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# COMMON FIXED POINTS FOR ASYMPTOTIC POINTWISE NONEXPANSIVE MAPPINGS

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**Abstract.** A common fixed point theorem for a commuting family of asymptotic pointwise nonexpansive mappings in a uniformly convex Banach space is proved. Weak and strong convergence of an iterative sequence defined by two of such mappings are also established. Our results generalize the results of Kirk and Xu [7], Khan and Takahashi [4] and Kozlowski [8].
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## 1. Introduction

A mapping $T$ on a subset $C$ of a Banach space $X$ is said to be asymptotic pointwise nonexpansive if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\|T^n(x) - T^n(y)\| \le \alpha_n(x)\|x - y\|, \tag{1.1}$$

where $\limsup_{n\to\infty} \alpha_n(x) \le 1$, for all $x, y \in C$. This class of mappings was introduced by Kirk and Xu [7], where it was shown that if $C$ is a bounded closed convex subset of a uniformly convex Banach space $X$, then every asymptotic pointwise nonexpansive mapping $T : C \to C$ has a fixed point. In 2009, Hussain and Khamsi [2] extended Kirk-Xu's result to the case of metric spaces, specifically to the so-called CAT(0) spaces. Recently, Khamsi and Kozlowski [3] proved an analogous result in the framework of modular function spaces. Moreover, Kozlowski [8] defined an iterative sequence for an asymptotic pointwise nonexpansive mapping $T : C \to C$ by

$$x_1 \in C,\ x_{k+1} = (1 - t_k)x_k + t_k T^{n_k}\left((1 - s_k)x_k + s_k T^{n_k}(x_k)\right),\ k \in \mathbb{N}, \tag{1.2}$$

where $\{n_k\}$ is an increasing sequence of natural numbers for which

$$\limsup_{k\to\infty} a_{n_k}(x_k) = 1.$$

He proved, under some suitable assumptions, that the sequence $\{x_k\}$ defined by (1.2) converges weakly to a fixed point of $T$ where $X$ is a uniformly convex Banach space which satisfies the Opial condition and $\{x_k\}$ converges strongly to a fixed point of

$T$ provided $T^m$ is a compact mapping for some $m \in \mathbb{N}$. In the latter case, no Opial condition is assumed for the uniformly convex space $X$.

In this paper, motivated by the results mentioned above, we ensure the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings. Furthermore, we obtain weak and strong convergence theorems of a sequence defined by two of such mappings.

## 2. Preliminaries

The notion of asymptotic contractions was introduced by Kirk [6] as the following statement.

Let $\Psi$ denote the class of all mappings $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(i) $\psi$ is continuous;

(ii) $\psi(s) < s$ for all $s > 0$.

**Definition 2.1.** Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be an asymptotic contraction ([6]) if

$$d(T^n(x), T^n(y)) \leq \psi_n(d(x, y)) \text{ for all } x, y \in X, \tag{2.1}$$

where $\psi_n : [0, \infty) \to [0, \infty)$ and $\psi_n \to \psi \in \Psi$ uniformly on the range of $d$.

$T$ is called a pointwise contraction ([7]) if there exists a mapping $\alpha : X \to [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha(x)d(x, y) \text{ for each } y \in X. \tag{2.2}$$

**Definition 2.2.** Let $(X, \|\cdot\|)$ be a Banach space. A mapping $T : X \to X$ is called an asymptotic pointwise mapping ([2]) if there exists a sequence of mappings $\alpha_n : X \to [0, \infty)$ such that

$$\|T^n(x) - T^n(y)\| \leq \alpha_n(x) \|x - y\| \text{ for any } y \in X. \tag{2.3}$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : X \to [0, 1)$, then $T$ is called an asymptotic pointwise contraction.

(ii) If $\limsup_{n\to\infty} \alpha_n(x) \leq 1$, then $T$ is called asymptotic pointwise nonexpansive.

(iii) If $\limsup_{n\to\infty} \alpha_n(x) \leq k$, with $0 < k < 1$, then $T$ is called strongly asymptotic pointwise contraction.

