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SOME RENORMINGS WITH THE STABLE FIXED POINT PROPERTY

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Dedicated to K. Goebel, on the occasion of his retirement, and to L. Ciric, W.A. Kirk and I.A. Rus on the occasion of their 75th birthday.

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Abstract. In this paper, we prove that for any number $\lambda < (\sqrt{33}-3)/2$, any separable space X can be renormed in such a way that X satisfies the weak fixed point property for non-expansive mappings and this property is inherited for any other isomorphic space Y such that the Banach-Mazur distance between X and Y is less than λ. We also prove that any, in general nonseparable, Banach space with an extended unconditional basis can be renormed to satisfy the w-FPP with the same stability constant.

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1. Introduction

A Banach space X is said to satisfy the fixed point property for non-expansive mappings (FPP) (respectively the weak-fixed point property for non-expansive mappings (w-FPP)) if every non-expansive mapping defined from a convex closed bounded (resp.: convex weakly compact) subset C of X into C has a fixed point. Many geometrical properties of X (uniform convexity, uniform smoothness, uniform convexity in every direction, uniform non-squareness, normal structure, etc) are known to imply either the FPP or the w-FPP for Banach spaces. Furthermore, some of these properties imply a certain stability of the FPP (w-FPP) in the sense that if X satisfies such a property and Y is another Banach space which is isomorphic to X and the Banach-Mazur distance between them is small enough, then Y also satisfies the FPP

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(w-FPP). In this case we say that X satisfies the stable FPP (stable w-FPP). (The monographs[1], [7] and [10] provide detailed information on this subject).

A relevant topic in the last years (see[5], [8],[9],[11] [12]) has been to determine whether a Banach space can be renormed to satisfy either the w-FPP or the FPP. More recently [6], the problem of existence of a renorming satisfying the stable FPP (stable w-FPP) is considered. In this paper we continue the study of this problem, proving that for any number $\lambda < (\sqrt{33}-3)/2$ any separable space X can be renormed in such a way that X satisfies the w-FPP and this property is inherited for any other isomorphic space Y such that the Banach-Mazur distance between X and Y is less than λ. The value $\lambda < (\sqrt{33}-2)/2$ first appeared in Metric Fixed Point Theory in a paper by P.K. Lin [13], where it is proved that any Banach space with unconditional basis satisfies the w-FPP when the unconditional basic constant is less than $(\sqrt{33}-3)/2$. As we will see in the next section, an easy consequence of this result is the following: every Banach space with unconditional basis X can be renormed to satisfy the w-FPP with stability constant $(\sqrt{33}-2)/2$. However, there are separable Banach spaces without any unconditional basis. In spite of this fact, we shall prove that the above stability property for a renorming still holds for every separable Banach space.

In the case of nonseparable Banach spaces, we can use the technique in [13] to prove that any Banach space with an extended unconditional basis can be renormed to satisfy the w-FPP with the same stability constant.

2. Stable renormings for separable spaces

We start proving the stability version of the Lin's result [13]. Recall that a Schauder basis $\{x_n\}$ of a Banach space X is said to be unconditional (see, for instance [2]) if every convergent series of the form $\sum_{n=1}^{\infty} t_n x_n$ is unconditionally convergent or, equivalently, for every convergent series $\sum_{n=1}^{\infty} t_n x_n$, and every sequence $\{\epsilon_n\}$ with $\epsilon_n = \pm 1$, the series $\sum_{n=1}^{\infty} \epsilon_n t_n x_n$ converges, or equivalently there exists a constant $K > 1$ such that if A and B are finite subsets of $\mathbb{N}$ with $A \subset B$, then for any sequence $\{t_n\}$ of scalars we have $\|\sum_{n\in A} t_n x_n\| \le K\|\sum_{n\in B} t_n x_n\|$. The smallest K satisfying this inequality is called the unconditional constant of $\{x_n\}$. The basis is called unconditionally monotone, if $K = 1$.

