frac{diam(C)}{r_C(C)} : C \text{ nonempty bounded convex subset of } X,\ diam(C) > 0\right\},$$

where $r_C(C) = \inf_{x\in C}\{\sup_{y\in C} \|x - y\|\}$ is the Chebyshev radius of C relative to itself and $diam(C) = \sup_{x,y\in C} \|x - y\|$ is diameter of C. The space X is said to have the *uniformly normal structure* if $N(X) > 1$. A weakly convergent sequence coefficient of X is defined (see [5]) by

$$WCS(X) = \sup\left\{k : k \limsup_{n\to\infty} \|x_n\| < diam_a(\{x_n\}) \text{ for all } \{x_n\} \text{ in } X \text{with } x_n \rightharpoonup 0\right\}.$$

It is proved in [7, Theorem 1] that

$$WCS(X) = \beta(X) := \inf\{D[\{x_n\}] : x_n \rightharpoonup 0, \|x_n\| \to 1\},$$

where $D[\{x_n\}] := \limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_m - x_n\|)$. It is readily seen that

$$1 \le N(X) \le WCS(X) \le 2.$$

The space X is said to have the *weak uniformly normal structure* if $WCS(X) > 1$.

A Banach space X is said to satisfy the *uniform Opial condition* ([18]) if for each $t > 0$, there exists an $r > 0$ such that

$$1 + r \le \liminf_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \ge t$ and each sequence $\{x_n\}$ in X such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \ge 1$. The *Opial modulus of* X, denoted by r_X, as follows:

$$r_X(t) = \inf\left\{\liminf_{n\to\infty} \|x_n + x\| - 1\right\},$$

where $t \ge 0$ and the infimum is taken over all $x \in X$ with $\|x\| \ge t$ and sequences $\{x_n\}$ in X such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \ge 1$. The function r_X is continuous and nondecreasing.

Next, we have the following inequality (see [1, 17]):

$$WCS(X) \ge 1 + r_X(1).$$

If X is a Banach space with a weakly sequentially continuous duality map J_μ associated with a gauge function μ which is continuous, strictly increasing, with $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$, then X has the uniform Opial property. Moreover, if X satisfies uniform Opial condition, then $WCS(X) > 1$ (see [17]).

The following lemmas will be needed in the sequel for the proof of our main results:

Lemma 2.1. ([1, Theorem 3.1.8]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in C such that $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$. If $\{y_m\}$ is a sequence in C such that $\lim_{m\to\infty} r_a(y_m, \{x_n\}) = r_a(C, \{x_n\})$, then $\lim_{m\to\infty} y_m = z$.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a Banach space and $T : C \to C$ a mapping such that $T^n u \to v$ as $n \to \infty$ for some $u, v \in C$. Suppose T is demicontinuous at v. Then v is a fixed point of T in C.*

Proof. By the demicontinuity of T at v, we have $w - \lim_{n\to\infty} T^{n+1}u = Tv \in C$. Note $\lim_{n\to\infty} T^n u = \lim_{n\to\infty} T^{n+1}u = v$. By the uniqueness of weak limits of $\{T^{n+1}u\}$, we have $Tv = v$. □

3. Pointwise Lipschitzian type mappings and fixed points

First, we introduce some wider classes of nonlinear mappings which include the classes of Lipschitzian and nearly Lipschitzian mappings.

Definition 3.1. Let C be a nonempty subset of normed space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is said to be

(a) *pointwise nearly Lipschitzian* if, for each $n \in \mathbb{N}$ and $x \in C$, there exist a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ and a function $\alpha_n(\cdot) : C \to (0, \infty)$ such that
$$\|T^n x - T^n y\| \leq \alpha_n(x)(\|x - y\| + a_n) \text{ for all } y \in C;$$

(b) *pointwise nearly uniformly $\alpha(\cdot)$-Lipschitzian* if, for each $n \in \mathbb{N}$ and $x \in C$, there exists a function $\alpha(\cdot) : C \to (0, \infty)$ such that
$$\|T^n x - T^n y\| \leq \alpha(x)(\|x - y\| + a_n) \text{ for all } y \in C;$$

(c) *asymptotic pointwise nearly Lipschitzian* if, for each $n \in \mathbb{N}$ and $x \in C$, there exist a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ and two functions $\alpha_n(\cdot), \alpha(\cdot) : C \to (0, \infty)$ with $\alpha_n \to \alpha$ pointwise such that
$$\|T^n x - T^n y\| \leq \alpha_n(x)(\|x - y\| + a_n) \text{ for all } y \in C.$$