A point $x \in X$ is called a fixed point of $T$ if $x = T(x)$. We shall denote by $F(T)$ the set of fixed points of $T$ and by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from $C$ into $C$. Let $S, T \in \mathcal{T}(C)$, without loss of generality, we can assume that there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that for all $x, y \in C$ and $n \in \mathbb{N}$,

$$\|S^n(x) - S^n(y)\| \leq \alpha_n(x)\|x - y\|, \ \|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|, \text{ and}$$
$$\limsup_{n\to\infty} \alpha_n(x) \leq 1.$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. Again, without loss of generality, we can assume that

$$\|S^n(x) - S^n(y)\| \leq a_n(x)\|x - y\|, \ \|T^n(x) - T^n(y)\| \leq a_n(x)\|x - y\|, \text{ and}$$

$$\lim_{n\to\infty} a_n(x) = 1, \ a_n(x) \geq 1, \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Define $b_n(x) = a_n(x) - 1$. Then, for each $x \in C$ we have $\lim_{n\to\infty} b_n(x) = 0$.

## 3. A common fixed point theorem

Before proving our fixed point theorem, we need the following lemma.

**Lemma 3.1.** *(See* [7, Theorem 3.5]*) Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $X$. Then every asymptotic pointwise nonexpansive mapping $T : C \to C$ has a fixed point. Moreover, $F(T)$ is closed and convex.*

The following result is a counterpart of [10, Theorem 6] and extends Theorem 3.5 of [7].

**Theorem 3.2.** *Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$. Then every commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on $C$ has a nonempty closed convex common fixed point set.*

*Proof.* Let $T_1, T_2, ..., T_n \in \mathcal{S}$. By Lemma 3.1, $F(T_1)$ is a nonempty closed and convex subset of $C$. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty closed and convex for some $k \in \mathbb{N}$ with $1 < k \leq n$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \leq j < k$, we have

$$T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x).$$

Thus $T_k(x)$ is a fixed point of $T_j$, which implies that $T_k(x) \in A$, therefore $A$ is invariant under $T_k$. Again, by Lemma 3.1, $T_k$ has a fixed point in $A$, i.e.,

$$\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \neq \emptyset.$$

Also, the set is closed and convex. By induction, $\bigcap_{j=1}^{n} F(T_j) \neq \emptyset$. This shows that the set $\{F(T) : T \in \mathcal{S}\}$ has the finite intersection property. We note that $C$ is weakly compact because $X$ is reflexive. Since $F(T)$ is weakly closed for every $T \in \mathcal{S}$, we have $\bigcap_{T\in\mathcal{S}} F(T) \neq \emptyset$. Obviously, the set is closed and convex. □

## 4. Convergence theorems

We now collect some basic definitions and lemmas.

**Lemma 4.1.** *(See* [1]*) Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly-index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \leq 0, \quad \textit{and} \quad r_{k+n} \leq r_k + d_{k,n}$$

*for each $k, n \in \mathbb{N}$. Then $\{r_k\}$ converges to an $r \in \mathbb{R}$.*

**Lemma 4.2.** *(See* [9, 11]*) Let $X$ be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences in $X$ such that*

$$\limsup_{n\to\infty} \|u_n\| \leq r, \ \limsup_{n\to\infty} \|v_n\| \leq r, \quad \textit{and} \quad \lim_{n\to\infty} \|t_n u_n + (1 - t_n) v_n\| = r,$$

*for some* $r \geq 0$. *Then* $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

**Lemma 4.3.** *(See* [8, Lemma 3.1]*) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$ and let $T \in \mathcal{T}(C)$ be such that $a_n$ is a bounded function for every $n \in \mathbb{N}$. If $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$ then for any $m \in \mathbb{N}$, $\lim_{n\to\infty} \|T^m(x_n) - x_n\| = 0$.*

**Lemma 4.4.** *(See* [8, Theorem 3.1]*) Let $X$ be a uniformly convex Banach space with the Opial property and let $C$ be a nonempty bounded closed convex subset of $X$. Let $T \in \mathcal{T}(C)$ be such that $a_n$ is a bounded function for every $n \in \mathbb{N}$. Then the conditions $\omega \in X$, $\{x_n\} \subset X$, $x_n \rightharpoonup \omega$, and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$, imply $\omega \in F(T)$.*