Theorem 2.1. *Let X be a Banach space which can be isomorphically embedded in a Banach space Z with an unconditional basis and $\lambda < (\sqrt{33}-2)/2$. Then, there exists an equivalent norm $|\cdot|$ on X such that if Y is an isomorphic Banach space and the the Banach-Mazur distance between $(X, |\cdot|)$ and Y is less than λ, then Y satisfies the w-FPP.*

Proof. Let $\{x_n\}$ be an unconditional basis of Z. For every $x = \sum t_n x_n \in Z$, define the equivalent norm

$$|x| = \sup\{\|\epsilon_n t_n x_n\| : \epsilon_n = \pm 1\}.$$

It is known [2] that $|\cdot|$ is equivalent to the original norm of Z and $\{x_n\}$ is unconditionally monotone for this new norm. Furthermore, if Y is isomorphic to $(X, |\cdot|)$ and the Banach-Mazur distance between Y and $(X, |\cdot|)$ is not greater than λ, we can assume that Y is the space X with a norm p which satisfies

$$(1/\lambda)|x| \leq p(x) \leq |x|$$

for every $x \in X$. By lemma 2.2 in [6] this norm can be extended to a norm on Z satisfying the same inequalities. Thus, $\{x_n\}$ is an unconditional basis for (X,p) with unconditional constant less than λ. Hence, (Z,p) satisfies the w-FPP and so does Y. □

However, there are separable Banach spaces which cannot embedded in Banach spaces with unconditional basis. Indeed, from Theorems 15.1, 15,2 15.4 in [16] (see also Proposition 4.1 in [14]) we can deduce the following:

Theorem 2.2. *The spaces $C([0,1]), L_1([0,1])$ and the James space J cannot be isomorphically embedded in a Banach space with unconditional basis.*

In spite of this fact, we shall prove that 2.1 still holds for every separable Banach space.

In the following, we will denote by $\ell_\infty(X)$ (respectively $c_0(X)$) the linear space of all bounded sequences (respectively all sequences convergent to zero) in the Banach space X. By $[X]$ we denote the quotient space $\ell_\infty(X)/c_0(X)$ endowed with the quotient norm $\|[z_n]\| = \limsup_n \|z_n\|$ where $[z_n]$ is the equivalent class of $(z_n) \in \ell_\infty(X)$. By identifying $x \in X$ with the class $[(x,x,...)]$ we can consider X as a subset of $[X]$. If C is a subset of X we can define the set $[C] = \{[z_n] \in [X] : z_n \in C \text{ for every } n \in N\}$. If T is a mapping from C into C, then $[T] : [C] \to [C]$ given by $[T]([x_n]) = [Tx_n]$ is a well defined mapping. If $\{S_n\}$ is a sequence of mappings from X into X, we will denote by $[S]$ the mapping from $[X]$ into $[X]$ defined by $[S][x_n] = [S_n(x_n)]$.

For two subsets A and B of $\mathbb{N}$ we write$A << B$ if $\max A < \min B$. As in [14], let X be a Banach space with a monotonous Schauder basis and $\mathcal{G}$ the set of all nondecreasing bounded sequences of nonnegative integers $g = \{p(n)\}$. For any $a \in (-1,0)$, consider an equivalent norm on X defined by $\|x\|_a = \sup\{\|g(x)\| : g \in \mathcal{G}\}$ where $g(x) := \sum_{n=1}^{\infty} a^{p(n)} t_n e_n$ for $g = \{p(n)\}$ and $x = \sum_{n=1}^{\infty} t_n e_n$. We will use the following lemma which is a particular case of Lemma 3.1 in [6].