We say that an asymptotic pointwise nearly Lipschitzian mapping is pointwise nearly asymptotically nonexpansive (pointwise asymptotically nonexpansive [15]) if $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \to 1$ pointwise ($a_n = 0$ and $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \to 1$ pointwise). Further, we say that an asymptotic pointwise nearly Lipschitzian mapping is asymptotic pointwise nearly contraction if $\alpha_n \to \alpha$ pointwise and $\alpha(x) \leq k < 1$ for all $x \in C$.

Before presenting main results, we discuss some properties of asymptotic pointwise nearly Lipschitzian mappings.

Proposition 3.2. *Let C be a nonempty subset of a normed space X and $T : C \to C$ a asymptotic pointwise nearly contraction. Suppose F(T) is nonempty. Then T has a unique fixed point.*

Proof. Suppose T is asymptotic pointwise nearly contraction mapping with sequence $\{(a_n, \alpha_n(\cdot))\}$ such that $\alpha_n \to \alpha$ pointwise. Suppose $u, v \in F(T)$ such that $u \neq v$. Observe that
$$\|u - v\| = \|T^n u - T^n v\| \leq \alpha_n(v)(\|u - v\| + a_n) \text{ for all } n \in \mathbb{N}.$$
Taking limit as $n \to \infty$, we have
$$\|u - v\| \leq \alpha(v)\|u - v\|.$$
Therefore, $u = v$. □

Proposition 3.3. *Let C be a nonempty bounded subset of a normed space X and $T : C \to C$ a pointwise nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \alpha_n(\cdot))\}$. Then T is a mapping of asymptotically nonexpansive type.*

Proof. By Definition 3.1(c), we have $\alpha_n(x) \geq 1$ and

$$\begin{aligned}\|T^n x - T^n y\| - \|x - y\| &= (\alpha_n(x) - 1)\|x - y\| + \alpha_n(x)a_n \\ &\leq (\alpha_n(x) - 1)diam(C) + \alpha_n(x)a_n\end{aligned}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \leq 0 \text{ for all } x \in C. \qquad \square$$

Note that a nearly asymptotically nonexpansive mapping is pointwise nearly asymptotically nonexpansive. It is immediately clear, from Proposition 3.3, that the classes of nearly asymptotically nonexpansive mappings, pointwise asymptotically nonexpansive mappings, pointwise nearly asymptotically nonexpansive mappings are intermediate classes lie between the class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type.

The next proposition shows that the asymptotic center of a bounded orbit of a pointwise asymptotically nonexpansive mapping is its fixed point in a uniformly convex Banach space.

Proposition 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a pointwise nearly asymptotically nonexpansive mapping. Assume that there exists a point $x_0 \in C$ such that $\{T^n x_0\}$ is bounded. Then:*

(a) *There exists the unique element $v \in \mathcal{Z}_a(C, \{T^n x_0\})$ such that $T^n v \to v$ as $n \to \infty$.*
(b) *v is a fixed point of T if T is a pointwise asymptotically nonexpansive mapping.*
(c) *v is a fixed point of T if T is demicontinuous at v.*

Proof. (a) By property (I), let $\mathcal{Z}_a(C, \{T^n x_0\}) = \{v\}$. We define a sequence $\{z_m\}$ in C by $z_m = T^m v, m \in \mathbb{N}$. For each $m \in \mathbb{N}$, we have

$$\begin{aligned}r_a(z_m, \{T^n x_0\}) &= \limsup_{n\to\infty} \|T^n x_0 - T^m v\| = \limsup_{n\to\infty} \|T^{n+m} x_0 - T^m v\| \\ &\leq \limsup_{n\to\infty} \alpha_m(x_0)(\|T^n x_0 - v\| + a_m) \\ &= \alpha_m(x_0)(r_a(C, \{T^n x_0\}) + a_m). \qquad (3.1)\end{aligned}$$

Hence $r_a(z_m, \{T^n x_0\}) \to r_a(C, \{T^n x_0\})$ as $m \to \infty$. It follows from Lemma 2.1 that $T^m v \to v$.