**Definition 4.5.** Let $S, T \in \mathcal{T}(C)$ and let $\{n_k\}$ be an increasing sequence of natural numbers. Let $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$. Define a sequence $\{x_k\}$ in $C$ as:

$$x_1 \in C,\ x_{k+1} = (1 - t_k)x_k + t_k S^{n_k}\left((1 - s_k)x_k + s_k T^{n_k}(x_k)\right),\ k \in \mathbb{N}. \tag{4.1}$$

We say that the sequence $\{x_k\}$ in (4.1) is well defined if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$.

As in [8], we observe that $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$. Hence we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well defined.

**Lemma 4.6.** *Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $X$ and let $S, T \in \mathcal{T}(C)$ be such that $\sum_{n=1}^{\infty} b_n(x) < \infty$ for each $x \in C$. Let $\omega \in F(S) \cap F(T)$, $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.1) is well-defined. Then $\lim_{k\to\infty} \|x_k - \omega\|$ exists.*

*Proof.* For each $k \in \mathbb{N}$, we let $y_k = (1 - s_k)x_k + s_k T^{n_k}(x_k)$. Then

$$\begin{aligned}
\|x_{k+1} - \omega\| &= \|(1 - t_k)x_k + t_k S^{n_k}(y_k) - \omega\| \\
&\leq (1 - t_k)\|x_k - \omega\| + t_k\|S^{n_k}(y_k) - \omega\| \\
&\leq (1 - t_k)\|x_k - \omega\| + t_k(1 + b_{n_k}(\omega))\|y_k - \omega\| \\
&\leq (1 - t_k)\|x_k - \omega\| \\
&\quad + t_k(1 + b_{n_k}(\omega))[(1 - s_k)\|x_k - \omega\| + s_k\|T^{n_k}(x_k) - \omega\|] \\
&\leq (1 - t_k)\|x_k - \omega\| \\
&\quad + t_k(1 + b_{n_k}(\omega))[(1 - s_k)\|x_k - \omega\| + s_k(1 + b_{n_k}(\omega))\|x_k - \omega\|] \\
&\leq (1 - t_k)\|x_k - \omega\| + t_k(1 + b_{n_k}(\omega))(1 + s_k b_{n_k}(\omega))\|x_k - \omega\| \\
&\leq (1 - t_k)\|x_k - \omega\| + t_k(1 + b_{n_k}(\omega))^2\|x_k - \omega\| \\
&\leq \|x_k - \omega\| + t_k(2b_{n_k}(\omega) + b_{n_k}^2(\omega))\|x_k - \omega\| \\
&\leq \|x_k - \omega\| + 3b_{n_k}(\omega)\|x_k - \omega\| \\
&\leq \|x_k - \omega\| + 3diam(C)b_{n_k}(\omega).
\end{aligned}$$

It follows that for each $n \in \mathbb{N}$,

$$\|x_{k+n} - \omega\| \leq \|x_k - \omega\| + 3diam(C) \sum_{i=k}^{k+n-1} b_{n_i}(\omega).$$

By assumption, $\limsup_{k\to\infty}\limsup_{n\to\infty}\sum_{i=k}^{k+n-1} b_{n_i}(\omega) = 0$. By Lemma 4.1, letting $r_k = \|x_k - \omega\|$ and $d_{k,n} = 3diam(C)\sum_{i=k}^{k+n-1} b_{n_i}(\omega)$, there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty}\|x_k - \omega\| = r$. □