Lemma 2.3. *Let X be a Banach space with a monotonous Schauder basis $\{x_n\}$ and $A_1 << A_2$ two finite intervals in $\mathbb{N}$. Denote by P_{A_i} the natural projections onto $\{x_n : n \in A_i\}$. Then, for $m = 1,2$ we have*

$$\|I - 2\sum_{i=1}^{m} P_{A_i}\|_a \leq 1 + 2m(1 - a^{2m}).$$

Theorem 2.4. *Let X be a separable Banach space and $\lambda < (\sqrt{33} - 3)/2$. Then, X can be equivalently renormed in such a way that if $|\cdot|$ is the new norm and Y is an isomorphic Banach space such that the Banach-Mazur distance between $(X, |\cdot|)$ and Y is less than λ, then Y satisfies the w-FPP*

Proof. We know that X can be isometrically embedded in a Banach space with a monotonous Schauder basis. Since the w-FPP is inherited by closed subspaces, we assume that X has a monotonous Schauder basis $\{e_n\}$. For any $a \in (-1,0)$, define $\|x\|_a$ as above. Assume that $\lambda < (\sqrt{33}-2)/2$ and choose $a \in (-1,0)$ such that

$$a^4 > 1 - \frac{1}{8}\left(\frac{\sqrt{33}-3}{2\lambda} - 1\right).$$

It is easy to check that the above inequality implies $\lambda < \dfrac{\sqrt{33}-3}{2\left(1+8(1-a^4)\right)}$.

Assume that Y is X with a norm $|\cdot|$ which satisfies $\|x\|_a \le |x| \le \lambda\|x\|_a$ for every $x \in X$ and that $(X, |\cdot|)$ fails the w-FPP. Hence, there exists a weakly compact convex subset K of X which is not a singleton and it is minimal invariant for a $|\cdot|$-non-expansive mapping T. By multiplication, we can assume that $\mathrm{diam}(K) = 1$. Let $\{x_n\}$ be an approximate fixed point sequence for T in K. By translation and passing to a subsequence, we can assume that $\{x_n\}$ is weakly null. Let $y_n = x_{2n}$ and $z_n = x_{2n+1}$. Then $\{y_n\}$ and $\{z_n\}$ are also approximate fixed point sequences for T. Passing to appropriated sequences and using the gliding hump method, we can find two sequences of finite intervals $\{I_n\}$ and $\{J_n\}$ in $\mathbb{N}$ satisfying $I_n \ll J_n \ll I_{n+1}$ and such that the natural projections P_n and Q_n onto I_n and J_n respectively satisfy $\lim_n P_n y_n = y_n, \lim_n Q_n z_n = z_n$, and $\lim_n P_n z_n = \lim_n Q_n y_n = 0$. We claim that

$$\limsup_n |y_n + z_n| \le \lambda(1 + 4(1-a^2)).$$

Indeed, by lemma 2.3 we have

$$\begin{aligned}
\limsup_n |y_n + z_n| &= \limsup_n |y_n - z_n - 2Q_n(y_n - z_n)| \\
&\le \lambda \limsup_n \|(I - 2Q_n)(y_n - z_n)\|_a \\
&\le \lambda(1 + 4(1-a^2)) \limsup_n |y_n - z_n| \\
&\le \lambda(1 + 4(1-a^2))
\end{aligned}$$

Let $[y] = [y_n]$, $[z] = [z_n]$ and the projections $[P] = [P_n]$ and $[Q] = [Q_n]$. Note that $[P]x = [Q]x = [0]$ for every $x \in X$ and moreover, $[P][y] = [y], [Q][z] = [z]$ and $[P][z] = [Q][y] = [0]$. Let

$$\begin{aligned}
[W] \;=\; &\Big\{[w] \in [K] : \text{ there exists } x \in K \text{ such that } \Big|[w] - [x]\Big| \le \frac{\lambda}{2}(1 + 4(1-a^2)), \\
&\Big|[w] - [y]\Big| \le \frac{1}{2} \text{ and } \Big|[w] - [z]\Big| \le \frac{1}{2}\Big\}.
\end{aligned}$$