(b) Since T is pointwise asymptotically nonexpansive mapping, we have

$$\|Tv - T^{m+1} v\| \leq \alpha_1(v)\|v - T^m v\|$$

Taking limit as $m \to \infty$, we obtain $v = Tv$.

(c) It follows from Lemma 2.2. $\square$

Our first existence result is a generalization of Banach contraction principle in the context of demicontinuous pointwise nearly Lipschitzian mappings. It also extends [15, Theorem 3.1], where it is assumed that the domain is weakly compact.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a demicontinuous pointwise nearly Lipschitzian mapping with sequence $\{(a_n, \alpha_n(\cdot))\}$. Then, for each $x \in C$ with $\limsup_{n\to\infty}[\alpha_n(x)]^{1/n} < 1$, we have*

(a) the sequence of Picard iterates, $\{T^n x\}$ converges strongly to, $x^* \in C$, a fixed point of T;

(b) $||T^n x - x^*|| \leq (||x - Tx|| + M)\sum_{i=n}^{\infty} \alpha_i(x)$ for all $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$;

(c) x^* is the unique fixed point of T.

Proof. (a) Fix $x_0 \in C$ such that $\limsup_{n\to\infty}[\alpha_n(x_0)]^{1/n} < 1$ and let $x_n = T^n x_0, n \in \mathbb{N}$. Set $d_n := ||x_n - x_{n+1}||$. Hence

$$d_n = ||T^n x_0 - T^{n+1} x_0|| \leq \alpha_n(x_0)(||x_0 - Tx_0|| + a_n).$$

Since $\lim_{n\to\infty} a_n = 0$, we have $\sum_{n=1}^{\infty} d_n \leq (d_0 + M)\sum_{n=1}^{\infty} \alpha_n(x_0)$. By the Root Test for convergence of series, if $\limsup_{n\to\infty}[\alpha_n(x_0)]^{1/n} < 1$, then $\sum_{n=1}^{\infty} \alpha_n(x_0) < \infty$. It follows that $\sum_{n=1}^{\infty} d_n < \infty$ and hence $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n\to\infty} x_n$ exists (say $x^* \in C$). By Lemma 2.2, x^* is fixed point of T.

(b) If $m \in \mathbb{N}$, we have

$$||T^n x_0 - T^{n+m} x_0|| \leq \sum_{i=n}^{n+m-1} ||T^i x_0 - T^{i+1} x_0|| \leq (||x_0 - Tx_0|| + M) \sum_{i=n}^{n+m-1} \alpha_i(x_0).$$

Letting limit as $m \to \infty$, we obtain the desired result.

(c) Note $\limsup_{n\to\infty}[\alpha_n(x^*)]^{1/n} < 1$. Suppose, for contradiction, that y^* is another fixed point of T. Then

$$\begin{aligned} \infty = \sum_{n=1}^{\infty} ||T^n x^* - T^n y^*|| &\leq \sum_{n=1}^{\infty} \alpha_n(x^*)(||x^* - y^*|| + a_n) \\ &\leq (||x^* - y^*|| + M)\sum_{n=1}^{\infty} \alpha_n(x^*) < \infty, \end{aligned}$$

a contradiction. □

Example 3.6. *Let $X = C = \mathbb{R}$ be equipped with usual norm $|\cdot|$. Define $T : X \to X$ by $Tx = x + a$ for all $x \in X$, where $a \neq 0$. Then $d(T^n x, T^n y) = \alpha_n(x) d(x, y)$ for all $n \in \mathbb{N}$ and $x, y \in X$, where $\alpha_n(x) = 1$. Clearly, there is no element $x_0 \in C$ such that $\limsup_{n\to\infty}[\alpha_n(x_0)]^{1/n} < 1$. Therefore, T has no fixed point in $\mathbb{R}$.*

The following result is a natural generalization of Theorem 1.1 and Kirk and Xu [15, Theorem 3.1].