**Lemma 4.7.** *Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $X$ and let $S, T \in \mathcal{T}(C)$ be such that $\sum_{n=1}^{\infty} b_n(x) < \infty$ for each $x \in C$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.1) is well-defined. Then*

$$\lim_{k\to\infty}\|S^{n_k}(y_k) - x_k\| = 0 \tag{4.2}$$

*and*

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0, \tag{4.3}$$

*where $y_k = (1 - s_k)x_k + s_k T^{n_k}(x_k)$, for each $k \in \mathbb{N}$.*

*Proof.* Let $\omega \in F(S) \cap F(T)$. Then there exists an $r \in \mathbb{R}$ from Lemma 4.6 such that

$$\lim_{k\to\infty}\|x_k - \omega\| = r. \tag{4.4}$$

Note that

$$\begin{aligned}
\limsup_{k\to\infty}\|S^{n_k}(y_k) - \omega\| &= \limsup_{k\to\infty}\|S^{n_k}(y_k) - S^{n_k}(\omega)\| \\
&\le \limsup_{k\to\infty} a_{n_k}(\omega)\,\|y_k - \omega\| \\
&\le \limsup_{k\to\infty} a_{n_k}(\omega)\,[(1 - s_k)\|x_k - \omega\| + s_k\|T^{n_k}(x_k) - \omega\|] \\
&\le \limsup_{k\to\infty}\left[a_{n_k}(\omega)\,(1 - s_k)\|x_k - \omega\| + s_k a_{n_k}^2(\omega)\|x_k - \omega\|\right] \\
&\le r,
\end{aligned} \tag{4.5}$$

and

$$\lim_{k\to\infty}\|(1 - t_k)(x_k - \omega) + t_k(S^{n_k}(y_k) - \omega)\| = \lim_{k\to\infty}\|x_{k+1} - \omega\| = r. \tag{4.6}$$

It follows from (4.4), (4.5), (4.6), and Lemma 4.2 that

$$\lim_{k\to\infty}\|S^{n_k}(y_k) - x_k\| = 0.$$

This, together with the construction formula for $x_{k+1}$, we also obtain that

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0.$$

□

**Lemma 4.8.** *Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $X$ and let $S, T \in \mathcal{T}(C)$ be such that $\sum_{n=1}^{\infty} b_n(x) < \infty$ for each $x \in C$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.1) is well-defined. Then*

$$\lim_{k\to\infty}\|S^{n_k}(x_k) - x_k\| = 0 = \lim_{k\to\infty}\|T^{n_k}(x_k) - x_k\|. \tag{4.7}$$

*Proof.* Let $\omega \in F(S) \cap F(T)$. Then there exists $r \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \|x_k - \omega\| \,=\, r. \tag{4.8}$$

Since

$$\|T^{n_k}(x_k) - \omega\| = \|T^{n_k}(x_k) - T^{n_k}(\omega)\| \le\, a_{n_k}(\omega)\|x_k - \omega\|,$$

then we get

$$\limsup_{k\to\infty} \|T^{n_k}(x_k) - \omega\| \le\, \limsup_{k\to\infty} a_{n_k}(\omega)\|x_k - \omega\| \,=\, r. \tag{4.9}$$

Now,

$$\begin{aligned}\|y_k - \omega\| \,&=\, \|(1-s_k)x_k + s_k T^{n_k}(x_k) - \omega\|\\ &\le\, (1-s_k)\,\|x_k - \omega\| + s_k\,\|T^{n_k}(x_k) - \omega\|\\ &\le\, (1-s_k)\,\|x_k - \omega\| + s_k\, a_{n_k}(\omega)\|x_k - \omega\|.\end{aligned}$$

This implies that

$$\limsup_{k\to\infty} \|y_k - \omega\| \le \limsup_{k\to\infty} \|x_k - \omega\| \,=\, r. \tag{4.10}$$

On the other hand

$$\begin{aligned}\|x_k - \omega\| \,&\le\, \|x_k - S^{n_k}(y_k)\| + \|S^{n_k}(y_k) - \omega\|\\ &\le\, \|x_k - S^{n_k}(y_k)\| + a_{n_k}(\omega)\|y_k - \omega\|.\end{aligned} \tag{4.11}$$

From (4.2) and (4.11), we get that

$$r \,=\, \liminf_{k\to\infty} \|x_k - \omega\| \,\le\, \liminf_{k\to\infty} \|y_k - \omega\|. \tag{4.12}$$