We have that $[W]$ is a nonempty bounded closed convex set because $\left[\dfrac{y+z}{2}\right] \in [W]$. Hence $[W]$ contains an approximate fixed point sequence for $[T]$. Assume that there exists an element $[w] \in [W]$ such that $|[w]| = 1$. Let $x \in K$ such that $\Big|[w] - [x]\Big| \le$

$\dfrac{\lambda}{2}(1+4(1-a^2))$ and let $[f] \in X^*$ with $[f]([w]) = 1 = \Big|[f]\Big|$. Then we have

$$1 - [f]([y]) = [f]([w]-[y]) \leq \Big|[w]-[y]\Big| \leq \frac{1}{2}$$

so $[f]([y]) \geq \dfrac{1}{2}$. Similarly, $[f]([z]) \geq \dfrac{1}{2}$. Since

$$1 - [f]([x]) = [f]([w]-[x]) \leq \Big|[w]-[x]\Big| \leq \frac{\lambda}{2}(1+4(1-a^2))$$

we have $[f]([x]) \geq 1 - \dfrac{\lambda}{2}(1+4(1-a^2))$.

Let $\alpha = [f]\Big(([I]-[P]-[Q])[w]\Big)$. Then

$$\begin{aligned} 1-\alpha &= [f]([w]) - [f]\Big(([I]-[P]-[Q])[w]\Big) \\ &= [f]\Big(([P]+[Q])[w]\Big) \\ &= [f]([P][w]) + [f]([Q][w]) \end{aligned}$$

so either $[f]([P][w]) \leq \dfrac{1-\alpha}{2}$ or $[f]([Q][w]) \leq \dfrac{1-\alpha}{2}$.

Assume that $[f]([P][w]) \leq \dfrac{1-\alpha}{2}$. From lemma 2.3, we have

$$\begin{aligned} 2(1-\alpha) - \frac{\lambda}{2}(1+8(1-a^4)) &\leq (2-2\alpha) - \frac{\lambda}{2}(1+4(1-a^2)) \\ &\leq 2[f]\Big(([P]+[Q])[w]\Big) - [f]\Big([w]-[x]\Big) \\ &= [f]\Big((2[P]+2[Q])[w]\Big) - [f]\Big([w]-[x]\Big) \\ &= [f]\Big((2[P]+2[Q])([w]-[x])\Big) - [f]\Big([w]-[x]\Big) \\ &= [f]\Big((2[P]+2[Q]-[I])([w]-[x])\Big) \\ &\leq \Big|[f]\Big|\Big|(2[P]+2[Q]-[I])([w]-[x])\Big| \\ &\leq \lambda\Big\|[I]-2[P]-2[Q]\Big\|_a\Big|[w]-[x]\Big| \\ &\leq \lambda\cdot(1+8(1-a^4))\cdot\frac{\lambda}{2}(1+4(1-a^2)) \\ &\leq \frac{\lambda^2}{2}(1+8(1-a^4))^2 \end{aligned}$$

and

$$\begin{aligned}
\alpha + \frac{1}{2} &= \frac{1}{2} + 1 - (1-\alpha) \\
&\leq [f]([y]) + [f]([w]) - 2[f]([P][w]) \\
&= [f]([w]-[y]) + 2[f]([y]) - 2[f]([P][w]) \\
&= [f]([w]-[y]) + 2[f]([P][y]) - 2[f]([P][w]) \\
&= [f]([w]-[y]) + 2[f]\Big([P]([y]-[w])\Big) \\
&= [f]\Big(([I]-2[P])([w]-[y])\Big) \\
&\leq \Big|[f]\Big|\Big|([I]-2[P])([w]-[y])\Big| \\
&\leq \lambda\Big\|[I]-2[P]\Big\|_a\Big|[w]-[y]\Big| \\
&\leq \lambda \cdot (1+4(1-a^2)) \cdot \frac{1}{2} \\
&\leq \frac{\lambda}{2}(1+8(1-a^4)).
\end{aligned}$$

Thus, we obtain that $\lambda \geq \dfrac{\sqrt{33}-3}{2\,(1+8(1-a^4))}$ which is a contradiction.