Theorem 3.7. *Let C be a weakly compact convex subset of a Banach space X and $T : C \to C$ a demicontinuous asymptotic pointwise nearly contraction. Then T has a unique fixed point $z \in C$, and for each $x \in C$, the sequence of Picard iterates, $\{T^n x\}$, converges strongly to z.*

Proof. Let $x \in C$. Then $\mathcal{Z}_a(C, \{T^n x\}) \neq \emptyset$. Let $v \in \mathcal{Z}_a(C, \{T^n x\})$. In the lines of (3.1), we have

$$r_a(T^m v, \{T^n x\}) \quad \leq \quad \alpha_m(x)(r_a(v, \{T^n x\}) + a_m) \text{ for all } m \in \mathbb{N}.$$

Note $v \in \mathcal{Z}_a(C, \{T^n x\})$ and $T^m v \in C$ for all $m \in \mathbb{N}$, we have

$$r_a(v, \{T^n x\}) \leq r_a(T^m v, \{T^n x\}) \quad \leq \quad \alpha_m(x)(r_a(v, \{T^n x\}) + a_m) \text{ for all } m \in \mathbb{N} \tag{3.2}$$

Taking limit as $m \to \infty$, the inequality (3.2) yields $r_a(v, \{T^n x\}) \leq \alpha(x) r_a(v, \{T^n x\})$. It follows that $T^n x \to v$. Therefore, from Lemma 2.2, we conclude that $v \in F(T)$. Uniqueness of v follows from Proposition 3.2. □

Following Proposition 3.4, we improve Goebel and Kirk [9, Theorem 1] and Kirk and Xu [15, Theorem 3.5] for the class of pointwise asymptotically nonexpansive mappings with unbounded domain.

Theorem 3.8. *Let C be a nonempty (not necessarily bounded) closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a pointwise asymptotically nonexpansive mapping. Then following statements are equivalent:*

(a) *T has a fixed point.*
(b) *There exists a point $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.*

Further, assume that there is $x_0 \in C$ such that $\{T^n x_0\}$ is bounded. Then $F(T)$ is nonempty, closed and convex.

Proof. $(a) \Rightarrow (b)$ follows easily.

$(b) \Rightarrow (a)$. It follows from Proposition 3.4(b).

Assume that $\{T^n x_0\}$ is bounded for some $x_0 \in C$.

Closedness of $F(T)$: Let $\{z_n\}$ be a sequence in $F(T)$ such that $z_n \to z$. Then it remains to show that $z \in F(T)$. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|z - Tz\| &\leq \|z - z_n\| + \|T^{m+1} z_n - Tz\| \\ &\leq \|z - z_n\| + \alpha_1(z)\|T^m z_n - z\| = (1 + \alpha_1(z))\|z_n - z\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, we conclude that $z \in F(T)$.

Convexity of $F(T)$: Let $x, y \in F(T)$ such that $x \neq y$. Let $z = (x + y)/2$. Then

$$\|T^n z - x\| = \|T^n z - T^n x\| \leq \alpha_n(z)\|z - x\| = \alpha_n(z)(\|x - y\|/2)$$

and

$$\|T^n z - y\| \leq \alpha_n(z)(\|x - y\|/2).$$

Thus,

$$\begin{aligned} \|T^n z - z\| &= \|(T^n z - x) + (T^n z - y)\|/2 \\ &\leq \alpha_n(z)(\|x - y\|/2)\{1 - \delta_X(2/\alpha_n(z))\} \text{ for all } n \in \mathbb{N}, \end{aligned}$$

where δ_X is modulus of convexity of X. It follows that $\lim_{n\to\infty} T^n z = z$ and hence

$$\|Tz - T^{n+1} z\| \quad \leq \quad \alpha_1(z)\|z - T^n z\| \to 0 \text{ as } n \to \infty.$$

Therefore, $z \in F(T)$. □

Using the idea of Theorem 3.8 and Sahu [19, Theorem 4.1], we prove existence theorem for fixed points of demicontinuous pointwise nearly asymptotically nonexpansive mappings.

Theorem 3.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a demicontinuous pointwise nearly asymptotically nonexpansive mapping. Then following statements are equivalent:*

(a) *T has a fixed point.*

(b) *There exists a point $x_0 \in C$ such that such that $\{T^n x_0\}$ is bounded.*

Further, assume that there is $x_0 \in C$ such that $\{T^n x_0\}$ is bounded. Then $F(T)$ is nonempty closed and convex.

We remark that results revelant to Theorems 3.7-3.9 have been obtained very recently in [21].