From (4.10) and (4.12), we have $\lim_{k\to\infty} \|y_k - \omega\| \,=\, r$. Thus

$$\lim_{k\to\infty} \|(1-s_k)(x_k - \omega) + s_k(T^{n_k}(x_k) - \omega)\| \,=\, \lim_{k\to\infty} \|y_k - \omega\| \,=\, r. \tag{4.13}$$

It follows from (4.8), (4.9), (4.13), and Lemma 4.2 that

$$\lim_{k\to\infty} \|T^{n_k}(x_k) - x_k\| \,=\, 0. \tag{4.14}$$

Since

$$\begin{aligned}\|S^{n_k}(x_k) - x_k\| &\le\, \|S^{n_k}(x_k) - S^{n_k}(y_k)\| + \|S^{n_k}(y_k) - x_k\|\\ &\le\, a_{n_k}(x_k)\|x_k - y_k\| + \|S^{n_k}(y_k) - x_k\|\\ &\le\, s_k a_{n_k}(x_k)\|x_k - T^{n_k}(x_k)\| + \|S^{n_k}(y_k) - x_k\|,\end{aligned}$$

it follows from (4.2) and (4.14) that

$$\lim_{k\to\infty} \|S^{n_k}(x_k) - x_k\| \,=\, 0.$$

Therefore the proof is complete. □

**Definition 4.9.** A strictly increasing sequence $\{n_i\} \subset \mathbb{N}$ is called quasi-periodic ([1]) if the sequence $\{n_{i+1} - n_i\}$ is bounded, or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of $p$ consecutive natural numbers must contain a term of the sequence $\{n_i\}$. The smallest of such numbers $p$ will be called a quasi-period of $\{n_i\}$.

**Lemma 4.10.** *Let $C$ be a nonempty bounded closed and convex subset of a uniformly convex Banach space $X$ and let $S, T \in \mathcal{T}(C)$ be such that $F(S) \cap F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} b_n(x) < \infty$ for every $x \in C$ and $a_n$ is bounded for every $n \in \mathbb{N}$. Let $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$, and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.1) is well-defined. If, in addition, the set of indices $\mathcal{J} = \{j : n_{j+1} = 1 + n_j\}$ is quasi-periodic, then*

$$\lim_{k\to\infty} \|S(x_k) - x_k\| = 0 = \lim_{k\to\infty} \|T(x_k) - x_k\|. \tag{4.15}$$

*Proof.* Assume that $\mathcal{J} = \{j : n_{j+1} = 1 + n_j\}$ is quasi-periodic with period $p$. For each $k \in \mathcal{J}$, we have

$$\begin{aligned}\|x_k - T(x_k)\| \leq\ & \|x_k - x_{k+1}\| + \|x_{k+1} - T^{n_{k+1}}(x_{k+1})\| \\ & + \|T^{n_{k+1}}(x_{k+1}) - T^{n_{k+1}}(x_k)\| + \|TT^{n_k}(x_k) - T(x_k)\| \\ \leq\ & \|x_k - x_{k+1}\| + \|x_{k+1} - T^{n_{k+1}}(x_{k+1})\| \\ & + a_{n_{k+1}}(x_{k+1})\|x_{k+1} - x_k\| + a_1(x_k)\|T^{n_k}(x_k) - x_k\|.\end{aligned}$$

This, together with (4.3) and (4.7), we can obtain that $\|T(x_k) - x_k\| \to 0$ as $k \to \infty$ through $\mathcal{J}$.

To prove that $\lim_{k\to\infty} \|T(x_k) - x_k\| = 0$ is similar to the proof of Lemma 4.3 of [8], therefore we omit it. Similarly, we also have $\lim_{k\to\infty} \|S(x_k) - x_k\| = 0$. □

The following theorem extends Theorem 1 of [4] and Theorem 5.1 of [8].