3. Unconditional uncountable basis

In the case of nonseparable spaces we can also obtain some renormings with the w-FPP by using extended basis. We recall [16] (Definition 17.5) that a family $\{x_i : i \in I\}$ of elements in a Banach space X is called an extended unconditional basis of X (or, an unconditional Enflo-Rosenthal set of X), if it is complete in X and if every countable subfamily of $\{x_i : i \in I\}$ is an unconditional basic sequence. This is equivalent ([16], Theorem 17.5) to say that for every $x \in X$ there exists a unique family of scalars $\{t_i : i \in I\}$ such that $\sum_{i\in I} t_i x_i = x$, i.e. for every $\epsilon > 0$ there exists a finite subset A of I such that for every finite subset B of I, $A \subset B$ we have $\| \sum_{i\in B} t_1 x_i - x\| < \epsilon$. We will denote $t_i = f_i(x)$, i.e. $\{f_i : i \in I\}$ are the functional coordinates for the basis. As in the separable case, it can be proved that there exists a constant M such that $\| \sum_{i\in A} t_i x_i\| \leq M\| \sum_{i\in B} t_i x_i\|$ if A and B are finite subsets of I and $A \subset B$. The smallest K satisfying this inequality is called the unconditional constant of $\{x_i : i \in I\}$. If the inequality holds for $M = 1$ we say that $\{x_i : i \in I\}$ is an extended unconditional monotonous basis.

Theorem 3.1. *Let X be a Banach space with an extended unconditional basis with constant $M < \frac{\sqrt{33}-2}{2}$. Then X enjoys the w-FPP.*

Proof. Otherwise there exists a nonexpansive mapping T and a T-minimal invariant convex weakly compact subset K of X. It is known that K must be separable (see [7], page 36). Thus, the set $A = \{i \in I : f_i(x) \neq 0 \text{ for some } x \in K\}$ is countable and $\{x_i : i \in A\}$ is a (countable) unconditional basis for span $\{K\}$ with unconditional constant M. From here, we can follow the same arguments as in [13] (Theorem 2) to prove the result. $\square$

Lemma 3.2. *Assume that $\{x_i : i \in I\}$ is an extended unconditional basis in X. For every $x = \sum_{i\in I} t_i x_i$, the expression $|x| = \sup\{\|\sum_{i\in A} \epsilon_i t_i x_i\| : A \subset I \text{ finite}\}$ where $\varepsilon_i = \pm 1$ defines an equivalent norm on X such that $\{x_i : i \in I\}$ is an extended unconditional monotonous basis for this norm*

Proof. Let A, B finite subsets of I with $A \subset B$. Denote $x = \sum_{i\in B} t_i x_i$, $u = \sum_{i\in A} \epsilon_i t_i x_i$ and $v = \sum_{i\in B\setminus A} \epsilon_i t_i x_i$. We have $|x| \geq \|u + v\|$ and $|x| \geq \|u - v\|$. Thus, $2\|u\| \leq \|u + v\| + \|u - v\| \leq 2|x|$ which implies that $|\sum_{i\in A} t_i x_i| \leq |\sum_{i\in B} t_i x_i|$. $\square$

Theorem 3.3. *Let X be a Banach space which can be isomorphically embedded in a Banach space Z with an extended unconditional basis and $\lambda < (\sqrt{33} - 2)/2$. Then, X has an equivalent norm $|\cdot|$ such that if Y is an isomorphic Banach space and the Banach-Mazur distance between $(X, |\cdot|)$ and Y is less than λ, then Y satisfies the w-fpp.*

Proof. It easily follows the same arguments used in Theorem 2.1. $\square$

Remark. It is known [3] that ℓ_∞ cannot be isomorphically embedded in a Banach space with an extended unconditional basis. This fact is also a consequence of the above theorem, because ℓ_∞ fails the w-FPP and every renorming of ℓ_∞ contains almost isometrically ℓ_∞ [15].

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