Example 3.6 shows that conclusion of Theorem 3.5 is not valid when T is pointwise nearly uniformly $\alpha(\cdot)$-Lipschitzian and it turn out our attention to deal with the problem of existence of fixed points of pointwise nearly uniformly $\alpha(\cdot)$-Lipschitzian which are not necessarily pointwise nearly asymptotically nonexpansive mappings. To prove the next result, we need the following preliminaries:

Let C be a nonempty subset of a Banach space X. A nonempty closed convex subset D of C is said to satisfy property (ω) with respect to a mapping $T : C \to C$ ([8]) if

(ω) $\omega_T(x) \subset D$ for every $x \in D$,

where $\omega_T(x)$ denotes the set of all weak subsequential limits of $\{T^n x : n \in \mathbb{N}\}$. Moreover, T is said to satisfy the (ω)-*fixed point property* ((ω)-fpp) if T has a fixed point in every nonempty closed convex subset D of C which satisfies property (ω).

As we know that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian, the (ω)-fixed point property of uniformly L-Lipschitzian mappings is closely related to the class of nonexpansive and asymptotically nonexpansive mappings. In this connection, a deep result of Casini and Maluta [6] was generalized by Lim and Xu [16]) as follows:

Theorem LX (Lim and Xu [16, Theorem 1]). *Let X be a Banach space with uniformly normal structure, C a nonempty bounded subset of X and $T : C \to C$ a uniformly L-Lipschitzian mapping with $L < \sqrt{N(X)}$. Then T satisfies the (ω)-fixed point property.*

Similar results for mappings of asymptotically nonexpansive type were studied in Kim and Xu [10] and Zeng [22].

We now prove a more general existence theorem for fixed points of asymptotic pointwise nearly Lipschitzian mappings in a Banach space with weak uniformly normal structure.

Theorem 3.10. *Let X be a Banach space with weak uniformly normal structure, C a nonempty weakly compact convex subset of X and $T : C \to C$ a mapping. Suppose* *that for each integer $n \in \mathbb{N}$ and $x \in C$, there exist a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ and two functions $\alpha_n(\cdot), \alpha(\cdot) : C \to (0,\infty)$ with $\alpha_n(x) \to \alpha(x)$ pointwise and $\sup_{x\in C} \alpha(x) < \sqrt{WCS(X)}$ such that*

$$\|T^n x - T^n y\| \leq \alpha_n(x)(\|x-y\| + a_n) \text{ for all } y \in C.$$

Also suppose that there exists a nonempty closed convex subset M of C which satisfies property (ω) with respect to T. Then:

(a) *For an arbitrary $x_0 \in M$, there exists an iterative sequence $\{x_m\}$ in M defined by*

$$x_m = w - \lim_{n\to\infty} T^n x_{m-1} \text{ for all } m \in \mathbb{N}. \tag{3.3}$$

(b) *If T is asymptotically regular on C, then there exists an element $v \in M$ such that $\{x_m\}$ converges strongly to $v \in M$ and further, if $a_n = 0$ for all $n \in \mathbb{N}$ or T is demicontinuous at v, then $v \in F(T)$.*

Proof. (a) Since one can easily construct a nonempty closed convex separable subset C_0 of C which is invariant under each T^n (i.e. $T^n(C_0) \subset C_0$ for $n = 1, 2, \ldots$), we may assume that C itself is separable.

The separability of C_0 makes it possible to select a subsequence $\{T^n x\}$ such that $\{T^n x\}$ is weakly convergent for every $x \in C$. For any $x_0 \in M \subset C$, consider a sequence $\{T^n x_0\}$ in C. Suppose $w - \lim_{n\to\infty} T^n x_0 = x_1 \in C$. Using property ($\omega$) we obtain that $x_1 \in M$. Inductively, we can construct a sequence $\{x_m\}$ in M defined by (3.3).

(b) The weak asymptotic regularity of T ensures that

$$x_m = w - \lim_{n\to\infty} T^{n+r} x_{m-1} \text{ for all } r \in \mathbb{N}.$$

We now show that $\{x_m\}$ converges strongly to a fixed point of T. Set

$$L := \sup_{x\in C} \alpha(x), \; D_m := \limsup_{n\to\infty} \|x_m - T^n x_m\|$$

and $R_m := \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\|$ for all $m = 0, 1, 2, \cdots$. By the property of $WCS(X)$, we have

$$R_m = \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\| \leq \frac{1}{WCS(X)} D[\{T^n x_m\}]. \tag{3.4}$$