**Theorem 4.11.** *Let $X$ be a uniformly convex Banach space with the Opial property and $C$ be a nonempty bounded closed convex subset of $X$. Let $S, T \in \mathcal{T}(C)$ be such that $F(S) \cap F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} b_n(x) < \infty$ for every $x \in C$ and $a_n$ is bounded for every $n \in \mathbb{N}$. Let $\{s_k\}, \{t_k\} \subset [a, b] \subset (0, 1)$. Let $\{n_k\}$ be such that the sequence $\{x_k\}$ in (4.1) is well defined. If the set $\mathcal{J} = \{j; n_{j+1} = 1 + n_j\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges weakly to a common fixed point of $S$ and $T$.*

*Proof.* We have by Lemma 4.6 that $\lim_{n\to\infty} \|x_k - \omega\|$ exists for every $\omega \in F(S) \cap F(T)$. We shall prove that $\{x_k\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. For this, we suppose that there are subsequences $\{x_{k_i}\}$ and $\{x_{k_j}\}$ of $\{x_k\}$ which converge weakly to $u$ and $v$, respectively. By Lemma 4.10, $\lim_{k\to\infty} \|S(x_k) - x_k\| = 0$. It follows from Lemma 4.4 that $S(u) = u$. Similarly, we can prove that $T(u) = u$. By using the same argument, we can prove that $v \in F(S) \cap F(T)$. Finally, we prove that $u = v$. Suppose not, then by the Opial property we get that

$$\begin{aligned}\lim_{k\to\infty} \|x_k - u\| &= \lim_{i\to\infty} \|x_{k_i} - u\| \\ &< \lim_{i\to\infty} \|x_{k_i} - v\| \\ &= \lim_{k\to\infty} \|x_k - v\| \\ &= \lim_{j\to\infty} \|x_{k_j} - v\| \\ &< \lim_{j\to\infty} \|x_{k_j} - u\| \\ &= \lim_{k\to\infty} \|x_k - u\|.\end{aligned}$$

This is a contradiction. Therefore the proof is complete. □

The following theorem extends Theorem 2 of [4] and Theorem 6.1 of [8].

**Theorem 4.12.** *Let $X$ be a uniformly convex Banach space and $C$ be a nonempty bounded closed convex subset of $X$. Let $S,T \in \mathcal{T}(C)$ be such that $F(S)\cap F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} b_n(x) < \infty$ for every $x \in C$ and $a_n$ is bounded for every $n \in \mathbb{N}$. Assume that there exists $m \in \mathbb{N}$ so that $S^m$ or $T^m$ is compact. Let $\{s_k\},\{t_k\} \subset [a,b] \subset (0,1)$. Let $\{n_k\}$ be such that the sequence $\{x_k\}$ in (4.1) is well defined. If the set $\mathcal{J} = \{j;\, n_{j+1} = 1+n_j\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges strongly to a common fixed point of $S$ and $T$.*

*Proof.* We will prove only the case that $S^m$ is compact (the proof for the other case is identical). Observe that by Lemma 4.10,

$$\lim_{k\to\infty} \|S(x_k) - x_k\| = 0.$$

By Lemma 4.3,

$$\lim_{k\to\infty} \|S^m(x_k) - x_k\| = 0. \tag{4.16}$$

Since $S^m$ is compact, we can find a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\lim_{j\to\infty} \|S^m(x_{k_j}) - x\| = 0 \text{ for some } x \in C. \tag{4.17}$$

Since

$$\|x_{k_j} - x\| \leq \|x_{k_j} - S^m(x_{k_j})\| + \|S^m(x_{k_j}) - x\|,$$

it follows from (4.16) and (4.17) that

$$\lim_{j\to\infty} \|x_{k_j} - x\| = 0. \tag{4.18}$$

Since $S$ and $T$ are continuous, then

$$\lim_{j\to\infty} S(x_{k_j}) = S(x) \text{ and } \lim_{j\to\infty} T(x_{k_j}) = T(x).$$

This, together with (4.15) and (4.18), we get that

$$\|S(x) - x\| = 0 = \|T(x) - x\|.$$

This means $x \in F(S)\cap F(T)$. Therefore $\{x_{k_j}\}$ converges strongly to $x \in F(S)\cap F(T)$. But $\lim_{k\to\infty} \|x_k - x\|$ exists, $\{x_k\}$ must itself converges to $x$. This completes the proof. □

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