By the asymptotic regularity of T and the w-l.s.c. of the norm $\|\cdot\|$, we have

$$\begin{aligned}
D[\{T^n x_m\}] &= \limsup_{n\to\infty}\big(\limsup_{r\to\infty}\|T^n x_m - T^r x_m\|\big)\\
&\leq \limsup_{n\to\infty}\big(\limsup_{r\to\infty}(\|T^n x_m - T^{n+r}x_m\| + \|T^{n+r}x_m - T^r x_m\|)\big)\\
&\leq \limsup_{n\to\infty}\big(\limsup_{r\to\infty}(\alpha_n(x_m)(\|x_m - T^r x_m\| + a_n))\big)\\
&= \alpha(x_m)\limsup_{r\to\infty}\|x_m - T^r x_m\|\\
&\leq \alpha(x_m)\limsup_{r\to\infty}\big(\limsup_{s\to\infty}\|T^s x_{m-1} - T^r x_m\|\big)\\
&\leq \alpha(x_m)\limsup_{r\to\infty}\big(\limsup_{s\to\infty}(\|T^s x_{m-1} - T^{r+s}x_{m-1}\|\\
&+ \|T^{r+s}x_{m-1} - T^r x_m\|)\big)\\
&\leq \alpha(x_m)\limsup_{r\to\infty}\big(\limsup_{s\to\infty}(\|T^s x_{m-1} - T^{r+s}x_{m-1}\|\\
&+ \alpha_r(x_m)(\|T^s x_{m-1} - x_m\| + a_r))\big)\\
&\leq \alpha(x_m)^2\limsup_{s\to\infty}\|T^s x_{m-1} - x_m\| = L^2 R_{m-1}.
\end{aligned}$$

Set $\lambda := \frac{L^2}{WCS(X)} < 1$. From (3.4), we obtain

$$R_m \leq \lambda R_{m-1} \leq \lambda^2 R_{m-2} \leq \cdots \leq \lambda^m R_0 \to 0 \text{ as } m \to \infty.$$

For any $m \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{m+1} - x_m\| &\leq \limsup_{n\to\infty}(\|x_{m+1} - T^n x_m\| + \|T^n x_m - x_m\|)\\
&\leq R_m + \limsup_{n\to\infty}\big(\limsup_{r\to\infty}\|T^n x_m - T^r x_{m-1}\|\big)\\
&\leq R_m + \limsup_{n\to\infty}\big(\limsup_{r\to\infty}(\|T^n x_m - T^{n+r}x_{m-1}\|\\
&+ \|T^{n+r}x_{m-1} - T^r x_{m-1}\|)\big)\\
&\leq R_m + \limsup_{n\to\infty}\big(\limsup_{r\to\infty}(\alpha_n(x_m)(\|x_m - T^r x_{m-1}\| + a_n))\big)\\
&\leq (\lambda + L)R_{m-1}\\
&\vdots\\
&\leq (\lambda + L)\lambda^{m-1}R_0.
\end{aligned}$$

One can easily see that $\{x_m\}$ is a Cauchy sequence in M and hence there exists an element $v \in M$ such that $\lim_{m\to\infty} x_m = v$. Observe that

$$\begin{aligned}
\|v - T^n v\| &\leq \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \|T^n x_m - T^n v\|\\
&\leq \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \alpha_n(x_m)(\|x_m - v\| + a_n).
\end{aligned}$$

Taking the limit superior as $n \to \infty$ on both sides we get

$$\limsup_{n\to\infty}\|v - T^n v\| \leq \|v - x_{m+1}\| + R_m + L\|x_m - v\| \to 0 \text{ as } m \to \infty.$$

Thus, $T^n v \to v$. Therefore, $v \in F(T)$ by Proposition 3.4(b) and (c). □

We remark that Theorem 3.10 is a more general result than those in the existing literature of this type. As $N(X) \leq WCS(X)$ and there are Banach spaces for which $N(X) = 1$ while $WCS(X) > 1$, the following result is an improvement on Casini and Maluta [6] and Lim and Xu [16, Theorem 1] even in the context of nearly uniformly L-Lipschitzian mappings.

Corollary 3.11. *Let X be a Banach space with weak uniformly normal structure, C a nonempty weakly compact convex subset of X and $T : C \to C$ a demicontinuous asymptotically regular nearly uniformly L-Lipschitzian mapping such that $L < \sqrt{WCS(X)}$. Then:*

(a) *T has a fixed point in C.*
(b) *T satisfies the (ω)-fixed point property.*

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